New companions to Gordon identities from commutative algebra

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Introduction

A partition of a positive integer n is a non-increasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_s)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_s = n$.

A famous family of partition identities which plays a central role in our paper [1], is due to Gordon:

Theorem 1: Gordon's identities ([6])

Let r and i be integers such that $r \geq 2$ and $1 \leq i \leq r$. Let $\mathcal{T}_{r,i}$ be the set of partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_s)$ where $\lambda_j - \lambda_{j+r-1} \geq 2$ for all j, and at most $i-1$ of the parts λ_j are equal to 1. Let $\mathcal{E}_{r,i}$ be the set of partitions whose parts are not congruent to $0, \pm i \mod (2r+1)$. Let n be a nonnegative integer, and let $T_{r,i}(n)$ (respectively $E_{r,i}(n)$) denote the number of partitions of n which belong to $\mathcal{T}_{r,i}$ (respectively $\mathcal{E}_{r,i}$). Then we have

$$T_{r,i}(n) = E_{r,i}(n).$$

We introduce a new companions $C_{r,i}(n)$ to these identities; This settles positively a conjecture made by the first author. By using the algebro-geometrical methods (see in [2], [3] and [9]) she defines the (i,ℓ)*-new part of* λ, note by $p_{i,\ell}(\lambda)$, and states:

Conjecture 1: (P. A. [2])

Let $r \geq 2$ and $1 \leq i \leq r$ be two integers. Let $\mathcal{C}_{r,i}$ be the set of partitions of the form $\lambda = (\lambda_1, \dots, \lambda_s)$, such that at most $i-1$ of the parts are equal to 1 and either $N_{r,i}(\lambda) < r-1$, or $N_{r,i}(\lambda) = r-1$ and $s \leq \sum_{j=1}^{r-1} p_{i,j}(\lambda) - (r-i)$. Let n be a nonnegative integer, and denote by $C_{r,i}(n)$ the number of partitions of n which belong to $\mathcal{C}_{r,i}$. Then we have

$$C_{r,i}(n) = T_{r,i}(n) = E_{r,i}(n).$$

Andrews–Gordon identities

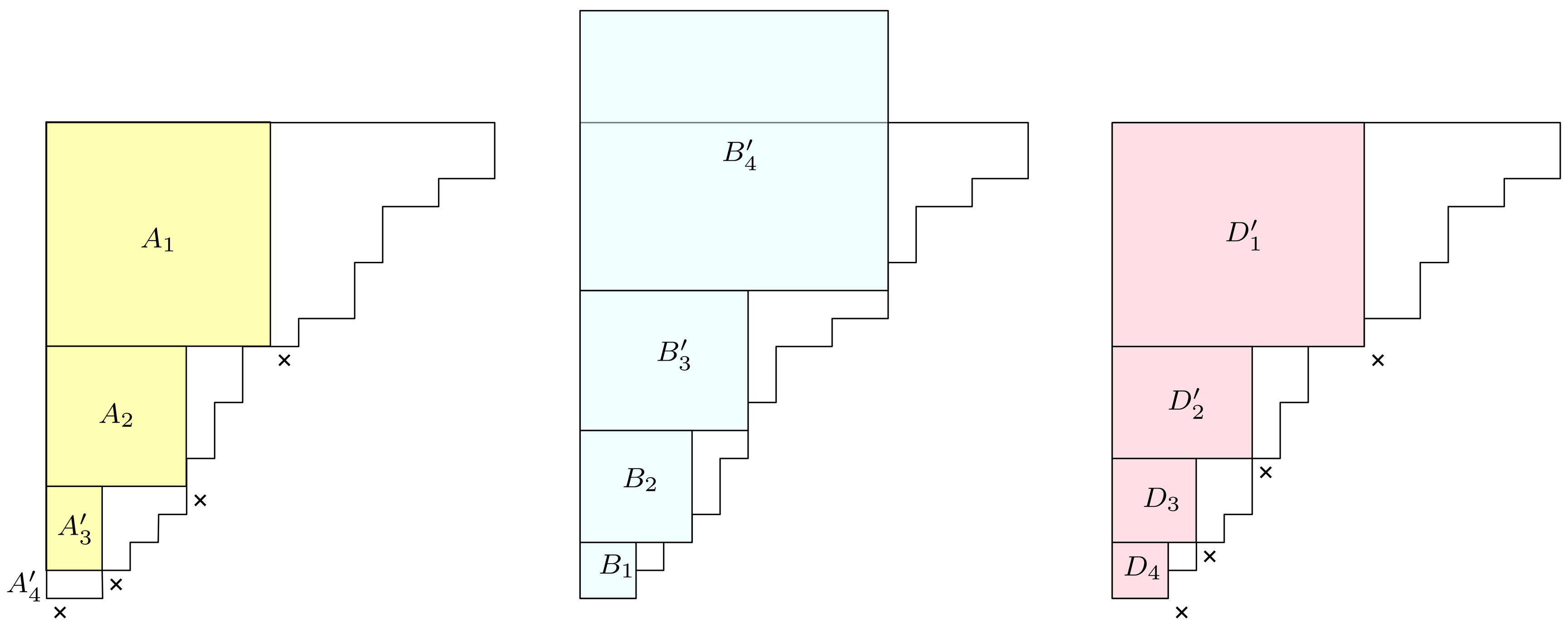

Figure 1: The three types of Durfee dissections

A *Durfee square* of a partition λ is the largest square of size $k \times k$ fitting in the top-left corner of the Young diagram of λ (see for instance A_1 in Figure 1). Similarly we can define its *vertical Durfee rectangle* to be the largest vertical rectangle of size $(k-1) \times k$ fitting in the top-left corner of its Young diagram.

When we choose to draw first $i-1$ Durfee squares, and then all the following ones the rectangles, the sequence of non-empty Durfee squares/rectangles in λ is uniquely defined and is called the *(vertical)* $(i-1)$*-Durfee dissection of* λ.

Theorem 2: Andrews [5]

Let $r \geq 2$ and $1 \leq i \leq r$ be two integers. Let $\mathcal{A}_{r,i}$ be the set of partitions such that in their vertical $(i-1)$-Durfee dissection, all vertical Durfee rectangles below A'_{r-1} are empty, and *such that the last row of each non-empty Durfee rectangle is actually a part of the partition*. For all nonnegative integers n, denote by $A_{r,i}(n)$ the number of partitions of n which belong to $\mathcal{A}_{r,i}$. Then we have

$$A_{r,i}(n) = E_{r,i}(n).$$

The partition in Figure 1 is not in $\mathcal{A}_{5,3}$. The analytic form of these identities can be stated for all integers $r \geq 2$ and $1 \leq i \leq r$ as:

$$\sum_{s_1 \geq \cdots \geq s_{r-1} \geq 0} \frac{q^{s_1^2+\cdots+s_{r-1}^2+s_i+\cdots+s_{r-1}}}{(q)_{s_1-s_2}\cdots(q)_{s_{r-2}-s_{r-1}}(q)_{s_{r-1}}} = \frac{(q^{2r+1}, q^i, q^{2r-i+1}; q^{2r+1})_\infty}{(q)_\infty}. \quad (1)$$

New Durfee-type dissections

In [1], we define the *bottom square* (resp. *bottom rectangle*) of a partition $\lambda = (\lambda_1, \dots, \lambda_s)$ to be the square of size $\lambda_s \times \lambda_s$ (resp. the horizontal rectangle of size $\lambda_s \times (\lambda_s - 1)$) whose bottom coincides with the bottom of the Young diagram of λ.

When we choose to draw first the $i-1$ bottom squares and then all following ones the rectangles, the sequence of non-empty bottom squares/rectangles in λ is uniquely defined and we call it the $(i-1)$*-bottom dissection* of λ (see for instance the middle of Figure 1 where the bottom rectangles above B'_4 are empty).

Let $\mathcal{B}_{r,i}$ be the set of partitions such that in their $(i-1)$-bottom dissection, all bottom rectangles above B'_{r-1} are empty. By definition of bottom squares/rectangles, for all $1 \leq i \leq r$, we have $\mathcal{B}_{r,i} = \mathcal{C}_{r,i}$ and so:

Conjecture 2: Reformulation of Conjecture 1 (P.A., J. D., F. J. & H. M. [1])

Let $r \geq 2$ and $1 \leq i \leq r$ be two integers. Then for all nonnegative integers n, we have $B_{r,i}(n) = A_{r,i}(n) = E_{r,i}(n)$.

Now define the *horizontal Durfee rectangle* of a partition λ to be the largest horizontal rectangle of size $k \times (k-1)$ fitting in the top-left corner of the Young diagram of λ.

When we choose to draw first k Durfee horizontal rectangles and then following all squares, the sequence of non-empty Durfee squares/rectangles in λ is uniquely defined and is called the k*-Durfee dissection* of λ.

Define $\mathcal{D}_{r,i}$ to be the set of partitions such that in their $(r-i)$-Durfee dissection, all Durfee squares below D_{r-1} are empty. For example, the partition in Figure 1 belongs to $\mathcal{D}_{5,3}$ but not to $\mathcal{D}_{4,3}$. We showed that the following holds.

Theorem 3: (P.A., J. D., F. J. & H. M. [1])

For $r \geq 2$ and $1 \leq i \leq r$ two integers, we have $\mathcal{B}_{r,i} = \mathcal{D}_{r,i}$.

Proof strategy for Conjecture 2

In [1], we computed the generating function for partitions in $\mathcal{D}_{r,i} = \mathcal{B}_{r,i}$ and we get (replacing $1 - q^{d_0}$ by 1)

$$\sum_{d_1 \geq \cdots \geq d_{r-1} \geq 0} \frac{q^{d_1^2+\cdots+d_{r-1}^2-d_1-\cdots-d_{r-i}}}{(q)_{d_1-d_2}\cdots(q)_{d_{r-2}-d_{r-1}}(q)_{d_{r-1}}}(1-q^{d_{r-i}}). \quad (2)$$

Our proof of Conjecture 2 actually consist in showing that the generating function (2) of $\mathcal{D}_{r,i}$ equals the infinite product which is the generating function for $\mathcal{E}_{r,i}$. This is done by proving the following theorem:

Theorem 4: (P.A., J. D., F. J. & H. M. [1])

For all integers $r > 0$ and $0 \leq i \leq r-1$, we have:

$$\sum_{s_1 \geq \cdots \geq s_{r-1} \geq 0} \frac{q^{s_1^2+\cdots+s_{r-1}^2-s_1-\cdots-s_i}(1-q^{s_i})}{(q)_{s_1-s_2}\cdots(q)_{s_{r-2}-s_{r-1}}(q)_{s_{r-1}}} = \frac{(q^{2r+1}, q^{r-i}, q^{r+i+1}; q^{2r+1})_\infty}{(q)_\infty}, \quad (3)$$

where for $i = 0$, the term $1 - q^{s_0}$ on the left-hand side is simply taken to be 1.

Indeed, the right-hand side (resp. left-hand side) of (3) is the generating series for $\mathcal{E}_{r,r-i}$ (resp. $\mathcal{D}_{r,r-i}$) obtained by taking $r-i$ instead of i in the right-hand side of (1) (resp. in (2)).

This shows that Conjecture 2 (and therefore Conjecture 1) is an immediate consequence of Theorem 4 and Theorem 3.

Proof of Theorem 4 via the Bailey lattice

Fix a formal indeterminate a. Recall [7] that a Bailey pair $(\alpha_n, \beta_n)_{n \geq 0}$ related to a is a pair of sequences satisfying:

$$\beta_n = \sum_{j=0}^{n} \frac{\alpha_j}{(q)_{n-j}(aq)_{n+j}} \qquad \forall\, n \in \mathbb{N}. \quad (4)$$

Theorem 5: Bailey lemma, special case

If (α_n, β_n) is a Bailey pair related to a, then so is (α'_n, β'_n), where

$$\alpha'_n = a^n q^{n^2} \alpha_n \quad \text{and} \quad \beta'_n = \sum_{j=0}^{n} \frac{a^j q^{j^2}}{(q)_{n-j}} \beta_j.$$

Theorem 6: Bailey lattice, special case

If (α_n, β_n) is a Bailey pair related to a, then (α'_n, β'_n) is a Bailey pair related to a/q, where $\alpha'_0 = \alpha_0$,

$$\alpha'_n = (1-a)a^n q^{n^2-n}\left(\frac{\alpha_n}{1-aq^{2n}} - \frac{aq^{2n-2}\alpha_{n-1}}{1-aq^{2n-2}}\right) \quad \text{and} \quad \beta'_n = \sum_{j=0}^{n} \frac{a^j q^{j^2-j}}{(q)_{n-j}} \beta_j.$$

In [7], the following unit Bailey pair (related to a) is considered:

$$\alpha_n^{(0)} = \frac{(-1)^n q^{n(n-1)/2}(1-aq^{2n})(a)_n}{(1-a)(q)_n}, \qquad \beta_n^{(0)} = \delta_{n,0}, \quad (5)$$

Iterating $r \geq 2$ times Theorem 5 for the unit Bailey pair (5) yields a new Bailey pair $(\alpha_n^{(r)}, \beta_n^{(r)})$ with

$$\alpha_n^{(r)} = a^{rn} q^{rn^2} \alpha_n^{(0)},$$

and

$$\beta_n^{(r)} = \sum_{n \geq s_1 \geq \cdots \geq s_r \geq 0} \frac{a^{s_1+\cdots+s_r} q^{s_1^2+\cdots+s_r^2}}{(q)_{n-s_1}(q)_{s_1-s_2}\cdots(q)_{s_{r-1}-s_r}} \beta_{s_r}^{(0)}.$$

Applying the definition of a Bailey pair to $(\alpha_n^{(r)}, \beta_n^{(r)})$, letting $n \to \infty$ and taking $a = 1$ gives

$$\sum_{s_1 \geq \cdots \geq s_{r-1} \geq 0} \frac{q^{s_1^2+\cdots+s_{r-1}^2}}{(q)_{s_1-s_2}\cdots(q)_{s_{r-1}}} = \frac{(q^{2r+1}, q^r, q^{r+1}; q^{2r+1})_\infty}{(q)_\infty}.$$

by taking $q \to q^{2r+1}$, $z \to q^r$ in the Jacobi triple product identity [8, Appendix, (II.28)]

$$\sum_{j \in \mathbb{Z}} (-1)^j z^j q^{j(j-1)/2} = (q, z, q/z; q)_\infty. \quad (6)$$

Therefore we get the $i = 0$ case of (3) (equivalently the $i = r$ instance of (1)). In the same way, one gets the $i = r-1$ case of (3) (equivalently the $i = 1$ instance of (1)) by choosing $a = q$ above.

For the other cases, we use Theorem 6 obtained in [4, Corollary 4.2] by iterating $r - i$ times Theorem 5, then using Theorem 6, and finally $i-1$ times Theorem 5 with a replaced by a/q, and at the end letting $n \to \infty$.

Taking at last $a = q$ as done in [4] gives (1), while for proving (3) we need to choose $a = 1$ and use a few computation tricks.

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