A description of minimal elements of Shi regions in classical Weyl groups

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Introduction

In this extended abstract, we show how a bijection between parking functions and regions of the Shi arrangement from [3] (in type A_n) and [4] (in type B_n, C_n, D_n) allows for the computation of the minimal elements of the Shi regions. This gives a combinatorial interpretation of these minimal elements: they can be seen as counting non-crossing arcs in non-nesting arc diagrams.

Affine Weyl groups

Let Φ be a irreducible crystallographic root system, Φ^+ the positive roots, W the associated Weyl group and $\forall(\alpha,k)\in\Phi^+\times\mathbb{Z}$, define:

$$s_{\alpha,k} = x - 2(\langle x|\alpha\rangle - k)\frac{\alpha}{\langle\alpha|\alpha\rangle} \text{ and}$$

$$H^s_{\alpha,k} = \{x \mid \operatorname{sign}(\langle x|\alpha\rangle) = s\}, s\in\{-,0,+\}.$$

The *affine Weyl group* $\widetilde{W}$ is the group generated by all $s_{\alpha,k}$. The *alcoves* are the connected components of the complement of $\bigcup_{\Phi^+\times\mathbb{Z}} H^0_{\alpha,k}$. If we decide that the only alcove in $\bigcap_{\Phi^+} H^+_{\alpha,0}$ touching the origin corresponds to the identity, there is a natural bijection between elements of $\widetilde{W}$ and the set of alcoves.

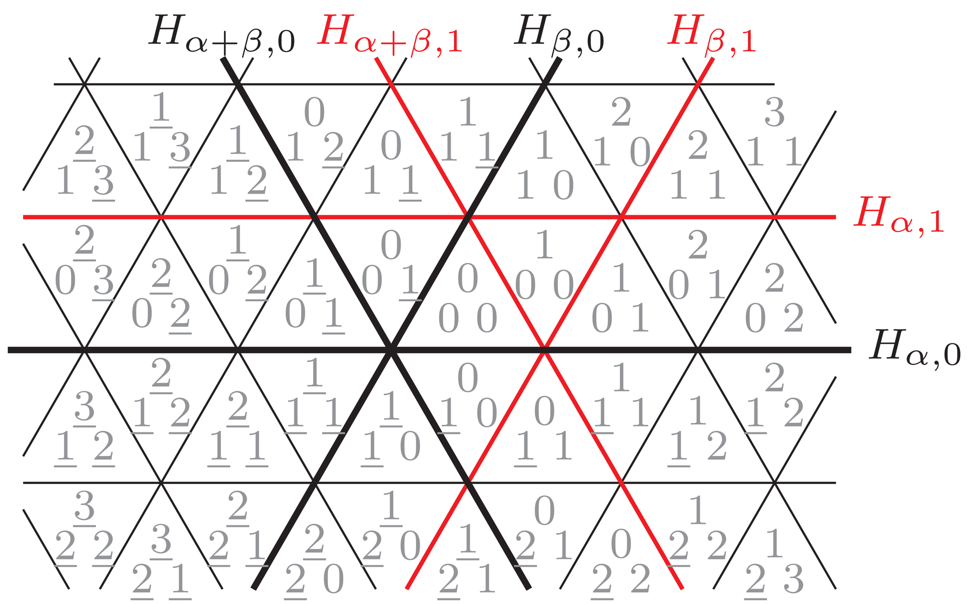


Figure 1: The case of $\widetilde{A}_2$: $\alpha = e_1 - e_2, \beta = e_2 - e_3$. The identity element corresponds to the alcove labeled by $0,0,0$. The underlined coefficients are negative.

Shi encoding

As the hyperplanes $H^0_{\alpha,k}$ are all parallels to $H^0_{\alpha,0}$, an element can be encoded by the number of hyperplanes separating its alcove A_w from the identity element (with a minus sign if $A_w \subset H^-_{\alpha,k}$), that is, by the integer vector:

$$K(w) = (\max(i\in\mathbb{Z}|A_w\subset H^+_{\alpha,k}))_{\alpha\in\Phi^+}$$

Theorem ([1]). *$K : \widetilde{W} \longmapsto \mathbb{Z}^{\Phi^+}$ is an injection with image the set of vectors v such that $\forall,\alpha,\beta,\gamma\in\Phi^+$, $\gamma=\alpha+\beta \implies \exists\varepsilon\in\{0,1\}$ such that $v_\gamma = v_\alpha + v_\beta + \varepsilon$.*

Shi arrangement

The *Shi arrangement* is $\mathcal{A}_{\widetilde{A}_n} = \bigcup_{\Phi^+\times\{0,1\}} H^0_{\alpha,k}$. The regions of the Shi arrangement can be seen as sets of elements of $\widetilde{W}$. A region R is caracterized by $\operatorname{sign}(R) = \operatorname{sign}(K(w))$ for any $w\in R$.

Theorem ([2]). *Given a region R there exist a unique element $\min(R)$ such that $\forall\alpha\in\Phi^+, K(\min(R))_\alpha = \min(K(w)_\alpha | w\in R)$.*

For instance, the minimum element of the region with sign $(+,-,-)$ is $(1,-1,-2)$.

Type A: The Athanasiadis-Linusson bijection

Following [3], a *type A_n parking function* is a permutation π of $[\![1,n+1]\!]$ along with a non-crossing partition P such that the blocks of P are sorted in π.

In type A_{n-1}, $\Phi^+ = \{e_i - e_j | 1\le i<j\le n+1\}$ thus if v is a vector indexed by Φ^+, we note $v_{i,j}$ instead of $v_{e_i-e_j}$.

Theorem ([3]). *The following procedure is a bijection between Shi regions of $\mathcal{A}_{\widetilde{A}_n}$ and type A_n parking functions. Given a sign type v, construct a permutation π such that (i,j) is an inversion if and only if $v_{i,j} = -$. Then for each (i,j) such that $v_{i,j} = +$, draw an arc between the values i and j in π. Remove any arc containing another to get P.*

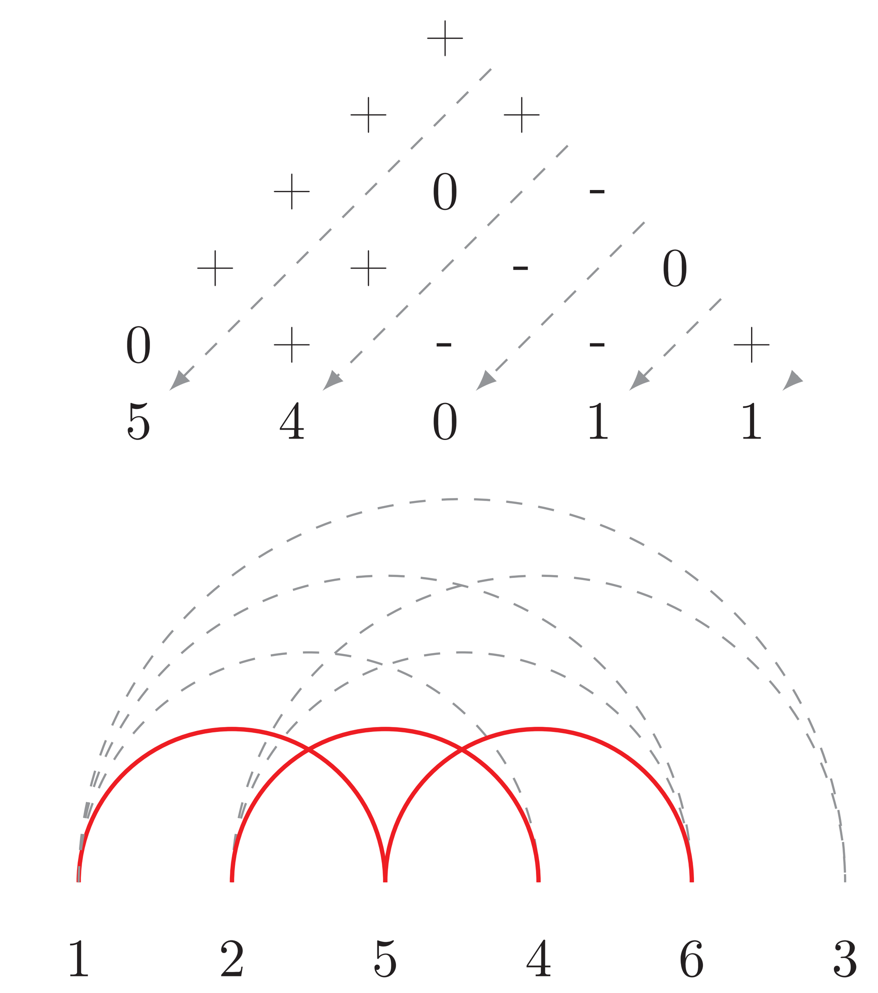
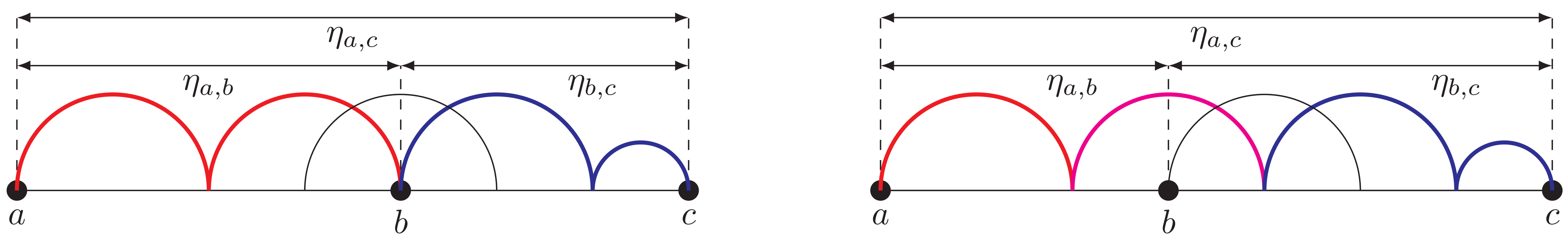


Figure 2: Example for A_5: the sign type v is given as a pyramid: $v_{i,j}$ is the i-th sign from the left on the $j-i$-th row from the bottom. For instance, the "middle 0" is $v_{2,5}$.

The main ingredient: an obvious lemma

Lemma. *Let P be a non-nesting partition and let $\eta_{i,j}$ be the maximal number of non-crossing arcs between a and b. Then for all i,j,k, there is a $\varepsilon\in\{0,1\}$ such that $\eta_{i,k} = \eta_{i,j} + \eta_{j,k} + \varepsilon$.*



Main result in type A

Proposition. *Let v be the sign of a Shi region R and (π,P) the parking function associated to it by [AL '99]. Then:*

$$\min(R)_{i,j} = \begin{cases} \eta_{i,j} & \text{if } v_{i,j}\in\{0,+\} \\ -(\eta_{i,j}+1) & \text{if } v_{i,j} = - \end{cases}$$

Proof. Check that permuting i,j,k in the lemma gives coefficients respecting the Shi relations. Prove they are minimal by induction on $|\pi^{-1}(i) - \pi^{-1}(j)|$. $\square$

Type W parking functions

In general, for a Weyl group W, a non-crossing partition is an antichain in the *root poset* $(\Phi^+,\le)$ where $\alpha\ge\beta$ if $\alpha-\beta\in\mathbb{N}\Phi^+$.

Theorem ([4]). *There is a bijection, similar to that of [3] between type $\widetilde{W}$ Shi regions and type W parking functions, that is pairs (π,P) with $\pi\in W$ and P a non-crossing partition such that for all $\alpha\in P, \pi(\alpha)\notin\Phi^+$.*

Morally, π encodes the position of R with respect to the $H^0_{\alpha,0}$ while P encodes the $H^0_{\alpha,1}$.

Generalizing to classical Weyl types

The classical Weyl group A_n, B_n, C_n, D_n can be realized as permutation groups, and the table below gives a way to associate non-crossing partitions with sets of arcs. We need to check that a non-crossing partition gives non-crossing arcs and that a version of the Lemma applies.

Root	$e_i - e_j$ $(ABCD)$	$e_i + e_j$ (BCD)	$2e_i$ (C)	e_i (B)
Extremities	i to j and $-j$ to $-i$	i to $-j$ and j to $-i$	i to $-i$	i to 0 and 0 to $-i$

In type B, C: write the permutation encoded by the signs in the format

$$\pi(1)\ \cdots\ \pi(n)\ 0\ \pi(-n)\ \cdots\ \pi(-1).$$

By convention, the identity written in this format is sorted, hence a parking function corresponds to a pair (π,P) with π in B_n/C_n and P with sorted blocks. The Lemma applies. Note that the use of the 0 for B_n but not for C_n stems from the fact that B_n and C_n don't have the same root poset.

In type D: write the permutation in the format:

$$\pi(1)\ \cdots\ \pi(n-1)\ \begin{matrix}\pi(n)\\ \pi(-n)\end{matrix}\ \pi(1-n)\ \cdots\ \pi(-1).$$

Let $\eta^+_{a,b}$ be as before except the count ignores $\pi(-n)$ (and similarly $\eta^-_{a,b}$). The Lemma applies to $\max(\eta^+_{a,b},\eta^-_{a,b})$.

In both case, the proof is the same as in type A.

References

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