Walks in simplices, cylindric tableaux, and asymmetric exclusion processes

Sergi Elizalde
Dartmouth College

FPSAC 2022, Bangalore (India)
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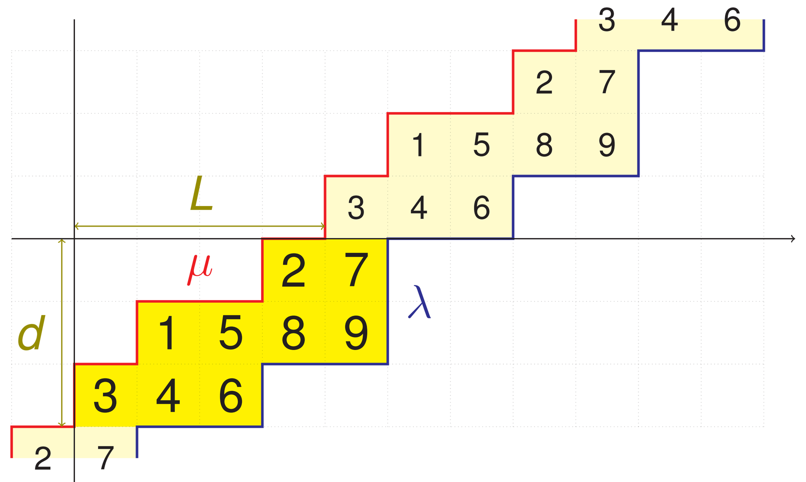
Standard cylindric tableaux (SCT)

An SCT of period $(d, L) = (3, 4)$ with inner shape $\mu = [3, 1, 0]$ and outer shape $\lambda = [5, 5, 3]$:



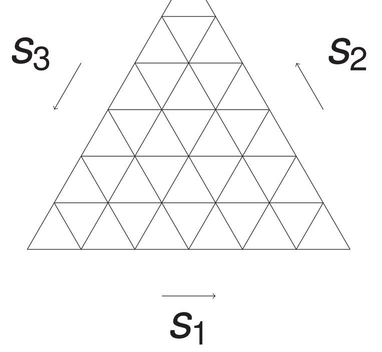
Cylindric partitions were introduced by Gessel and Krattenthaler [2], and semistandard cylindric tableaux have been studied by Postnikov [6] in connection to Gromov–Witten invariants, and by Neyman [5] in connection to RSK. The resulting cylindric Schur functions have been further studied by McNamara [3].

Walks in simplicial regions

Consider walks in

$$\Delta_{d,L} = \{(x_1, x_2, \ldots, x_d) \in \mathbb{N}^d : x_1 + x_2 + \cdots + x_d = L\}$$

with steps $s_i = e_{i+1} - e_i$ for $1 \le i \le n$, with the convention $e_{d+1} := e_1$.



Theorem 1 (Mortimer–Prellberg [4])

The number of n-step walks in $\Delta_{3,L}$ starting at $(L, 0, \ldots, 0)$ equals the number of certain Motkin paths of bounded height.

A complicated bijective proof is given in [1], along with the following.

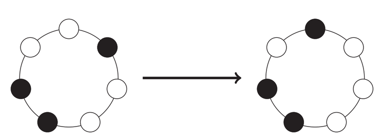
Theorem 2 (Courtiel–Elvey Price–Marcovici [1])

For any $\mathbf{x} \in \Delta_{d,L}$, there is a bijection

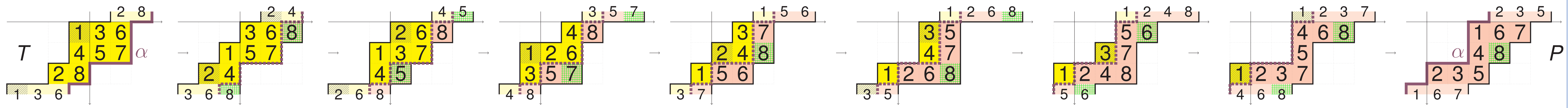
$$\{n\text{-step walks starting at } \mathbf{x}\} \longleftrightarrow \{n\text{-step walks ending at } \mathbf{x}\}$$

Totally asymmetric simple exclusion process (TASEP)

States of the TASEP on the cycle are binary words with d ones (representing particles) and L zeros. Each particle can jump counterclockwise if the adjacent site is empty. Let $\mathcal{E}_{d,L}$ be the underlying graph.

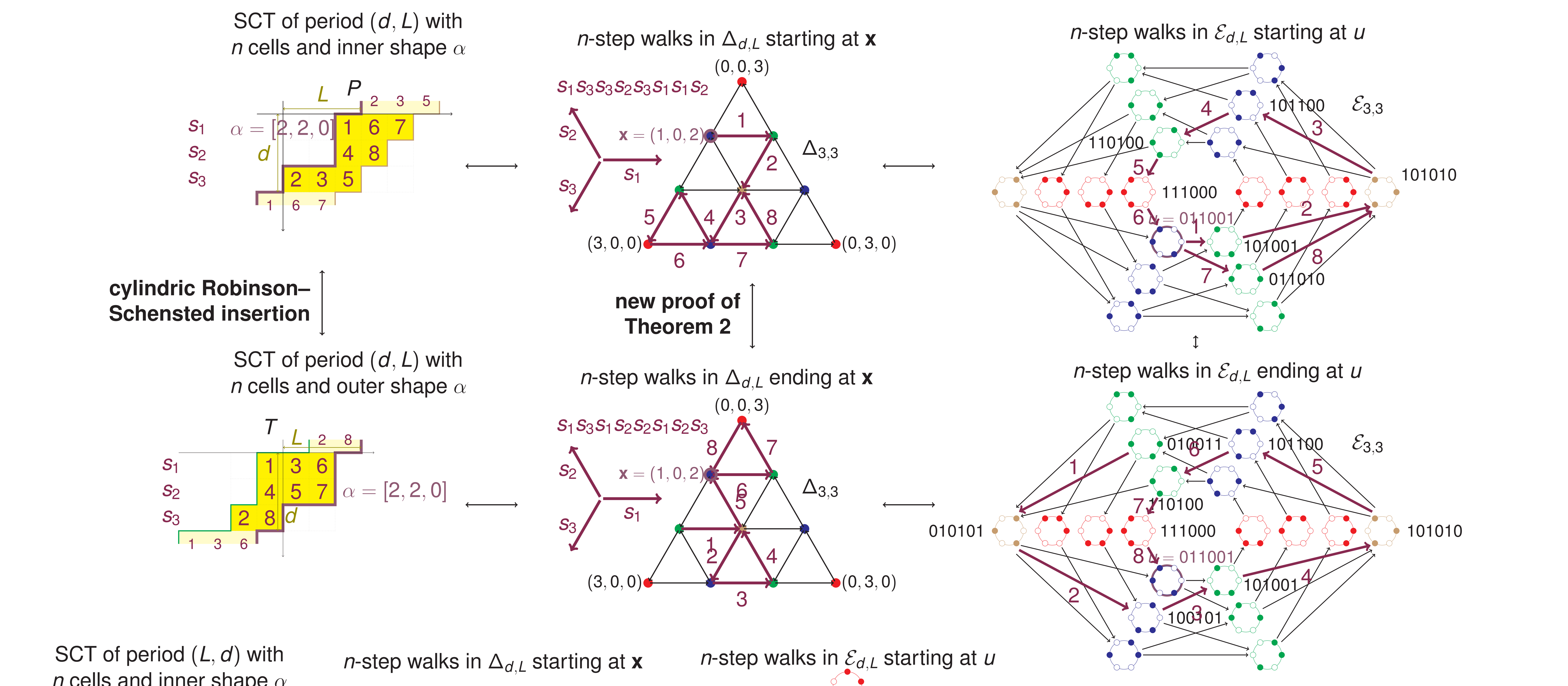


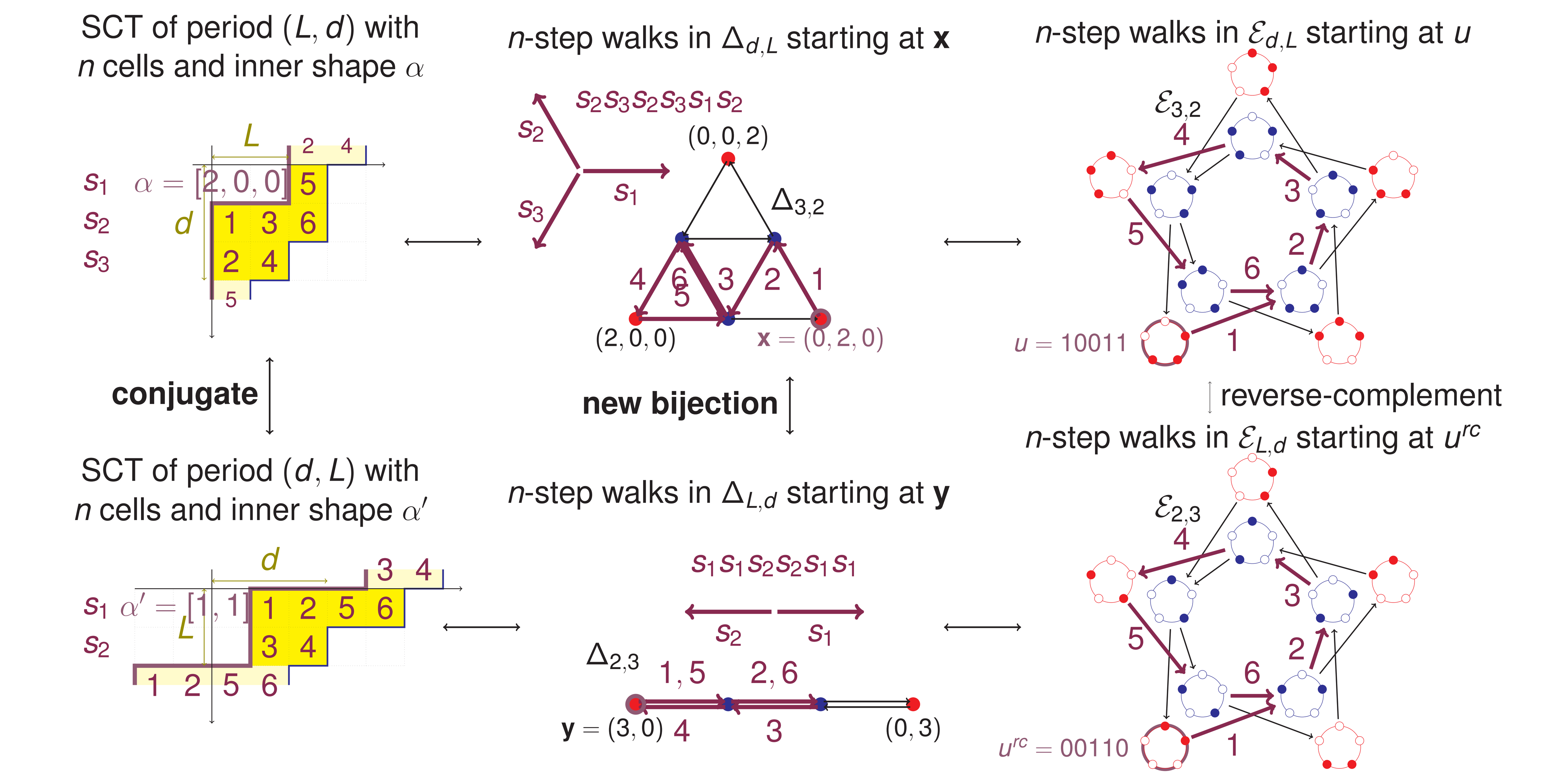
Cylindric Robinson–Schensted insertion [5]



Our bijections

Let α be a cylindric shape if period (d, L), let $\mathbf{x} = (x_1, x_2, \ldots, x_d) \in \Delta_{d,L}$ where $x_i = \alpha_{i-1} - \alpha_i$ for $1 \le i \le d$, and let $u = 0^{x_1}10^{x_2}1\ldots0^{x_d}1$.
Let α' be the conjugate of α, let $\mathbf{y} = (y_1, y_2, \ldots, y_d) \in \Delta_{d,L}$ where $y_j = \alpha'_{j-1} - \alpha_j$ for $1 \le j \le L$, and let $u^{rc} = 01^{x_d}01^{x_{d-1}}\ldots01^{x_1}$.





References

[1] J. Courtiel, A. Elvey Price and I. Marcovici, Bijections between walks inside a triangular domain and Motzkin paths of bounded amplitude, *Electron. J. Combin.* 28, #2.6 (2021).

[2] I. M Gessel and C. Krattenthaler, Cylindric partitions, *Trans. Amer. Math. Soc.* 349, 429–479 (1997).

[3] P. McNamara, Cylindric skew Schur functions, *Adv. Math.* 205, 275–312 (2006).

[4] P. Mortimer and T. Prellberg, On the number of walks in a triangular domain, *Electron. J. Combin.* 22, #1.64 (2015).

[5] E. Neyman, Cylindric Young Tableaux and their Properties, *arXiv:1410.5039 [math]*.

[6] A. Postnikov, Affine approach to quantum Schubert calculus, *Duke Math. J.*, 128, 473–509 (2005).

Department of Mathematics, Dartmouth College, 6188 Kemeny Hall, Hanover, NH 03784, USA
E-mail: sergi.elizalde@dartmouth.edu
http://www.math.dartmouth.edu/~sergi