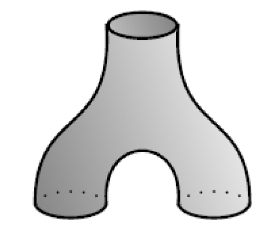


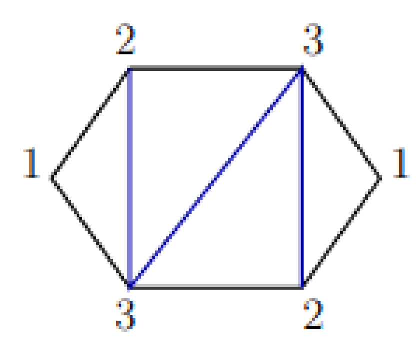
Friezes for a pair of pants

Ana Garcia Elsener[1*] (*elsener@mdp.edu.ar*), I. Canakci[2], A. Felikson[3], P. Tumarkin[3]
[1] Universidad Nacional de Mar del Plata [2] Vrije Universiteit Amsterdam [3] Durham University

1 Generalizing Conway-Coxeter friezes

Frieze patterns ([2, 3]) are arrangemnts of numbrers such that the *diamond rule* $ab - cd = 1$ holds for each diamond.

	1	1	1	1	1	1	1	
…	1	3	2	1	3	2	…	
	1	2	5	1	2	5	1	
…	1	3	2	1	3	2	…	
	1	1	1	1	1	1	1	



The numbers coincide with possible evaluations from a cluster algebra $\mathcal{A}(Q)$ of type $\mathbb{A}$ to the positive integers! We use a more general definition of (positive) frieze inspired in the theory of cluster algebras.

Definitions

A cluster algebra $\mathcal{A}(Q)$ is a subalgebra of an ambient field $\mathbb{Q}(x_1, \ldots, x_n)$ generated by combinatorially defined elements called *cluster variables* x, which are grouped into overlapping sets called *clusters* X of constant cardinality. Different clusters are obtained from each other by sequences of mutations, starting from a pair (X, Q) called initial seed.

1. A **frieze** associated to $\mathcal{A}(Q)$ is a **ring homomorphism** $\lambda : \mathcal{A}(Q) \to \mathbb{Z}$.
2. A frieze λ is **positive** if for any cluster variable $x \in \mathcal{A}(Q)$, its image $\lambda(x)$ is in $\mathbb{Z}_+$.
3. A positive frieze λ is **unitary** if there exists a cluster X in $\mathcal{A}(Q)$ such that every cluster variable $x_i \in X$ is mapped to 1 by λ.

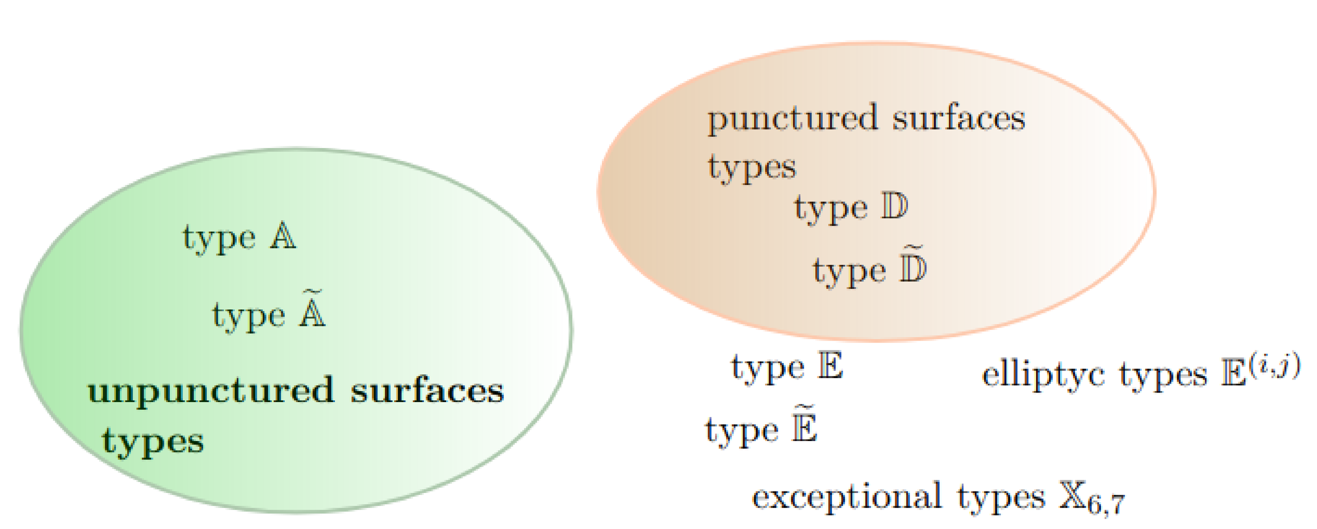
The The Theorem by Conway-Coxeter relating friezes and triangulations of the polygon implies that:

Theorem: [3] All (positive) friezes from cluster algebras of type $\mathbb{A}$ are unitary.

2 Known results on positive friezes

Now that we have a generalized notion of frieze and a property that holds for friezes of type $\mathbb{A}$, it is natural to ask if other cluster algebra types will have the same properties.

Finite mutation (simply laced) cluster types:



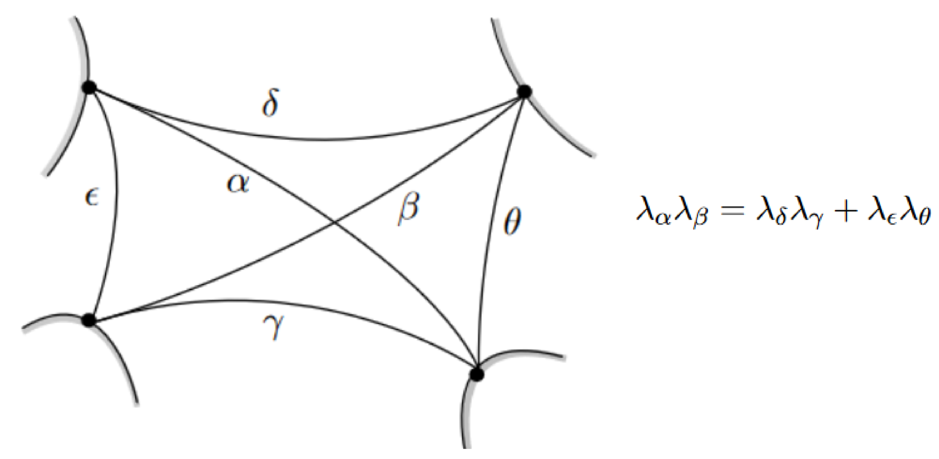
1. Baur-Marsh [1] (and Thomas) (2008) study positive friezes **in type $\mathbb{D}$. In particular, there are non-unitary examples.**
2. Fontaine-Plamondon [5] (2014) count **non-unitary friezes in type $\mathbb{E}_6$**
3. Gunawan-Schiffler [6] (2020) prove that **in type $\widetilde{\mathbb{A}}$ (unpunctured annulus) all friezes are unitary.**
4. Gunawan-Schiffler [6] (2020) examples of **non-unitary friezes in types $\widetilde{\mathbb{D}}, \widetilde{\mathbb{E}}$**

3 Friezes for a pair of pants.

From the information above it is natural to ask: Are there other unpunctured surface types such that all friezes are unitary?

Cluster algebras arising form surfaces are very related to hyperbolic geometry and Teichmüller theory. ([4, 7])

λ-lengths of arcs $\Longleftrightarrow$ cluster variables
triangulations $\Longleftrightarrow$ clusters
Ptolemy relations $\Longleftrightarrow$ algebraic relations in $\mathcal{A}(Q)$

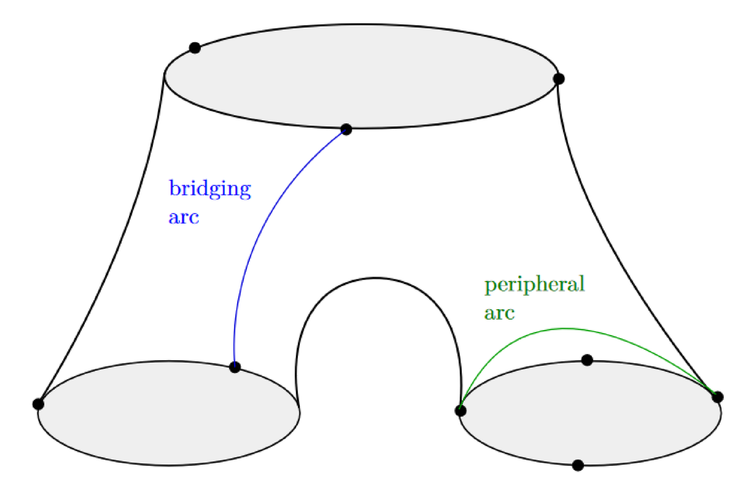


Theorem: All (positive) friezes defined from the cluster algebra arising from the pair of pants are unitary.

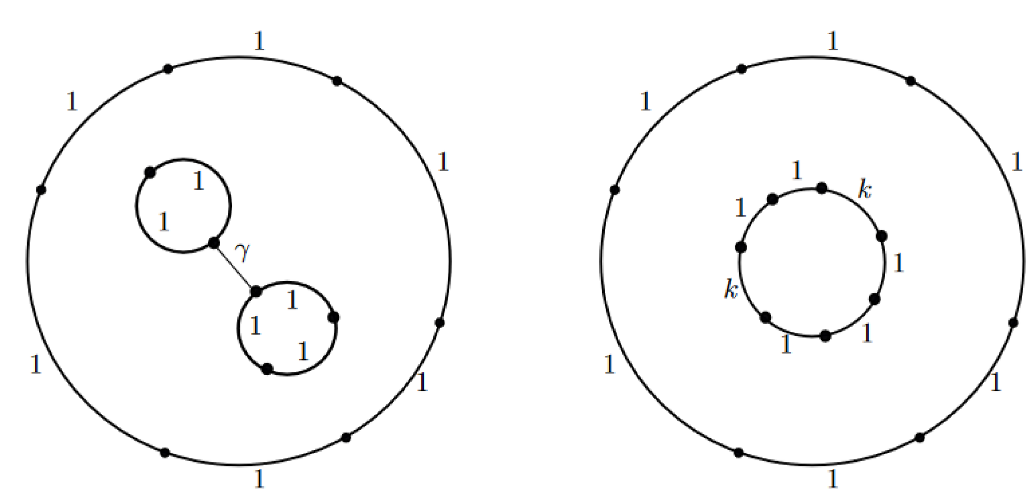
Strategy of the proof:

All arcs have λ-length in $\mathbb{Z}^+$ and all boundary segment have λ-length 1. We assume there is no triangulation with all λ-lengths 1.

1) We can reduce *peripheral arcs* with length 1.
2) Take the *bridging arc* γ with minimal λ-length k (we can suppose $k > 1$, if it is not the case we can find a triangulation with all lengths 1). Use γ to cut the surface. It *becomes an annulus* where some boundary segments have length k.



3) Define a triangulation by bridging arcs for this special annulus. Start by selecting an *arc α_0 with minimal length* a_0 and continue recursively, until you have a triangulation.



4) The triangulation constructed for the annulus will have a sequence of arcs $\alpha_0, \alpha_1, \ldots, \alpha_t$ with an associated *sequence of lengths*

$$a_0 \leq a_1 \leq \cdots \leq a_t$$

but we know, by simple arithmetic, that this sequence has to have a strict inequality! This will produce an absurd. This starts from assuming $k > 1$, so $k = 1$ and we can find a triangulation with all lengths equal to 1.

Remark: This proof cannot be directly extrapolated to other unpunctured surfaces.

**But this theorem rises the question:
are all friezes from unpunctured surfaces unitary?**

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Acknowledgements

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