Combinatorics of Newell-Littlewood numbers

Shiliang Gao[1], Gidon Orelowitz[1], Nicolas Ressayre[2] and Alexander Yong[1]

[1]University of Illinois Urbana-Champaign, [2]Institut Camille Jordan (ICJ), Université Claude Bernard Lyon I

Abstract

We give an exposition of recent developments in the study of Newell-Littlewood numbers. These are the tensor product multiplicities of Weyl modules in the stable range. They are also the structure coefficients of the Koike-Terada basis of the ring of symmetric functions. Two types of combinatorial results are exhibited, those obtained combinatorially starting from the definition of the numbers, and those that also employ geometric and/or representation theoretic methods.

Introduction

The *Newell-Littlewood numbers* are defined by

$$N_{\mu,\nu,\lambda} = \sum_{\alpha,\beta,\gamma} c^{\mu}_{\alpha,\beta} c^{\nu}_{\alpha,\gamma} c^{\lambda}_{\beta,\gamma},$$

where the indices are in Par_n, partitions of at most n parts, and $c^{\mu}_{\alpha,\beta}$ is the *Littlewood-Richardson coefficient*.

Let W be a complex vector space with a nondegenerate symplectic or orthogonal form ω. Let G be the subgroup of $\mathsf{SL}(W)$ preserving ω. Then $\mathsf{G} = \mathsf{SO}_{2n+1}, \mathsf{Sp}_{2n}$ or SO_{2n}. These are groups in the B_n, C_n, D_n series of the Cartan-Killing classification, respectively.

If $\lambda \in \mathsf{Par}_n$, H. Weyl's construction gives a G-module $\mathbb{S}_{[\lambda]}(W)$. These modules are irreducible, except in type D_n, where irreducibility holds if $\lambda_n = 0$.

$N_{\mu,\nu,\lambda}$ as tensor product multiplicities

In the *stable range* $\ell(\mu) + \ell(\nu) \leq n$,

$$\mathbb{S}_{[\mu]}(W) \otimes \mathbb{S}_{[\nu]}(W) \cong \bigoplus_{\lambda \in \mathsf{Par}_n} \mathbb{S}_{[\lambda]}(W)^{\oplus N_{\mu,\nu,\lambda}}.$$

In particular, $N_{\mu,\nu,\lambda}$ is independent of G.

The Schur functions s_λ form a basis of the ring Λ of symmetric functions. It is the "universal character" of $\mathbb{S}_\lambda(V)$ for GL. In a similar fashion, Koike-Terada establish universal characters of $\mathbb{S}_{[\lambda]}(W)$ for Sp.

$N_{\mu,\nu,\lambda}$ from symmetric functions

This *Koike-Terada basis* $\{s_{[\lambda]}\}$ of Λ satisfies

$$s_{[\mu]} s_{[\nu]} = \sum_{\lambda} N_{\mu,\nu,\lambda} s_{[\lambda]}.$$

Tableaux combinatorics and shape of $s_{[\mu]}s_{[\nu]}$

Let $\mu\Delta\nu = (\mu\backslash\nu)\cup(\nu\backslash\mu)$ be the symmetric difference of the Young diagrams of λ and μ. Define Par to be the set of all integer partitions.

Shape of $s_{[\mu]}s_{[\nu]}$

Fix $\mu, \nu \in \mathsf{Par}$.

- Let $k \in \mathbb{Z}_{\geq 0}$. There exists $\lambda \in \mathsf{Par}$ with $|\lambda| = k$ and $N_{\mu,\nu,\lambda} > 0$ if and only if
  $k \equiv |\mu\Delta\nu| \pmod 2$ and $|\mu\Delta\nu| \leq k \leq |\mu| + |\nu|$.
- If $N_{\mu,\nu,\lambda} > 0$ with $|\lambda| > |\mu\Delta\nu|$, there exists $\lambda^{\downarrow\downarrow}$ such that $N_{\mu,\nu,\lambda^{\downarrow\downarrow}} > 0$, $\lambda^{\downarrow\downarrow} \subset \lambda$ and $|\lambda^{\downarrow\downarrow}| = |\lambda| - 2$.
- If $N_{\mu,\nu,\lambda} > 0$ with $|\lambda| < |\mu| + |\nu|$, there exists $\lambda^{\uparrow\uparrow}$ such that $N_{\mu,\nu,\lambda^{\uparrow\uparrow}} > 0$, $\lambda \subset \lambda^{\uparrow\uparrow}$ and $|\lambda^{\uparrow\uparrow}| = |\lambda| + 2$.

This theorem is proved in [1] using Young tableau combinatorics based on a *demotion* procedure. In [2] it is further studied in connection to the Robinson-Schensted-Knuth correspondence.

Newell-Littlewood polytope

Fix $\lambda, \mu, \nu \in \mathsf{Par}_n$.

$N_{\mu,\nu,\lambda}$ as lattice points

There is a polytope $\mathcal{P}_{\mu,\nu,\lambda} \subset \mathbb{R}^{3n^2}$ such that

$$N_{\mu,\nu,\lambda} = \#(\mathcal{P}_{\mu,\nu,\lambda} \cap \mathbb{Z}^{3n^2}).$$

Define the *Newell-Littlewood function* to be $\mathfrak{N}_{\mu,\nu,\lambda} : \mathbb{Z}_{\geq 1} \to \mathbb{N}$ by $k \mapsto N_{k\mu,k\nu,k\lambda}$.

Non-polynomiality

There exist λ, μ, ν such that $\mathfrak{N}_{\mu,\nu,\lambda}(k)$ is not a polynomial in k.

A special case of $\mathfrak{N}_{\mu,\nu,\lambda}$ is the *Littlewood-Richardson polynomial* $\mathfrak{c}^{\lambda}_{\mu,\nu}$. That is, when $|\nu| = |\lambda| + |\mu|$, $\mathfrak{N}_{\mu,\nu,\lambda} = \mathfrak{c}^{\lambda}_{\mu,\nu}$ is in fact a polynomial.

Geometric aspects of $N_{\mu,\nu,\lambda}$

We now turn to the results of [2], whose proofs rely on a mix of geometry and combinatorics. Fix $n \in \mathbb{N}$. Let $\text{NL-semigroup}(n) = \{(\lambda, \mu, \nu) \in (\text{Par}_n)^3 : N_{\lambda,\mu,\nu} > 0\}$. Indeed, NL-semigroup is a finitely generated semigroup. An approximation of it is the saturated cone:

$$\text{NL-sat}(n) = \{(\lambda, \mu, \nu) \in (\text{Par}_n^{\mathbb{Q}})^3 : \exists t > 0\ N_{t\lambda,t\mu,t\nu} \neq 0\}$$

In [3], three of the authors conjectured a description of $\text{NL-semigroup}(n)$ using *extended Horn inequalities*.

Conjectural description of $\text{NL-semigroup}(n)$

$(\lambda, \mu, \nu) \in \text{NL-semigroup}(n)$ if and only if $|\lambda| + |\mu| + |\nu|$ is even and (λ, μ, ν) satisfy the extended Horn inequalities.

In [2], we proved saturated version of the conjecture:

A recursive description of $\text{NL-sat}(n)$

$(\lambda, \mu, \nu) \in \text{NL-sat}(n)$ if and only if they satisfy extended Horn inequalities.

Similar to the Horn inequalities, the recursive nature of the extended Horn inequalities comes from indexing sets satisfying Horn inequalities for smaller n. As a result, we obtain a description of $\text{NL-sat}(n)$ that only involves inequalities rather than tensor product multiplicities. In fact, this is a generalization of Klyachko's result on the description of $\text{LR-sat}(n)$.

The extended Horn inequalities, like the Horn inequalities, are not minimal. The work of Knutson-Tao-Woodward gives a set of minimal inequalities that defines $\text{LR-sat}(n)$. Our next result is an NL-generalization:

Minimal inequalities

We gave a minimal set of inequalities that defines $\text{NL-sat}(n)$.

The proof use ideas of Belkale-Kumar on their de- formation of the cup product on flag manifolds, as well as the third author's work on GIT-semigroups/cones.

Connection to matrix eigenvalues

Famously, Klyachko gives a relation between $\text{LR-sat}(n)$ and the *Hermitian eigencone*. Here we relate $\text{NL-sat}(n)$ with the *Symplectic eigencone*. For a complex Hermitian matrix M, let $\lambda(M) \in \{(\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n) : \lambda_i \in \mathbb{R}\}$ be its eigenvalues in weakly decreasing order. For $\lambda \in \mathsf{Par}_n$, let $\hat{\lambda} := (\lambda_1, \ldots, \lambda_n, -\lambda_n, \ldots, -\lambda_1)$.

Eigencone interpretation

$(\lambda, \mu, \nu) \in \text{NL-sat}(n)$ if and only if there exists

$$M_1, M_2, M_3 \in \left\{ \begin{pmatrix} A & B \\ \bar{B}^T & -A^T \end{pmatrix} : \bar{A}^T = A, B^T = B \right\}$$

such that $M_1 + M_2 = M_3$ and $(\hat{\lambda}, \hat{\mu}, \hat{\nu}) = (\lambda(M_1), \lambda(M_2), \lambda(M_3))$.

References

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Acknowledgements

SG, GO and AY were partially supported by NSF RTG grant DMS 1937241. SG was partially supported by an NSF Graduate Research Fellowship. AY was partially supported by a Simons Collaboration Grant, and the UIUC Center for Advanced Study

Contact Information

- Email: sgao23@illinois.edu, gidono2@illinois.edu, ressayre@math.univ-lyon1.fr, ayong@illinois.edu