A q-deformation of enriched P-partitions

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FPSAC, July 18–22, 2022, Indian Institute of Science, Bengaluru, India

Weighted posets and enriched P-partitions

Let $[n] = \{1, 2, \ldots, n\}$ and $\mathbb{P} = \{1, 2, 3, \ldots\}$. A **labelled poset** $P = ([n], <_P)$ is an arbitrary partial order $<_P$ on the set $[n]$. A P-**partition** is a map $f : [n] \longrightarrow \mathbb{P}$ that satisfies the two following conditions:

(i) If $i <_P j$, then $f(i) \leq f(j)$.

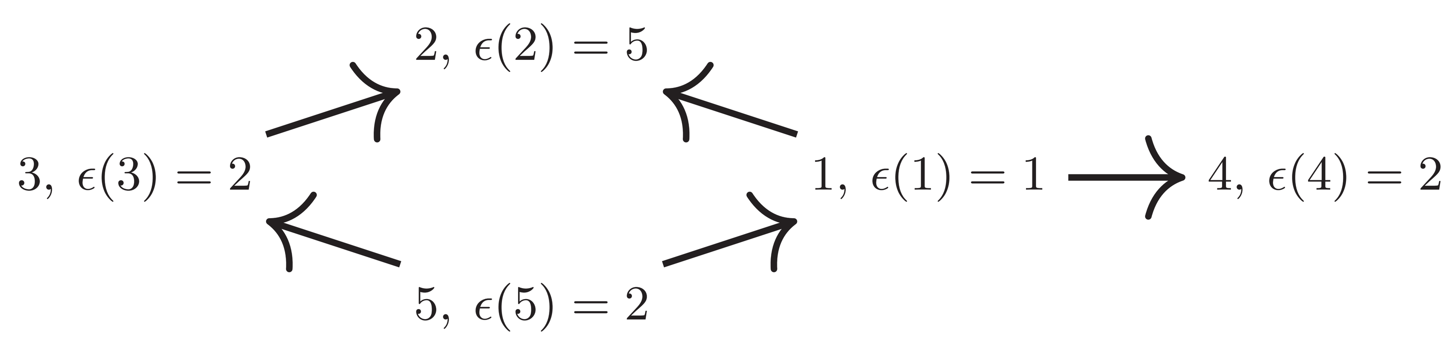
(ii) If $i <_P j$ and $i > j$, then $f(i) < f(j)$.

We denote $\mathcal{L}_{\mathbb{P}}(P)$ the set of P-partitions. Let $\mathbb{P}^{\pm} = -\mathbb{P} \cup \mathbb{P}$ be the set of positive and negative integers totally ordered by $-1 < 1 < -2 < 2 < -3 < 3 < \ldots$. An **enriched** P**-partition** is a map $f : [n] \longrightarrow \mathbb{P}^{\pm}$ that satisfies the two following conditions:

(i) If $i <_P j$ and $i < j$, then $f(i) < f(j)$ or $f(i) = f(j) \in \mathbb{P}$.

(ii) If $i <_P j$ and $i > j$, then $f(i) < f(j)$ or $f(i) = f(j) \in -\mathbb{P}$.

A **labelled weighted poset** is a triple $P = ([n], <_P, \epsilon)$ where $([n], <_P)$ is a labelled poset and $\epsilon : [n] \longrightarrow \mathbb{P}$ is a map (called the **weight function**). Each node of a labelled weighted poset is marked with its label and weight.



Let $X = \{x_1, x_2, x_3, \ldots\}$. For $\mathcal{Z} \in \{\mathbb{P}, \mathbb{P}^{\pm}\}$, define the **generating function** $\Gamma_{\mathcal{Z}}([n], <_P, \epsilon)$ corresponding to labelled weighted poset $([n], <_P, \epsilon)$ by

$$\Gamma_{\mathcal{Z}}([n], <_P, \epsilon) = \sum_{f \in \mathcal{L}_{\mathcal{Z}}([n], <_P)} \prod_{1 \leq i \leq n} x_{|f(i)|}^{\epsilon(i)}. \tag{1}$$

Quasisymmetric functions

Let S_n be the symmetric group on $[n]$. Given a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with n entries, and a permutation $\pi = \pi_1 \ldots \pi_n$ in S_n, we let $P_{\pi,\alpha} = ([n], <_\pi, \alpha)$ be the labelled weighted poset on $[n]$, where $\pi_i <_\pi \pi_j$ if and only if $i < j$ and α is the weight function sending the vertex labelled π_i to α_i. For $\mathcal{Z} \in \{\mathbb{P}, \mathbb{P}^{\pm}\}$, its generating function

$$U_{\pi,\alpha}^{\mathcal{Z}} = \Gamma_{\mathcal{Z}}([n], <_\pi, \alpha)$$

is called the **universal quasisymmetric function** [5] indexed by π and α.

$$\pi_1,\ \alpha_1 \longrightarrow \pi_2,\ \alpha_2 \longrightarrow \cdots\cdots\cdots \longrightarrow \pi_n,\ \alpha_n$$

Let $[1^n]$ denote the composition with n entries equal to 1. For each $\pi \in S_n$, let $L_\pi = U_{\pi,[1^n]}^{\mathbb{P}}$ and $K_\pi = U_{\pi,[1^n]}^{\mathbb{P}^{\pm}}$. The power series L_π (resp. K_π) are **Gessel's fundamental [1] (resp. Stembridge's peak [2]) quasisymmetric functions** indexed by π. They are related to the **descent set** $\mathrm{Des}(\pi) = \{1 \leq i \leq n-1 | \pi(i) > \pi(i+1)\}$ and the **peak set** $\mathrm{Peak}(\pi) = \{2 \leq i \leq n-1 | \pi(i-1) < \pi(i) > \pi(i+1)\}$ statistics.

$$L_\pi = \sum_{\substack{i_1 \leq \cdots \leq i_n; \\ j \in \mathrm{Des}(\pi) \Rightarrow i_j < i_{j+1}}} x_{i_1} \ldots x_{i_n}, \qquad K_\pi = \sum_{\substack{i_1 \leq \cdots \leq i_n; \\ j \in \mathrm{Peak}(\pi) \Rightarrow i_{j-1} < i_{j+1}}} 2^{|\{i_1, \ldots, i_n\}|} x_{i_1} \cdots x_{i_n}.$$

L_π (K_π) depends only on n and $\mathrm{Des}(\pi)$ ($\mathrm{Peak}(\pi)$) and we may write $L_{n,\mathrm{Des}(\pi)}$ ($K_{n,\mathrm{Peak}(\pi)}$) instead of L_π (K_π) or even $L_{\mathrm{Des}(\pi)}$ ($K_{\mathrm{Peak}(\pi)}$) if n is clear from context. Let id_n and $\overline{id_n}$ denote the permutations in S_n given by $id_n = 1\ 2\ 3 \ldots n$ and $\overline{id_n} = n\ n-1 \ldots 1$. Given a composition $\alpha = (\alpha_1, \ldots, \alpha_n)$ of n entries, define the **monomial** M_α [1], **essential** E_α [4] and **enriched monomial** η_α [3, 5] quasisymmetric functions

$$M_\alpha = U_{\overline{id_n},\alpha}^{\mathbb{P}} = \sum_{i_1 < \cdots < i_n} x_{i_1}^{\alpha_1} \ldots x_{i_n}^{\alpha_n}, \qquad E_\alpha = U_{id_n,\alpha}^{\mathbb{P}} = \sum_{i_1 \leq \cdots \leq i_n} x_{i_1}^{\alpha_1} \ldots x_{i_n}^{\alpha_n},$$

$$\eta_\alpha = U_{id_n,\alpha}^{\mathbb{P}^{\pm}} = \sum_{i_1 \leq \cdots \leq i_n} 2^{|\{i_1, \ldots, i_n\}|} x_{i_1}^{\alpha_1} \ldots x_{i_n}^{\alpha_n}.$$

As compositions α such that $|\alpha| = s$ are in bijection with subsets $I \subseteq [s-1]$, one may also reindex these quasisymmetric functions with sets and get M_I, E_I and η_I.

References

[1] I. Gessel, Multipartite P-partitions and inner products of skew Schur functions, 1984

[2] J. Stembridge, Enriched P-partitions, 1997

[3] S. K. Hsiao, Structure of the peak Hopf algebra of quasisymmetric functions, 2007

[4] M. E. Hoffman, Quasi-symmetric functions and mod p multiple harmonic sums, 2015

[5] D. Grinberg and E. Vassilieva, Weighted posets and the enriched monomial basis of QSym, 2021

A q-deformed generating function for P-partitions

Let $\omega(i, f) = x_{|f(i)|}^{\epsilon(i)}$ be the contributing monomial in the generating function Γ of (1) for vertex i and P-partition f. As per Stembridge, its value does not depend on the sign of f. We break this assumption and write for an additional parameter q:

$$\omega(i, f, q) = x_{f(i)}^{\epsilon(i)} \text{ if } f(i) \in \mathbb{P}, \qquad \omega(i, f, q) = q x_{-f(i)}^{\epsilon(i)} \text{ if } f(i) \in -\mathbb{P}.$$

Def. Let $q \in \mathbf{k}$ (the base ring of the power series). The q-**generating function** for enriched P-partitions on the weighted poset $([n], <_P, \epsilon)$ is

$$\Gamma_q([n], <_P, \epsilon) = \sum_{f \in \mathcal{L}_{\mathbb{P}^{\pm}}([n], <_P)} \prod_{1 \leq i \leq n} \omega(i, f, q) = \sum_{f \in \mathcal{L}_{\mathbb{P}^{\pm}}([n], <_P)} \prod_{1 \leq i \leq n} q^{[f(i) < 0]} x_{|f(i)|}^{\epsilon(i)}.$$

This definition covers the cases of Gessel ($q = 0$) and Stembridge ($q = 1$). We further define the q-**universal quasisymmetric function** $U_{\pi,\alpha}^q = \Gamma_q([n], <_\pi, \alpha)$.

Prop. *Let $q \in \mathbf{k}$, $\pi \in S_n$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a composition with n entries.*

$$U_{\pi,\alpha}^q = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ j \in \mathrm{Peak}(\pi) \Rightarrow i_{j-1} < i_{j+1}}} q^{|\{j \in \mathrm{Des}(\pi) | i_j = i_{j+1}\}|} (q+1)^{|\{i_1, i_2, \ldots, i_n\}|} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \ldots x_{i_n}^{\alpha_n}.$$

Prop. *The product of two q-universal quasisymmetric functions is given by*

$$U_{\pi,\alpha}^q U_{\sigma,\beta}^q = \sum_{(\tau,\gamma) \in (\pi,\alpha) ⧢ (\sigma,\beta)} U_{\tau,\gamma}^q.$$

Enriched q-monomials

The **enriched** q-**monomials** generalise essential and enriched monomials. For $q \in \mathbf{k}$ and α composition with n entries, define $\eta_\alpha^{(q)} = U_{id_n,\alpha}^q$. One has $\eta_\alpha^{(0)} = E_\alpha$ and $\eta_\alpha^{(1)} = \eta_\alpha$.

Prop. *Let $q \in \mathbf{k}$, and let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a composition with n entries. Then,*

$$\eta_\alpha^{(q)} = \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} (q+1)^{|\{i_1, i_2, \ldots, i_n\}|} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \ldots x_{i_n}^{\alpha_n}.$$

Thm. *Express the quasisymmetric function $U_{\pi,\alpha}^q$ in terms of enriched q-monomials:*

$$U_{\pi,\alpha}^q = \sum_{\substack{I \subseteq \mathrm{Des}(\pi) \\ J \subseteq \mathrm{Peak}(\pi) \\ I \cap J = \emptyset}} (-q)^{|J|} (q-1)^{|I|} \eta_{\alpha \downarrow I \downarrow\downarrow J}^{(q)}.$$

Cor. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_m)$ be two compositions. Let $S_\beta(\gamma)$ be the set of the positions of the entries of β in a composition $\gamma \in \alpha ⧢ \beta$.

$$\eta_\alpha^{(q)} \eta_\beta^{(q)} = \sum_{\substack{\gamma \in \alpha ⧢ \beta; \\ I \subseteq S_\beta(\gamma) \\ J \subseteq (S_\beta(\gamma) \setminus (S_\beta(\gamma) - 1)) \setminus \{1\} \\ I \cap J = \emptyset}} (q-1)^{|I|} (-q)^{|J|} \eta_{\gamma \downarrow I \downarrow\downarrow J}^{(q)}.$$

Reindex with sets $I \subseteq [s-1]$ such that $\eta_I^{(q)} = \sum_{\substack{i_1 \leq \cdots \leq i_s \\ j \in I \Rightarrow i_j = i_{j+1}}} (q+1)^{|\{i_1, \ldots, i_s\}|} x_{i_1} \ldots x_{i_s}$.

Thm. *Let $q \in \mathbf{k}$ be such that $q+1$ is invertible. The family of enriched q-monomial quasisymmetric functions $\left(\eta_{s,I}^{(q)}\right)_{s \geq 0, I \subseteq [s-1]}$ is a basis of* QSym. *Furthermore*

$$\eta_I^{(q)} = \sum_{I \subseteq J} (q+1)^{s-|J|} M_J, \quad (q+1)^{s-|J|} M_J = \sum_{J \subseteq I} (-1)^{|I \setminus J|} \eta_I^{(q)}.$$

$$\eta_I^{(q)} = (q+1) \sum_{J \subseteq [s-1]} (-1)^{|J|} (-q)^{|J \setminus I|} L_J, \quad (q+1)^s L_J = \sum_{I \subseteq [s-1]} (-1)^{|I|} (-q)^{|I \setminus J|} \eta_I^{(q)}.$$

q-fundamental quasisymmetric functions

The q-**fundamental quasisymmetric functions** interpolate between Gessel's fundamental and Stembridge peak quasisymmetric functions. Define for $\pi \in S_n$ and $q \in \mathbf{k}$ $L_{n,\mathrm{Des}(\pi)}^{(q)} = U_{\pi,[1^n]}^q$. For $q = 0$, $L_{n,I}^{(0)} = L_{n,I}$ while for $q = 1$, $L_{n,I}^{(1)} = K_{n,\mathrm{Peak}(I)}$.

Prop. *Let $I \subseteq [n-1]$ and $q \in \mathbf{k}$.*

$$L_I^{(q)} = \sum_{\substack{J \subseteq I \\ K \subseteq \mathrm{Peak}(I) \\ J \cap K = \emptyset}} (-q)^{|K|} (q-1)^{|J|} \eta_{J \cup (K-1) \cup K}^{(q)}.$$

Thm. *The family of q-fundamental quasisymmetric functions $(L_{n,I}^{(q)})_{n \geq 0, I \subseteq [n-1]}$ is a basis of* QSym *iff q is not a root of unity.*