# A combinatorial model for the transition matrix between the Specht and web bases


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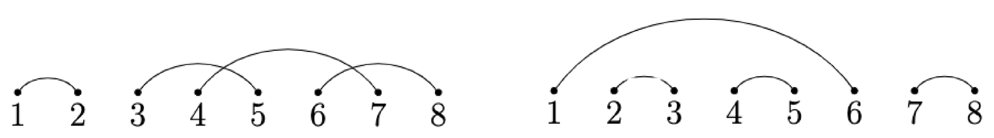
## The transition matrix $A = (a_{MM'})$

A **(perfect) matching** on $[2n]$ is a set partition of $[2n]$ such that each block has size 2. Let $\mathrm{Mat}_{2n}$ ($\mathrm{NC}_{2n}$ and $\mathrm{NN}_{2n}$, respectively) stand for the set of (**noncrossing** and **nonnesting**, respectively) matchings on $[2n]$.



$M \in \mathrm{NN}_8$  $M' \in \mathrm{NC}_8$

There are two bases of irreducible $\mathfrak{S}_{2n}$-representation of shape $(n,n)$:

- the Specht basis $\{v_M \in \mathcal{S}^{(n,n)} : M \in \mathrm{NN}_{2n}\}$ and
- the web basis $\{\Delta_M \in W_n : M \in \mathrm{NC}_{2n}\}$.

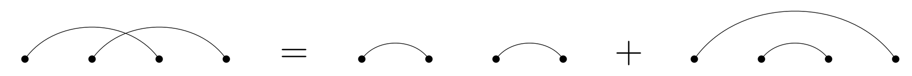
There is a unique (up to scalar) isomorphism $\varphi : W_n \to \mathcal{S}^{(n,n)}$ and let $w_{M'} := \varphi(\Delta_{M'})$ for each $M' \in \mathrm{NC}_{2n}$. The **transition matrix** $A = (a_{MM'})$ is defined by

$$v_M = \sum_{M' \in \mathrm{NC}_{2n}} a_{MM'} w_{M'}$$

for all $M \in \mathrm{NN}_{2n}$.

**Theorem** (H. Russell and J. Tymoczko, 2019)**.** *The transition matrix $A$ is unitriangular with respect to an appropriate order on the bases.*

**Theorem** (B. Rhoades, 2019)**.** *The entries $a_{MM'}$ of $A$ are nonnegative integers. In fact, $a_{MM'}$ is the appearances of $M'$ by resolving all crossings in $M$ into the rule:*



(1)

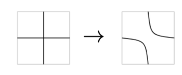
**Problem.** Find an explicit combinatorial interpretation of $a_{MM'}$.

## Grid configurations

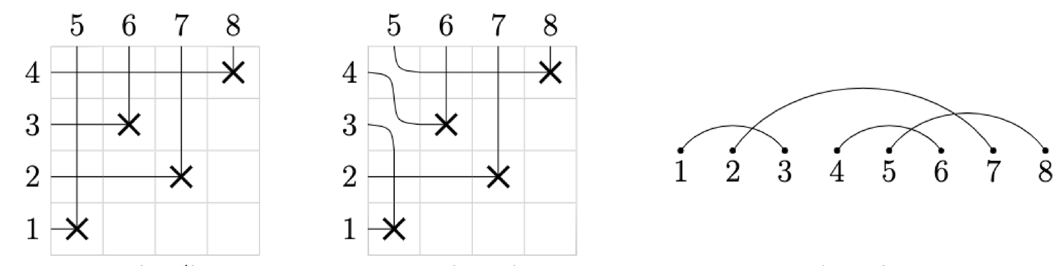
We define grid configurations which represent matchings in a 'rigid' setting.

Let $\sigma \in \mathfrak{S}_n$. Define a **grid configuration** $G(\sigma, E)$ as follows:

1. Mark the cell $(i, \sigma(i))$ and draw a horizontal line to the left and a vertical line to the top from the marked cell.
2. Define $\mathrm{Cr}(\sigma)$ to be the set of crossings and let $E \subseteq \mathrm{Cr}(\sigma)$.
3. Replace each crossings in $E$ with the 'elbow' cell as shown below:



Define $M(\sigma, E)$ to be the matching associated to $G(\sigma, E)$. For instance, if $\sigma = 1324$ and $E = \{(1,3),(1,4)\}$, then we get the following:

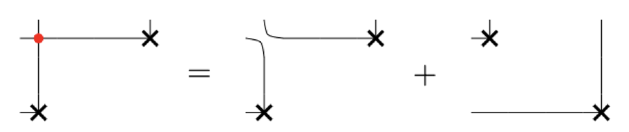


$G(\sigma, \emptyset)$  $G(\sigma, E)$  $M(\sigma, E)$

We can write the relation (1) in terms of grid configurations as

$$G(\sigma, E) = G(\sigma, E \cup \{c\}) + G(\sigma \cdot (i, \sigma^{-1}(j)), E), \tag{2}$$

where $c = (i,j)$ is the northwest-most crossing in $G(\sigma, E)$. Equivalently,

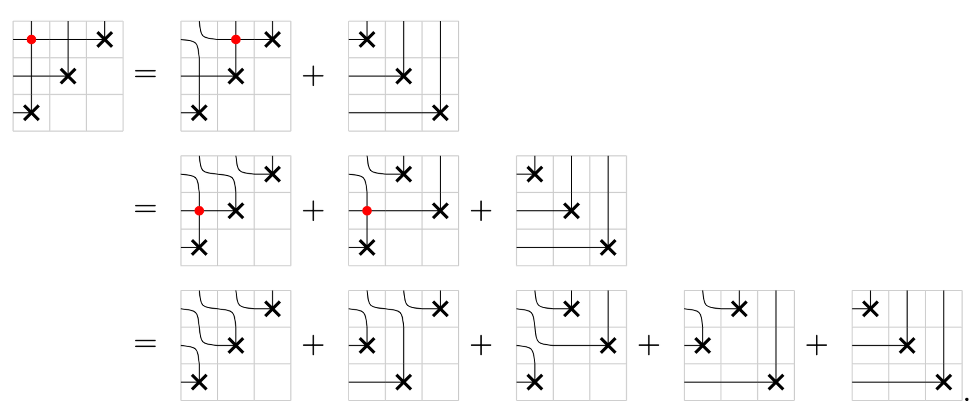


## Web permutations

From the grid configuration $G(id, \emptyset)$, we obtain two grid configurations by resolving a northwest-most crossing into the rule 2. By resolving crossings until there is no crossing left, we get grid configurations of the form $G(\sigma, \mathrm{Cr}(\sigma))$. For each remaining grid configuration $G(\sigma, \mathrm{Cr}(\sigma))$, the permutation $\sigma$ is called a **web permutation** of $[n]$ and we denote the set of web permutations of $[n]$ by $\mathrm{Web}_n$. In other words, we have

$$G(id, \emptyset) = \sum_{\sigma \in \mathrm{Web}_n} G(\sigma, \mathrm{Cr}(\sigma)). \tag{3}$$

For example, starting from the grid configuration $G(id, \emptyset)$ for $n = 3$, we have



Therefore, $\mathrm{Web}_3 = \{123, 213, 132, 231, 321\}$. The following proposition justifies that web permutations are well-defined.

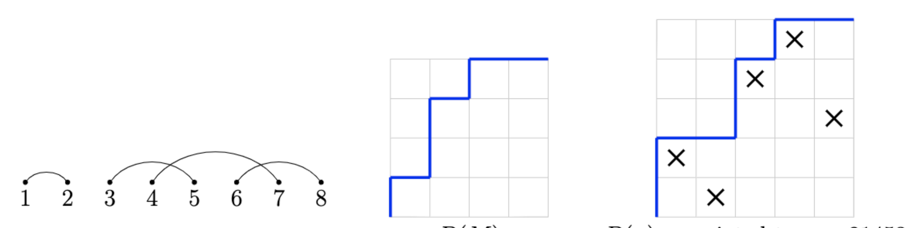
**Proposition 1.** *The expansion in (3) is unique. In other words, the grid configurations appearing in (3) does not depend on the order of resolving procedure (choice of maximal crossings). In addition, the permutations $\sigma$ in (3) are all distinct.*

## Main result

1. **A map $D$**

   For a matching $M$, record N for openers and E for closers reading $M$ from left to right. This gives the Dyck path $D(M)$ in the $n$ by $n$ grid.

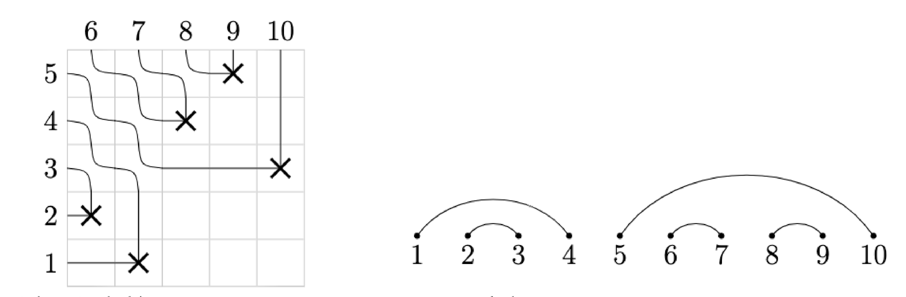
   To a permutation $\sigma$, we associate the minimum Dyck path $D(\sigma)$ where every cell $(i, \sigma(i))$ lies below the path.



$M$  $D(M)$  $D(\sigma)$ associated to $\sigma = 21453$

2. **A map $M$**

   We write $M(\sigma) = M(\sigma, \mathrm{Cr}(\sigma))$.



$G(\sigma, \mathrm{Cr}(\sigma))$ where $\sigma = 21453$  $M(\sigma)$ where $\sigma = 21453$

**Theorem 2.** *For matchings $M \in \mathrm{NN}_{2n}$ and $M' \in \mathrm{NC}_{2n}$, the entry $a_{MM'}$ is equal to the number of web permutations $\sigma \in \mathfrak{S}_n$ such that $D(\sigma) \subseteq D(M)$ and $M(\sigma) = M'$.*

## Characterization

**Definition.** For a permutation $w = w_1 w_2 \cdots w_n$ with $n \geq 2$, let $w_k$ be the smallest letter in $w$. Then $w$ is an **André permutation** if both $w_1 \cdots w_{k-1}$ and $w_{k+1} \cdots w_n$ are André permutations and $\max\{w_1, \ldots, w_{k-1}\} < \max\{w_{k+1}, \ldots, w_n\}$.

**Example.** A word 547239 is an André permutation.

**Definition 3.** Let $C = (a_1, \ldots, a_k)$ be a cycle with $a_1 = \min\{a_1, \ldots, a_k\}$. We say that $C$ is an **André cycle** if the permutation $a_2 \cdots a_k$ is an André permutation.

**Example.** A cycle $C = (2,3,9,1,5,4,7)$ is an André cycle since $C = (1,5,4,7,2,3,9)$ and the permutation 547239 is an André permutation.

**Theorem 4** (Characterization I)**.** *A permutation $\sigma \in \mathfrak{S}_n$ is a web permutation if and only if each cycle of $\sigma$ is an André cycle.*

**Corollary 5.** *A 312-avoiding permutation is a web permutation.*

**Corollary 6** (RT19, IZ21)**.** *Let $M \in \mathrm{NN}_{2n}$ and $M' \in \mathrm{NC}_{2n}$. Then $a_{MM'} > 0$ if and only if $D(M') \subseteq D(M)$. In particular, the transition matrix $(a_{MM'})$ is upper-triangular. Moreover, there are ones along the diagonal of the transition matrix, and 312-avoiding permutations contribute to the ones.*

## Enumerative properties

**Definition.** For $\sigma \in \mathfrak{S}_n$, the **canonical cycle notation** of $\sigma$ is a cycle notation of $\sigma$ such that its cycles are sorted based on the smallest elements of the cycles and the smallest element of each cycle is written in the last place of the cycle.

**Definition** (**Foata transformation** $\widehat{\ } : \mathfrak{S}_n \to \mathfrak{S}_n$)**.** For $\sigma \in \mathfrak{S}_n$, we define $\widehat{\sigma}$ to be the permutation obtained by dropping the parentheses in the canonical cycle notation of $\sigma$.

**Definition 7** (A map $\phi : \mathfrak{S}_n \to \mathfrak{S}_{n+2}$)**.** For a permutation $\sigma \in \mathfrak{S}_n$, define the one-cycle permutation $\phi(\sigma) \in \mathfrak{S}_{n+2}$ by

$$\phi(\sigma) := (1, \widehat{\sigma}_1 + 1, \ldots, \widehat{\sigma}_n + 1, n+2).$$

**Theorem 8** (Characterization II)**.** *For $n \geq 1$, let $\mathrm{AC}_{n+2} \subset \mathfrak{S}_{n+2}$ be the set of André cycles consisting of $[n+2]$. Then we have $\phi(\mathrm{Web}_n) = \mathrm{AC}_{n+2}$. In particular, the number of web permutations of $[n]$ is equal to the number of André cycles consisting of $[n+2]$.*

The **Euler numbers** $E_n$ are defined via the exponential generating function

$$E(z) := \sum_{n \geq 0} E_n \frac{x^n}{n!} = \sec z + \tan z.$$

The Euler number $E_n$ counts André permutations of $[n]$.

**Corollary 9.** *The Euler number $E_{n+1}$ enumerates the number of web permutations of $[n]$.*

There are numbers that refine Euler numbers, called **Entringer numbers**, in the following sense: For $n \geq 1$,

$$\sum_{k=1}^{n} E_{n,k} = E_{n+1}.$$

We have a counterpart of this refinement.

**Corollary 10.** *The Entringer number $E_{n,k}$ enumerates the number of web permutations $\sigma$ of $[n]$ with $\sigma_1 = n+1-k$.*

## (Selected) References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