An involution on derangements preserving excedances and right-to-left minima

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Notations and Background

Let $\mathfrak{S}_n$ be the set of all permutations, $\mathfrak{S}_n^e$ ($\mathfrak{S}_n^o$) be the set of all even (odd) permutations, $\mathfrak{D}_n$ be the set of all derangements, and $\mathfrak{D}_n^e$ ($\mathfrak{D}_n^o$) be the set of all even (odd) derangements of $[n]$.

For any function $g : [n] \longrightarrow [n]$, let

$$\begin{aligned} \mathrm{EXCi}(g) &:= \{j \in [n] : g(j) > j\}, \\ \mathrm{EXCv}(g) &:= \{g(j) : j \in \mathrm{EXCi}(g)\}, \\ \mathrm{RLMi}(g) &:= \{i \in [n] : g(i) < g(j) \text{ for all } j \in \{i+1, \dots, n\}\}, \\ \mathrm{RLMv}(g) &:= \{g(i) : i \in \mathrm{RLMi}(g)\}, \\ \mathrm{FIX}(g) &:= \{i \in [n] : g(i) = i\}, \end{aligned}$$

Moreover, $\mathrm{exc}(g) := |\mathrm{EXCi}(g)|$ and $\mathrm{rlm}(g) := |\mathrm{RLMi}(g)| = |\mathrm{RLMv}(g)|$.

Note that, $|\mathrm{EXCv}(\sigma)| = |\mathrm{EXCi}(\sigma)| = \mathrm{exc}(\sigma)$, for any $\sigma \in \mathfrak{S}_n$

A *subexcedant function* f on $[n]$: $f : [n] \longrightarrow [n]$ such that

$$1 \leq f(i) \leq i, \text{ for all } 1 \leq i \leq n.$$

$\mathcal{F}_n$: the set of all subexcedant functions on $[n]$. And $\mathrm{IM}(f) := \{f(i) : i \in [n]\}$ is the *image* of $f \in \mathcal{F}_n$. The bijection $\mathtt{sefToPerm} : \mathcal{F}_n \longrightarrow \mathfrak{S}_n$, from [2], is defined as:

$$\mathtt{sefToPerm}(f) := (n\ f(n)) \cdots (2\ f(2))(1\ f(1)).$$

For $\sigma \in \mathfrak{S}_n$ and $j \in [n]$, the j^{th} entry of $\mathtt{sefToPerm}^{-1}(\sigma)$ is

$$\mathtt{sefToPerm}^{-1}(\sigma)_j := \begin{cases} \sigma(n) \text{ if } j = n, \\ \mathtt{sefToPerm}^{-1}\left((n\ \sigma(n)) \circ \sigma\right)_j & \text{otherwise.} \end{cases}$$

For example, the corresponding subexcedant function of $\sigma = 612935487 \in \mathfrak{S}_9$ is $f_\sigma = 112435487 \in \mathcal{F}_9$.

An involution

A subexcedant function f is *matchless* if it is of the form

$$f := 1\,1\,2\,3\,4 \dots k{-}1\ k\ k \dots k \quad \text{for } 1 \leq k \leq n-1.$$

There are $n-1$ matchless subexcedant functions of length n.

$\mathcal{DF}_n$: the set of subexcedant functions corresponding to derangements of $[n]$.

Define $\Psi : \mathcal{DF}_n \longrightarrow \mathcal{DF}_n$ below, where $f_\tau := \Psi(f_\sigma)$. First, if f_σ is matchless, we set $f_\tau := f_\sigma$. Now we assume that f_σ is non-matchless and let

$$\mathrm{IM}(f_\sigma) = \{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3, \dots, \mathbf{m}_\ell\}.$$

Now define two auxiliary maps, $\mathtt{fix}_i$, $\mathtt{unfix}_i$ on subexcedant functions. For $i \in \{2, \dots, \ell\}$,

$$\mathtt{fix}_i(f_\sigma)(\mathbf{m}_i) := \mathbf{m}_i, \qquad \mathtt{unfix}_i(f_\sigma)(\mathbf{m}_i) := \mathbf{m}_{i-1}$$

while the remaining entries of f_σ are untouched. For $i \in \{2, \dots, \ell\}$, we say that f_σ satisfies $\circledast_i$ if the three conditions

$$f_\sigma(\mathbf{m}_i) < \mathbf{m}_i < \mathbf{m}_\ell, \quad f_\sigma^{-1}(1) = \{1, 2\}, \text{ and } \{\mathbf{m}_i + 1\} \subsetneq f_\sigma^{-1}(\mathbf{m}_i), \qquad (\circledast_i)$$

hold. Now let $i \in \{2, \dots, \ell\}$ be the *smallest* element satisfying one of the cases below, and let f_τ be given as described in each case.

- $\heartsuit_i$: If $f_\sigma(\mathbf{m}_i) = \mathbf{m}_i$, then $f_\tau := \mathtt{unfix}_i(f_\sigma)$.
- $\spadesuit_i$: If $f_\sigma(\mathbf{m}_i) < \mathbf{m}_i$ and $|f_\sigma^{-1}(1)| \geq 3$, then $f_\tau := \mathtt{fix}_i(f_\sigma)$.
- $\diamondsuit_i$: If $\circledast_i$ holds and $f_\sigma(\mathbf{m}_{i+1}) = \mathbf{m}_{i+1}$, then $f_\tau := \mathtt{unfix}_{i+1}(f_\sigma)$.
- $\clubsuit_i$: If $\circledast_i$ holds and $f_\sigma(\mathbf{m}_{i+1}) < \mathbf{m}_{i+1}$, then $f_\tau := \mathtt{fix}_{i+1}(f_\sigma)$.

Examples of the involution

1. Let $f_\sigma = 1133535$. Then f_σ is in case $\heartsuit_2$ and $f_\tau = \mathtt{unfix}_2(f_\sigma) = 1113535$.
2. Now let $f_\sigma = 1121355$. Since $f_\sigma(2) < 2$ and $|f_\sigma^{-1}(1)| = 3$, then f_σ is in case $\spadesuit_2$. Thus, $f_\tau = \mathtt{fix}_2(f_\sigma) = 1221355$.
3. Suppose that $f_\sigma = 1123535$. f_σ is in case $\diamondsuit_3$ and $f_\tau = \mathtt{unfix}_{i+1}(f_\sigma) = \mathtt{unfix}_4(f_\sigma) = 1123335$.
4. Now take $f_\sigma = 1123445$. It is in $\clubsuit_4$ and $f_\tau = \mathtt{fix}_5(f_\sigma) = 1123545$.

Properties of the involution

1. The image is preserved, $\mathrm{IM}(f_\sigma) = \mathrm{IM}(\Psi(f_\sigma))$.
2. If $f_\tau = \Psi(f_\sigma)$, then $\mathrm{EXCv}(\sigma) = \mathrm{EXCv}(\tau)$.
3. The set of right-to-left minima is preserved, $\mathrm{RLMv}(f_\sigma) = \mathrm{RLMv}(\Psi(f_\sigma))$.
4. Ψ changes the parity of a non-matchless subexcedant function.

We now have an involution on derangements $\hat{\Psi} : \mathfrak{D}_n \to \mathfrak{D}_n$ by setting

$$\hat{\Psi}(\sigma) := (\mathtt{sefToPerm} \circ \Psi \circ \mathtt{sefToPerm}^{-1})(\sigma), \text{ for } \sigma \in \mathfrak{D}_n,$$

with properties:

1. The excedance value set is preserved, $\mathrm{EXCv}(\hat{\Psi}(\sigma)) = \mathrm{EXCv}(\sigma)$.
2. The set of right-to-left minima is preserved, $\mathrm{RLMv}(\hat{\Psi}(\sigma)) = \mathrm{RLMv}(\sigma)$.
3. Whenever σ is a *non-matchless derangement* (the corresponding f_σ is non-matchless), $\hat{\Psi}$ changes the parity of σ.

Consequences of the involution

Theorem 1: We have that

$$\sum_{\pi \in \mathfrak{D}_n} (-1)^{\mathrm{inv}(\pi)} \left(\prod_{j \in \mathrm{RLMv}(\pi)} x_j\right)\left(\prod_{j \in \mathrm{EXCv}(\pi)} y_j\right) = (-1)^{n-1} \sum_{j=1}^{n-1} x_1 \cdots x_j y_{j+1} \cdots y_n. \tag{1}$$

Moreover,

$$\sum_{\pi \in \mathfrak{D}_n} (-1)^{\mathrm{inv}(\pi)} \left(\prod_{j \in \mathrm{RLMi}(\pi)} x_j\right)\left(\prod_{j \in \mathrm{EXCi}(\pi)} y_j\right) = (-1)^{n-1} \sum_{j=1}^{n-1} y_1 \cdots y_j x_{j+1} \cdots x_n. \tag{2}$$

Corollary 2: By letting $x_j \to 1$ and $y_j \to t$, we have that

$$\sum_{\pi \in \mathfrak{D}_n} (-1)^{\mathrm{inv}(\pi)} t^{\mathrm{exc}(\pi)} = (-1)^{n-1}(t + t^2 + \dots + t^{n-1}). \tag{3}$$

By comparing coefficients of t^k, we get

$$|\{\pi \in \mathfrak{D}_n^e : \mathrm{exc}(\pi) = k\}| - |\{\pi \in \mathfrak{D}_n^o : \mathrm{exc}(\pi) = k\}| = (-1)^{n-1}, \tag{4}$$

for every $n \geq 1$ and $1 \leq k \leq n-1$. Equation (4) studied by R. Mantaci and F. Rakotondrajao in [3].

Similarly,

$$\sum_{\pi \in \mathfrak{D}_n} (-1)^{\mathrm{inv}(\pi)} t^{\mathrm{rlm}(\pi)} = (-1)^{n-1}(t + t^2 + \dots + t^{n-1}). \tag{5}$$

A proof using generating functions

Mantaci, in [1], proved Proposition 3 by introducing a bijection on $\mathfrak{S}_n$ that preserves the set of excedances and changes the sign of non-fixed elements of the bijection.

Proposition 3: Let $n \geq 1$, then

$$\sum_{\pi \in \mathfrak{S}_n} (-1)^{\mathrm{inv}(\pi)} \left(\prod_{j \in \mathrm{EXCi}(\pi)} x_j\right) = \prod_{j \in [n-1]} (1 - x_j) \sum_{E \subseteq [n-1]} (-1)^{|E|} \mathbf{x}_E. \tag{6}$$

In particular, by setting all x_i equal to t, we have

$$\sum_{\pi \in \mathfrak{S}_n^e} t^{\mathrm{exc}(\pi)} - \sum_{\pi \in \mathfrak{S}_n^o} t^{\mathrm{exc}(\pi)} = (1-t)^{n-1}.$$

Proposition 4: Let $n \geq 1$ and let $T \subseteq [n]$. Let $m \leq n$ be the largest integer not in T and set $E = \{1, 2, \dots, m-1\} \setminus T$. Then

$$\sum_{\substack{\pi \in \mathfrak{S}_n \\ T \subseteq \mathrm{FIX}(\pi)}} (-1)^{\mathrm{inv}(\pi)} \left(\prod_{j \in \mathrm{EXCi}(\pi)} x_j\right) = \prod_{j \in E} (1 - x_j), \tag{7}$$

where the empty product has value 1.

Setting all x_i to be t, we have

$$\sum_{\substack{\pi \in \mathfrak{S}_n^e \\ T \subseteq \mathrm{FIX}(\pi)}} t^{\mathrm{exc}(\pi)} - \sum_{\substack{\pi \in \mathfrak{S}_n^o \\ T \subseteq \mathrm{FIX}(\pi)}} t^{\mathrm{exc}(\pi)} = \begin{cases} 1 & \text{if } |T| = n \\ (1-t)^{n-1-|T|} & \text{otherwise.} \end{cases} \tag{8}$$

Using inclusion-Exclusion and Proposition 4, our main theorem is obtained.

Theorem 5: Let $n \geq 1$. Then

$$\sum_{\pi \in \mathfrak{D}_n} (-1)^{\mathrm{inv}(\pi)} \left(\prod_{j \in \mathrm{EXCi}(\pi)} y_j\right) = (-1)^{n-1} \sum_{j=1}^{n-1} x_1 x_2 \cdots x_j. \tag{9}$$

A right-to-left minima analog

We defined a bijection $\kappa : \mathfrak{S}_n \to \mathfrak{S}_n$ that has the following properties:

1. κ is an involution,
2. κ preserves the number of right-to-left minima,
3. κ changes sign of non-fixed elements,
4. For each subset $T \in [n] \cap \{2, 4, 6, \dots\}$, there is a unique fixed element with $\{1, 3, 5, \dots\} \cup T$ as right-to-left minima set.
5. There are $\binom{\lfloor n/2 \rfloor}{k - \lceil n/2 \rceil}$ fixed elements with exactly k right-to-left minima, and they all have sign $(-1)^{n-k}$.

Proposition 6: We have that for any $n \geq 1$

$$\sum_{\pi \in \mathfrak{S}_n} (-1)^{\mathrm{inv}(\pi)} \left(\prod_{j \in \mathrm{RLMv}(\pi)} x_j\right) = \Big(\prod_{\substack{i \in [n] \\ i \text{ odd}}} x_i\Big)\Big(\prod_{\substack{j \in [n] \\ j \text{ even}}} (x_j - 1)\Big). \tag{10}$$

In particular, for any $k = 1, \dots, n$ we have that

$$|\{\pi \in \mathfrak{S}_n^e : \mathrm{rlm}(\pi) = k\}| - |\{\pi \in \mathfrak{S}_n^o : \mathrm{rlm}(\pi) = k\}| = (-1)^{n-k} \binom{\lfloor n/2 \rfloor}{k - \lceil n/2 \rceil}. \tag{11}$$

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