Planar Tanglegram Layouts & Single Edge Insertion

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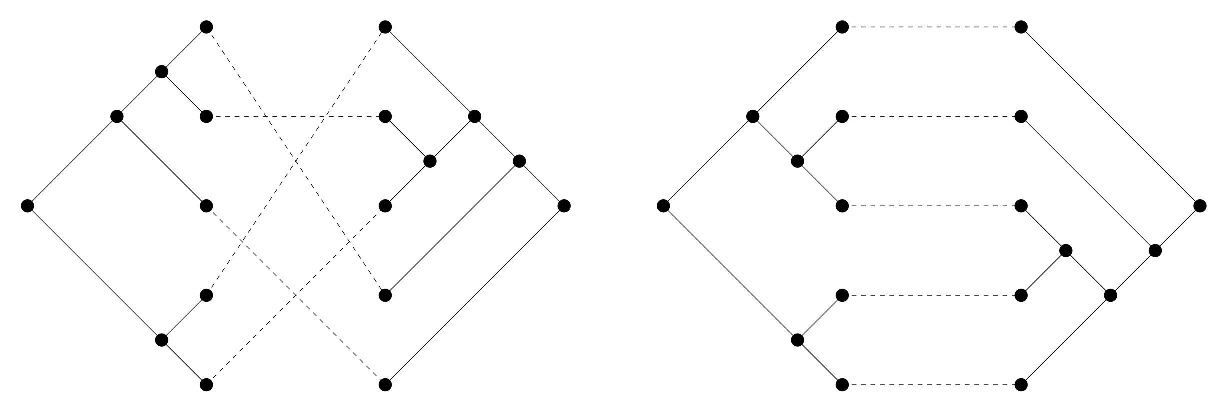
Tanglegrams

1. A *tanglegram* (T, S, ϕ) is formed from a pair of rooted binary trees T and S with a bijection ϕ matching their leaves.
2. A *layout* of a tanglegram draws T, S, and the edges $(t_i, s_{\phi(i)})$ in the plane such that
   - T is planarly embedded left of the line $x = 0$ with all leaves on $x = 0$,
   - S is planarly embedded right of the line $x = 1$ with all leaves on $x = 1$, and
   - the matching ϕ is drawn using straight lines.
3. A tanglegram is *planar* if it has a planar layout, i.e., a layout with no crossings.

Layouts for the same planar tanglegram

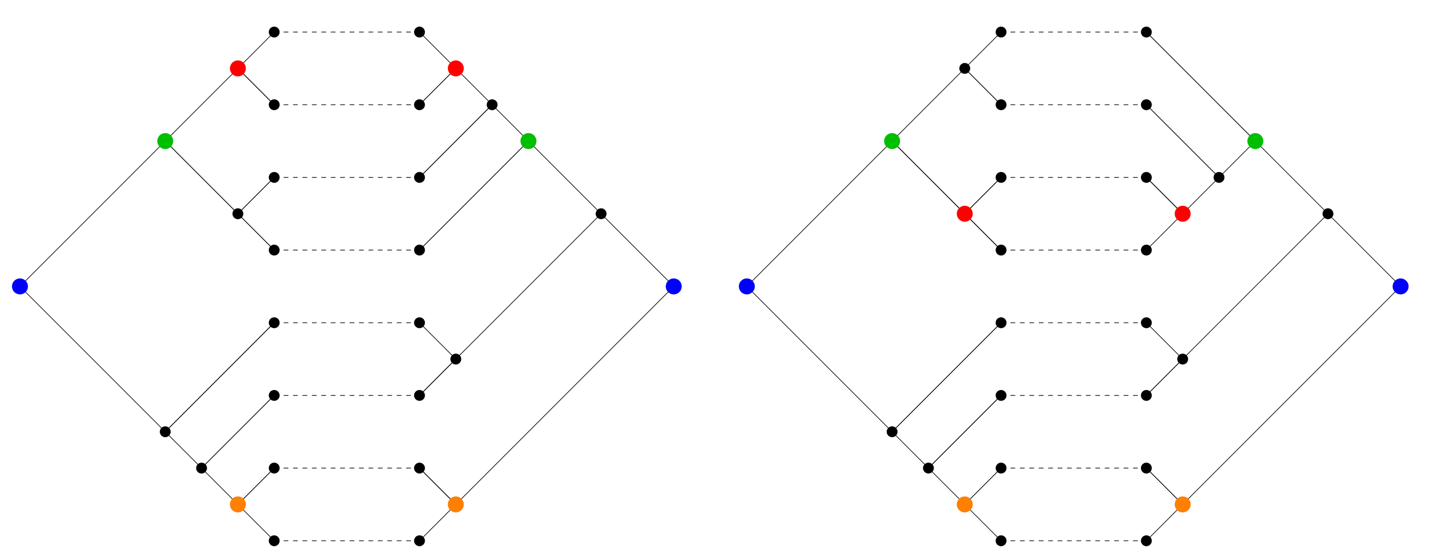


Full paper

https://arxiv.org/abs/2201.10533

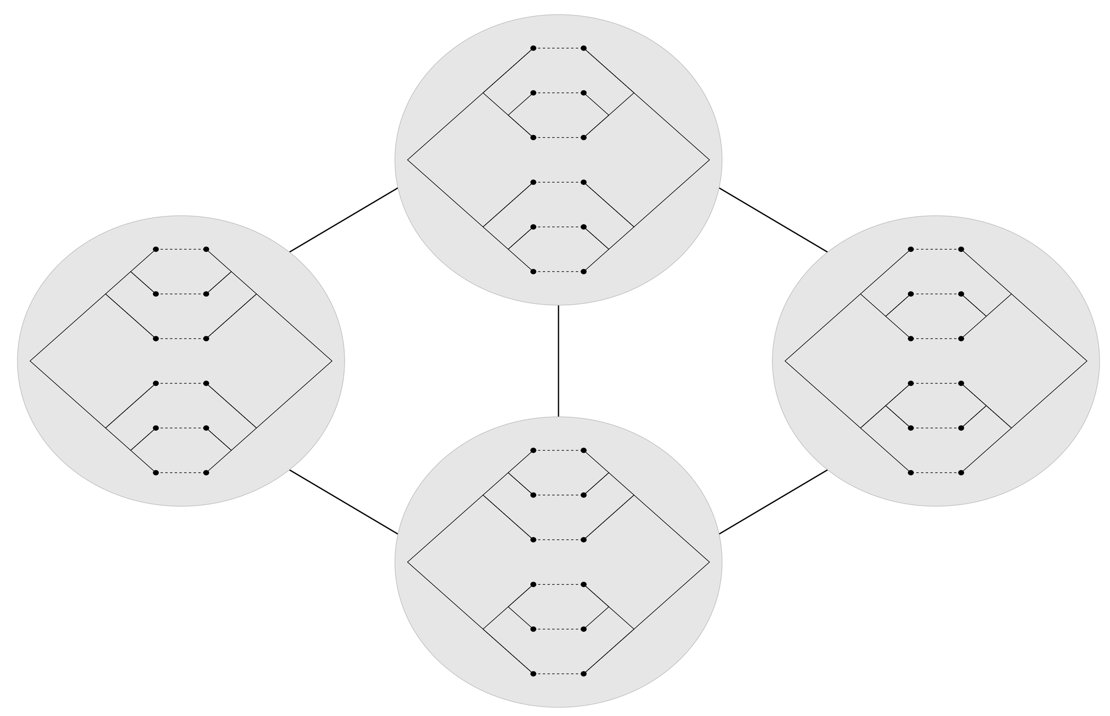
Leaf-matched pairs

A *leaf-matched pair* of a planar tanglegram (T, S, ϕ) is a pair of internal vertices $u \in T$ and $v \in S$ whose descendant leaves are matched by ϕ. A *paired flip* at a leaf-matched pair (u, v) is a flip at u and a flip at v, as shown below for the leaf-matched pair in green.



Results on planar tanglegram layouts

Let (T, S, ϕ) be a planar tanglegram. Starting with a single planar layout of (T, S, ϕ) and performing sequences of paired flips at leaf-matched pairs of vertices generates all possible planar layouts of (T, S, ϕ).



Results on enumeration

The *size* of a tanglegram is the common number of leaves in the component trees. The table below shows the number of tanglegrams of size n with k leaf-matched pairs.

n, k	1	2	3	4	5	6	7	total
2	1							1
3	1	1						2
4	5	4	2					11
5	34	28	11	3				76
6	273	239	102	29	6			649
7	2436	2283	1045	325	73	11		6173
8	23391	23475	11539	3852	968	181	23	63429

See OEIS Sequence A349409 for more terms and connections to other sequences.

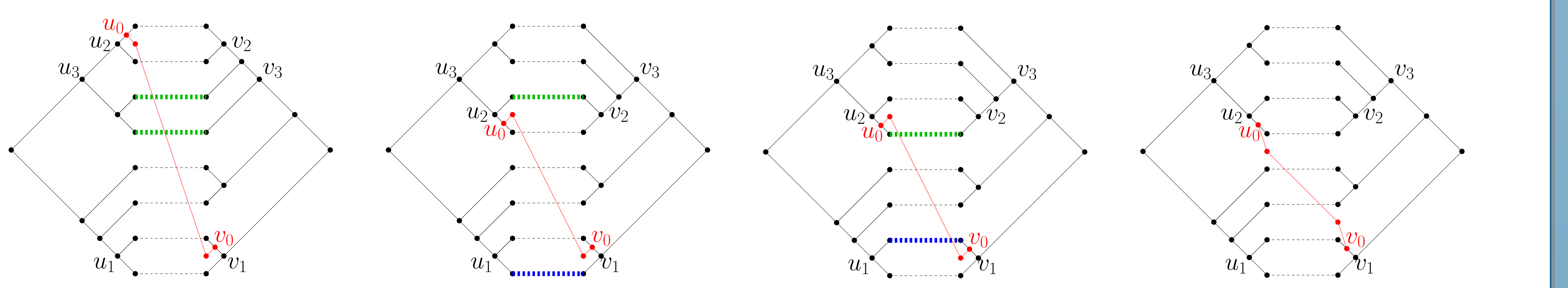
Results on single edge insertion

What if we insert a new pair of matched leaves into a planar tanglegram to form a new tanglegram, and we want to both draw the planar subtanglegram with a planar layout and minimize crossings in the inserted edge? A general approach:

1. consider leaf-matched pairs of the planar subtanglegram and parents of the inserted leaves,
2. only consider operations at a vertex or leaf-matched pair if operations at all ancestors have been considered,
3. only perform operations that preserve planarity of the subtanglegram and minimize crossings that cannot be affected by future operations.

Single edge insertion example

We consider operations in the order $(u_3, v_3), (u_2, v_2), (u_1, v_1), u_0$, and v_0. Crossings that can be affected at a vertex or leaf-matched pair and not at descendants are color-coded.



(u_3, v_3) $\qquad$ (u_2, v_2) and (u_1, v_1) $\qquad$ u_0 and v_0 $\qquad$ final layout

Acknowledgements

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References

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