Shuffle Lattices and Bubble Lattices

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Shuffle lattice

A shuffle word is a subword of a shuffle of $x_1x_2\cdots x_m$ and $y_1y_2\cdots y_n$ for some fixed $m,n\geq 1$. The shuffle lattice is the reflexive, transitive closure of the indel relation on shuffle words: $u\to v$ if v is obtained from u either by deleting an x_i or inserting a y_j. The shuffle lattice is bounded and graded of rank $m+n$.

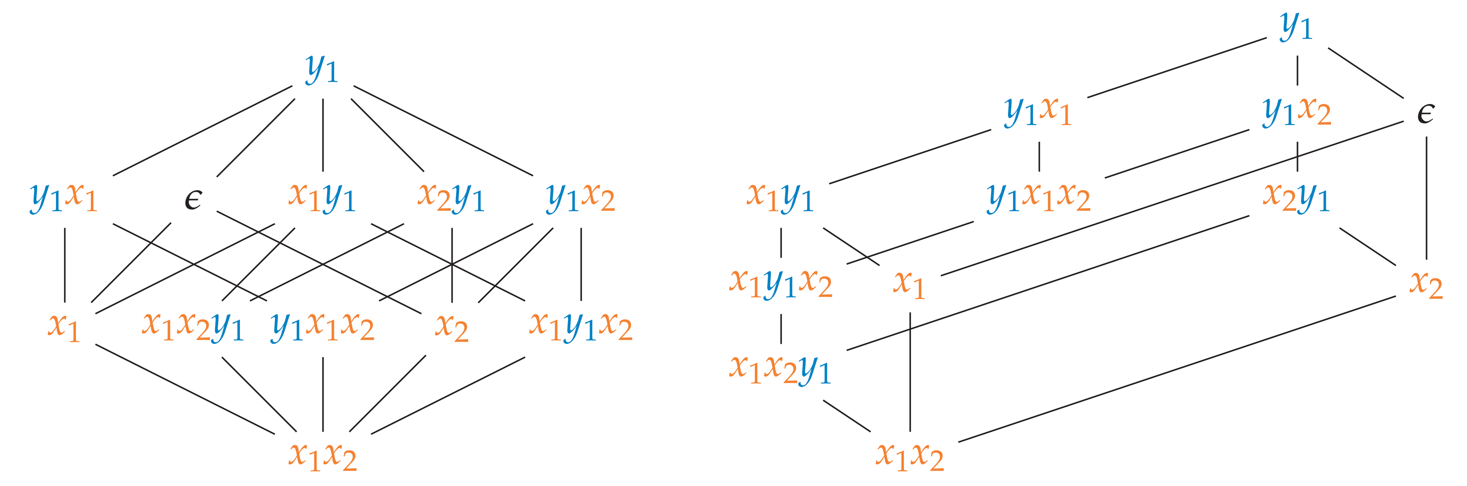

Figure: The shuffle lattice (left) and the bubble lattice (right) for $m=2$ and $n=1$.

Enumeration

Greene proved that many enumerative invariants of the shuffle lattice, such as the rank generating polynomial and the characteristic polynomial, are closely linked. We consider refinements of these two polynomials, called the $\mathbf{H}$-triangle and $\mathbf{M}$-triangle, respectively.
The interface of a shuffle word u is the set of pairs $\{x_i,y_j\}$ for which x_i appears immediately after y_j in u. Let $i(u)$ be the size of the interface of u.

Proposition

For $m,n\geq 0$,

$$\mathbf{H}_{m,n}(p,q)=\sum_u p^{\mathrm{rk}(u)}q^{\mathrm{rk}(u)-i(u)}=\sum_{a\geq 0}\binom{m}{a}\binom{n}{a}p^a(1+pq)^{m+n-2a}.$$

Conjecture

For $m,n\geq 0$,

$$\mathbf{M}_{m,n}(p,q)=\sum_{u,v}\mu(u,v)p^{\mathrm{rk}(u)}q^{\mathrm{rk}(v)}=\sum_{a\geq 0}\binom{m}{a}\binom{n}{a}(p-1)^aq^a(1-q)^a(1-q+pq)^{m+n-2a}.$$

Example: M for $m=2,n=1$

$$\mathbf{M}_{2,1}(p,q)=p^3q^3-5p^2q^3+5p^2q^2+7pq^3-12pq^2-3q^3+5pq+7q^2-5q+1.$$

Coxeter–Catalan enumeration

Chapoton introduced $\mathbf{F},\mathbf{H},\mathbf{M}$-triangles in the Coxeter–Catalan setting, coming from cluster complexes, nonnesting partitions, and noncrossing partitions associated with a finite Coxeter group W. He conjectured that these polynomials are related by certain transformations. These relations were later proved by several authors.

Noncrossing bipartite complex

Fix $m,n\geq 0$. Place letters $x_0,\ldots,x_m$ at the bottom of a horizontal strip, and place $y_0,\ldots,y_n$ at the top. The noncrossing bipartite complex $\Delta(m,n)$ is the simplicial complex whose faces are collections of edges of the form $\{x_i,y_j\}$ and loops that are pairwise noncrossing. We delete the edge $\{x_0,y_0\}$ as well as the loops $\{x_0\}$ and $\{y_0\}$ from this complex.

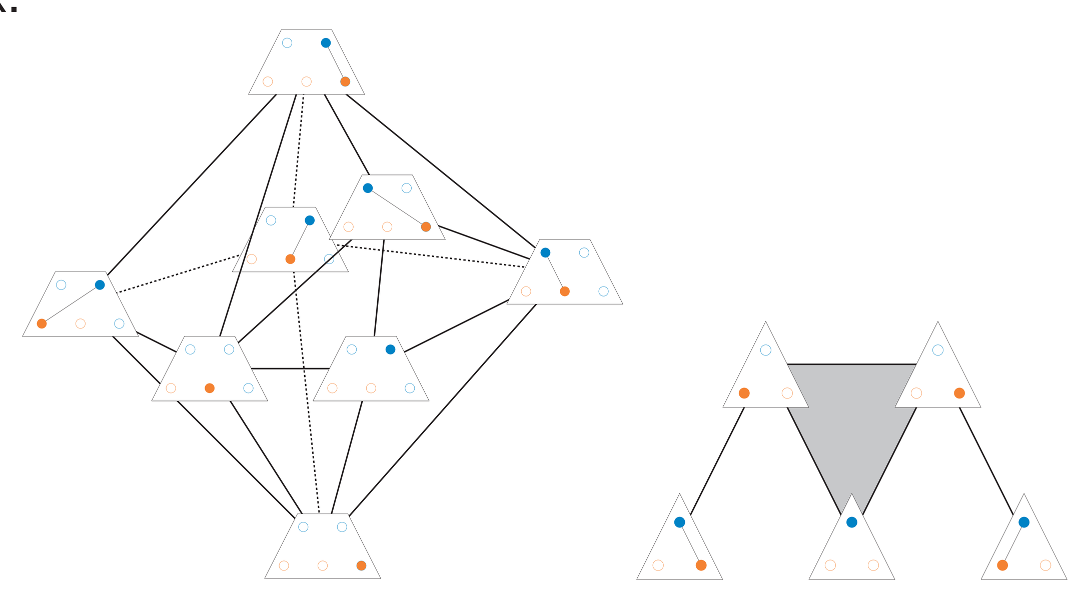

Figure: The noncrossing bipartite complex (left) and the noncrossing matching complex (right) for $m=2$ and $n=1$.

Proposition

The noncrossing bipartite complex is a shellable pseudomanifold homeomorphic to a sphere of dimension $m+n-1$.

Face enumeration

Let $\mathcal{E}$ and $\mathcal{L}$ be the sets of edges and loops, respectively. We consider a refinement of the f-polynomial of $\Delta(m,n)$ known as the $\mathbf{F}$-triangle.

Proposition

For $m,n\geq 0$,

$$\mathbf{F}_{m,n}(p,q)=\sum_\sigma p^{|\sigma\cap\mathcal{E}|}q^{|\sigma\cap\mathcal{L}|}=\sum_{a\geq 0}\binom{m}{a}\binom{n}{a}p^a(p+1)^a(p+q+1)^{m+n-2a}.$$

Example: F for $m=2,n=1$

$$\mathbf{F}_{2,1}(p,q)=3p^3+5p^2q+3pq^2+q^3+7p^2+8pq+3q^2+5p+3q+1.$$

Theorem (Athanasiadis, Thiel)

Let r be the rank of a finite Coxeter group W. Then,

$$\mathbf{F}_W(p,q)=p^r\mathbf{H}_W\left(\frac{p+1}{p},\frac{q+1}{p+1}\right)$$
$$\mathbf{M}_W(p,q)=(1-q)^r\mathbf{H}_W\left(\frac{q(p-1)}{1-q},\frac{p}{p-1}\right).$$

Noncrossing matching complex

The noncrossing matching complex $\Gamma(m,n)$ is the subcomplex of $\Delta(m,n)$ with x_0 and y_0 deleted and with each vertex having degree at most 1. A vertex with a loop has degree 1 by convention. The $\mathbf{H}$-triangle has an alternate interpretation in terms of this complex, namely

$$\mathbf{H}_{m,n}(p,q)=\sum_\sigma p^{|\sigma|}q^{|\mathcal{L}\cap\sigma|}.$$

Bubble lattice

The bubble lattice extends the shuffle lattice by including transpositions: $u\Rightarrow v$ if v is obtained from u by swapping x_i with an adjacent y_j on its right.

Theorem

The bubble lattice is an orientation of the dual graph of the noncrossing bipartite complex. Any linear extension of the bubble lattice is a shelling order of this complex.

Lattice structure

A finite lattice is congruence–uniform if it can be obtained from a 1–element lattice by a sequence of interval doublings. Such lattices come equipped with two additional structures: the canonical join complex and the core label order.

Theorem

The bubble lattice is a congruence–uniform lattice. Its canonical join complex is isomorphic to the noncrossing matching complex.

When $m\leq 1$ or $n\leq 1$, the core label order of the bubble lattice is isomorphic to the shuffle lattice. This is not true if $m,n\geq 2$, but we conjecture the following in general.

Conjecture

The core label order of the bubble lattice is a graded poset that refines the shuffle lattice and shares its rank function.

Example: H for $m=2,n=1$

$$\mathbf{H}_{2,1}(p,q)=p^3q^3+3p^2q^2+2p^2q+3pq+2p+1.$$

Chapoton triangles for bubble lattices

The $\mathbf{F},\mathbf{H},\mathbf{M}$-triangles introduced in our work are conjecturally related by the same transformations, where the parameter r is replaced by $m+n$. Our main goal is to find an appropriate generalization of both theories that explains why these relations occur.