Peaks are Preserved under Run-Sorting

Surprising consequences of the ***Run-sort*** *function!*

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Background

☞ By Run-sort, we mean rearranging the runs of a permutation $\sigma \in S_n$ in lexicographical order.

Example 1.

$$\mathrm{runsort}(29\ 7\ 368\ 5\ 14) = 14\ 29\ 368\ 5\ 7$$

☞ We study permutations whose runs are run-sorted, i.e., run-sorted permutations (resp. $\mathcal{RSP}(n)$).

☞ A bijection on permutations over $[n] = \{1, 2, \dots, n\}$, which keeps track of the peak-values before and after applying the run-sort function.

☞ We further show that the descent generating polynomials, $A_n(t)$ for $\mathcal{RSP}(n)$ are real rooted, and satisfy an interlacing property.

Definition 2. A peak of a permutation $\sigma \in S_n$, is an integer i, $1 < i < n$ such that $\sigma(i-1) < \sigma(i) > \sigma(i+1)$ and the corresponding $\sigma(i)$ is a peak-value of σ.

Given a permutation σ, we let $\mathrm{runsort}(\sigma)$ denote the permutation obtained by rearranging the runs of σ lexicographically. Hence, if $\sigma \in S_n$, then $\mathrm{runsort}(\sigma) \in \mathcal{RSP}(n)$. We let $\mathrm{PKV}(\sigma)$ denote the set of peak-values of the permutation σ, and $\mathrm{SPV}(\sigma) \coloneqq \mathrm{PKV}(\mathrm{runsort}(\sigma))$.

A recursion for permutations

We recursively construct permutations of length n, from those of length $n-1$, by inserting n somewhere.

Let $a \in \{\emptyset, 1, 2, \dots, n-1\}$, $\sigma \in S_{n-1}$. Then

$$\mathrm{Stay}_a(\sigma) = \begin{cases} \text{insert } n \text{ after } a \\ \text{insert } n \text{ at the start if } a = \emptyset \end{cases}$$

Note: We never consider $a = \emptyset$ as an entry in the permutation. It is just a label.

Then we have the map $\mathcal{B} : \{\emptyset, 1, 2, \dots, n-1\} \times S_{n-1} \to S_n$, i.e., $f(a, \sigma) = \mathrm{Stay}_a(\sigma)$.

Question: Can we track peak-values?

Recursively constructing permutations while tracking peaks

For simplicity, we set $\pi' \coloneqq \mathrm{Stay}_a(\pi)$ and we let k be the value immediately succeeding a in π (unless a is the last entry in π). Then we have the following choices.

(1) $a = \emptyset$, so $\mathrm{PKV}(\pi') = \mathrm{PKV}(\pi)$.

(2) a is the last entry of π, so $\mathrm{PKV}(\pi') = \mathrm{PKV}(\pi)$.

(3) $a \in \mathrm{PKV}(\pi)$. Then $\mathrm{PKV}(\pi') = (\mathrm{PKV}(\pi) \setminus \{a\}) \cup \{n\}$.

(4) $k \in \mathrm{PKV}(\pi)$. Then $\mathrm{PKV}(\pi') = (\mathrm{PKV}(\pi) \setminus \{k\}) \cup \{n\}$.

(5) Otherwise $\mathrm{PKV}(\pi') = \mathrm{PKV}(\pi) \cup \{n\}$.

Example 3. Consider a permutation $\pi = 21574368 \in S_8$. Then

☞ $\mathrm{Stay}_{\emptyset}(\pi) = 921574368$

☞ $\mathrm{Stay}_8(\pi) = 215743689$

☞ $\mathrm{Stay}_7(\pi) = 215794368$

☞ $\mathrm{Stay}_5(\pi) = 215974368$.

Runsort function

- Let runsort : $S_n \to \mathcal{RSP}(n)$ (not injective!)
- Let $\mathrm{SPV}(\sigma) \coloneqq \mathrm{PKV}(runsort(\sigma))$
- Surprise! (see title)

$$\sum_{\sigma \in S_n} X_{\mathrm{PKV}(\sigma)} = \sum_{\sigma \in S_n} X_{SPV(\sigma)}$$

- Next we describe the bijection $\eta : S_n \to S_n$, where $\mathrm{PKV}(\sigma) = \mathrm{SPV}(\eta(\sigma))$.

For any $n \geq 1$, there is a bijection

$$\mathcal{C} : \{\emptyset, 1, 2, \dots, n-1\} \times S_{n-1} \to S_n$$

which has the following properties ($\mathcal{C}$ is not $Stay_a(\sigma)$ in general!)

(1) If $a = \emptyset$, then $\mathrm{SPV}(\pi') = \mathrm{SPV}(\pi)$.

(2) If a is the last entry of $\mathrm{runsort}(\pi)$, then $\mathrm{SPV}(\pi') = \mathrm{SPV}(\pi)$.

(3) If $a \in \mathrm{SPV}(\pi)$, then
$$\mathrm{SPV}(\pi') = (\mathrm{SPV}(\pi) \setminus \{a\}) \cup \{n\}.$$

(4) If $k \in \mathrm{SPV}(\pi)$, then
$$\mathrm{SPV}(\pi') = (\mathrm{SPV}(\pi) \setminus \{k\}) \cup \{n\}.$$

(5) If a is not the last entry of $\mathrm{runsort}(\pi)$, and neither a or k are in $\mathrm{SPV}(\pi)$, then
$$\mathrm{SPV}(\pi') = \mathrm{SPV}(\pi) \cup \{n\}.$$

Table showing examples of how η works

σ	$\eta(\sigma)$	σ	$\eta(\sigma)$	σ	$\eta(\sigma)$	σ	$\eta(\sigma)$
12	12	1234	1234	2314	2413	3412	3124
21	21	1243	1243	2341	2134	3421	3241
123	123	1324	1324	2413	2314	4123	4123
132	132	1342	1342	2431	2143	4132	4132
213	231	1423	1423	3124	3412	4213	4231
231	213	1432	1432	3142	3142	4231	4213
312	312	2134	2341	3214	3421	4312	4312
321	321	2143	2431	3241	3214	4321	4321

Coopman and Rubey, see [CR21] have recently found more properties of permutations using the runsort function.

Probabilistic statements

Let $\sigma \in S_n$ be a uniformly chosen permutation, and let $\sigma' \coloneqq \mathrm{runsort}(\sigma)$. As $n \to \infty$ does this curve approach some limit curve?

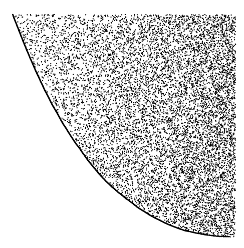

Figure 1: A random permutation matrix σ' after runsort, for $n = 20000$. The entries equal to 1 are shaded black.

This question has recently been answered in the affirmative by Alon, Defant and Kravitz, see [ADK22].

Real-rootedness and interlacing roots

Definition 4 ([Wag92]). Let g be a polynomial of degree n with non-positive roots $g_1 \leq g_2 \leq \cdots \leq g_n$. If f is a degree $n-1$ polynomial with non-positive roots $f_1 \leq f_2 \leq \cdots \leq f_{n-1}$, we say that the roots of f interlace those of g, if

$$g_1 \leq f_1 \leq g_2 \leq f_2 \leq \cdots \leq f_{n-1} \leq g_n \leq 0.$$

we show that the polynomials

$$A_n(t) \coloneqq \sum_{\sigma \in \mathcal{RSP}(n)} t^{\mathrm{des}(\sigma)}$$

are real-rooted. Moreover, the roots of $A_{n-1}(t)$ *interlace* the roots of $A_n(t)$.

We let $f_{n,k}$ be the number of run-sorted permutations of $[n]$ having k runs. In [NRB20], it was proved that the numbers $f_{n,k}$ satisfy the recurrence relation

$$f_{n,k} = k f_{n-1,k} + (n-2) f_{n-2,k-1} \text{ whenever } 1 \leq k < n.$$

Hence we have that

$$tA_n(t) = \sum_{\pi \in \mathcal{RSP}(n)} t^{\mathrm{des}(\pi)+1} = \sum_{k \geq 1} t^k f_{n,k}. \quad (1)$$

From Equation 1, let us set $R_n(t) \coloneqq tA_n(t)$.

Lemma 5. *$R_n(t)$ satisfies the recurrence*

$$R_n(t) = tR'_{n-1}(t) + t(n-2)R_{n-2}(t), R_1(t) = R_2(t) = t. \quad (2)$$

From Equation 2, we then prove Theorem 6 using a result by Wagner, See, [Wag92, Sec. 3] as a main tool.

Theorem 6. *The polynomials*

$$R_n(t) = \sum_{\pi \in \mathcal{RSP}(n)} t^{\mathrm{des}(\pi)+1}$$

satisfy $R_{n-1} \ll R_n$ for all $n \geq 1$. In particular, they are all real-rooted.

Finally, we end with a recursion for a multivariate extension of $A_n(t)$.

Theorem 7. *For all integers $n \geq 1$, let*

$$A_n(\mathbf{x}) \coloneqq \sum_{\pi \in \mathcal{RSP}(n)} \prod_{j \in \mathrm{DES}(\pi)} x_{n-j}.$$

Then

$$A_n(\mathbf{x}) = 1 + \sum_{i=1}^{n-2} \left(\binom{n-1}{i} - 1 \right) x_i A_i(\mathbf{x})$$

by indexing of the descent set from the end.

This solves the problem posed in [NRB20] for the exponential generating function of the $A_n(t)$.

Below, we illustrate $A_5(\mathbf{x})$ where $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)$ for all $\pi \in \mathcal{RSP}(5)$.

We have that

$$A_5(\mathbf{x}) = 1 + 3x_3 + 5x_2 + 3x_1 + 3x_3x_1$$

.

1	12345
x_3	13245, 14235, 15234
x_2	12435, 12534, 13425, 13524, 14523
x_1	12354, 12453, 13452
x_3x_1	13254, 14253, 15243

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