Generalized permutahedra and positive flag Dressians

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Permutahedra

The **(regular) permutahedron** is the convex hull of the orbit of the point $(1, \dots, n)$ under the action of Sym_n under permuting coordinates.

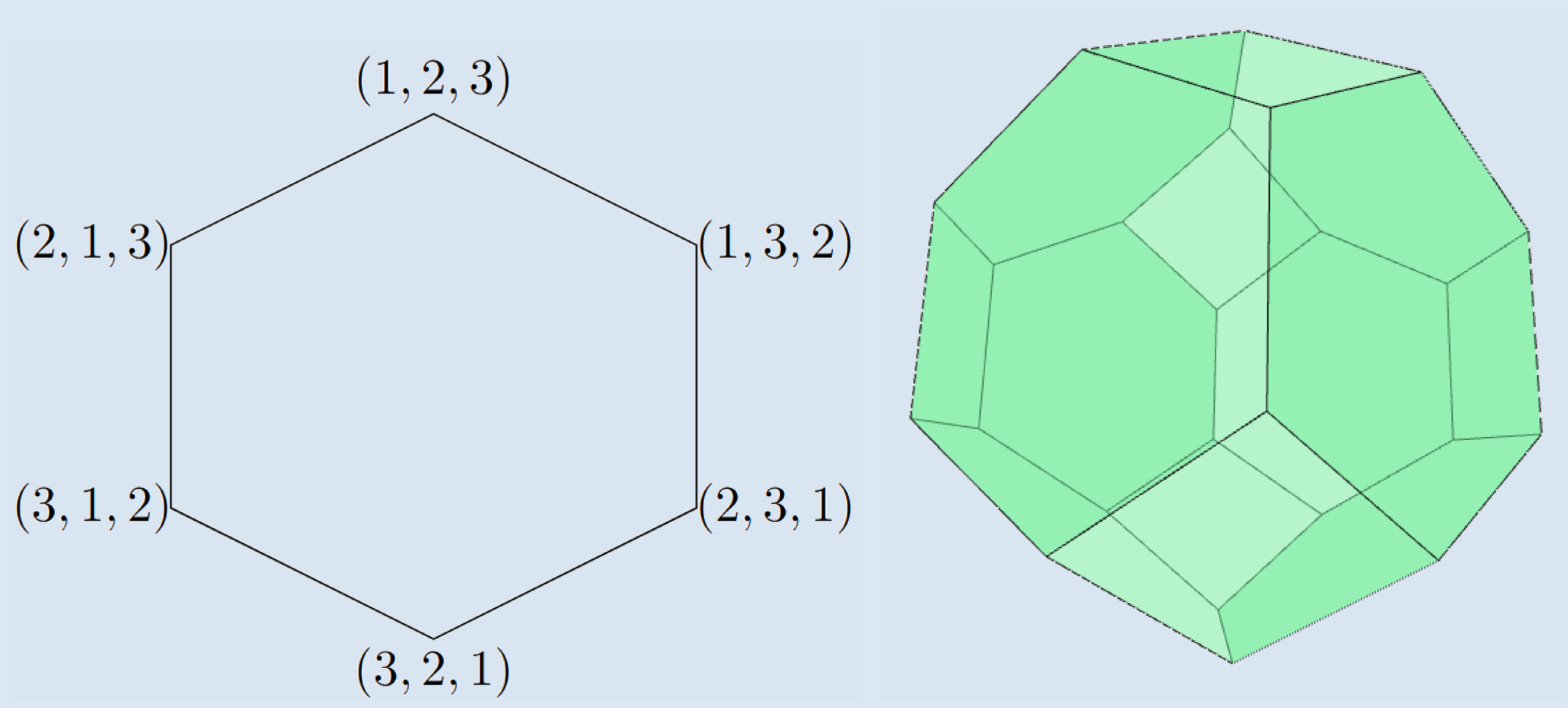

Figure 1: Π_3 (left) and Π_4 (right).

A **generalized permutahedron** is a polytope with all edges parallel to an edge of the permutahedron.

Flag matroids

A **matroid polytope** P_M is the convex hull of the indicator vectors of bases $\mathcal{B}(M)$ of a matroid M. A **matroid quotient** $M_1 \leftarrow M_2$ if every flat of M_1 is a flat of M_2. A **flag matroid** $\mathcal{M} = (M_1, \dots, M_n)$ is a sequence of matroid quotients. The polytope of a flag matroid is the Minkowski sum

$$P_{\mathcal{M}} := P_{M_1} + \cdots + P_{M_n}.$$

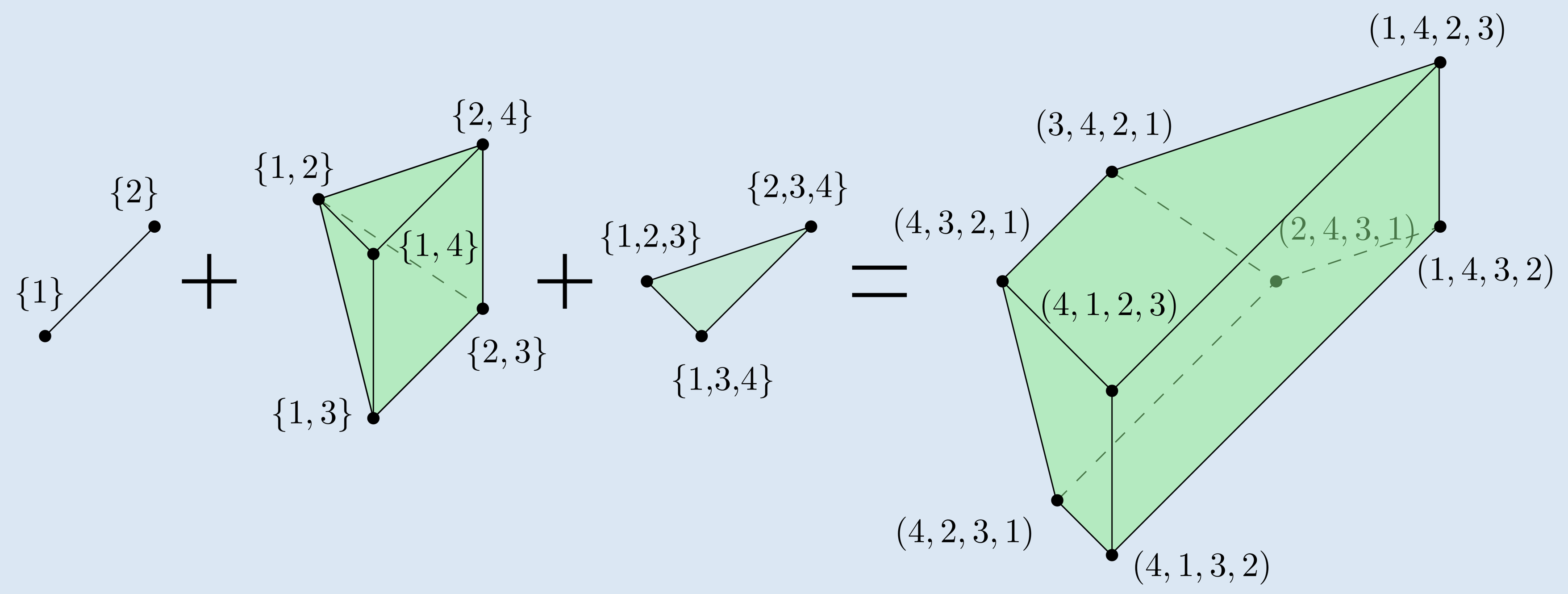

Figure 2: A flag matroid polytope and its matroid summands.

Flag matroid polytopes are exactly the subpolytopes of Π_n which are generalized permutahedra [2].

Bruhat polytopes

The **(strong) Bruhuat order** on Sym_n is the partial order where $\sigma \leq \tau$ if σ can be expressed as a subword of reduced expression for τ. A **Bruhat (interval) polytope** is the convex hull of a Bruhat interval.

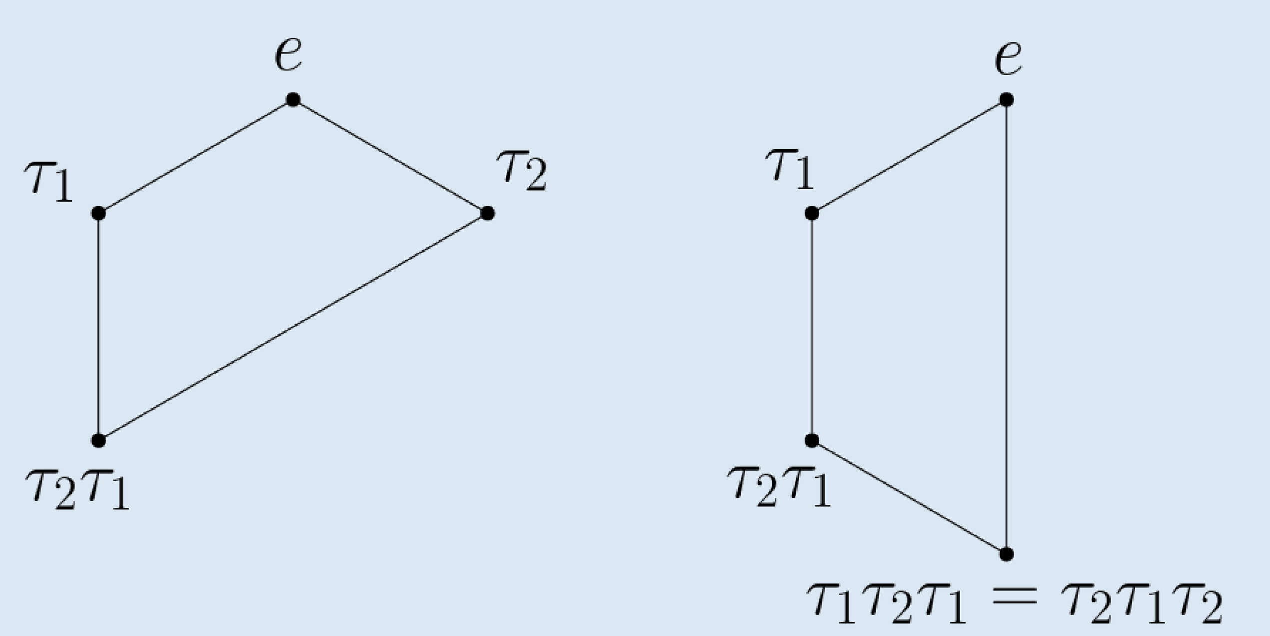

Figure 3: The polytope corresponding to the Bruhat interval $[e, \tau_2\tau_1]$ (left) and a flag matroid polytope which is not Bruhat (right).

The flag variety

The **flag variety** $\mathrm{Fl}(n)$ is the space of all flags $L_1 \subset \cdots \subset L_n$ of linear subspaces. Given a flag $\mathcal{L} = L_1 \subset \cdots \subset L_n$, then $\mathcal{M}(\mathcal{L}) := (M(L_1), \dots, M(L_n))$ is a flag matroid. The **totally non-negative (TNN) part** of $\mathrm{Fl}(n)$ consists of all flags $\mathcal{L}$ where $P_{\mathcal{M}(\mathcal{L})}$ is a Bruhat polytope [5].

Permutahedral subdivisions

A **permutahedral subdivision** of a generalized permutahedron P is a subdivision into generalized permutahedra.

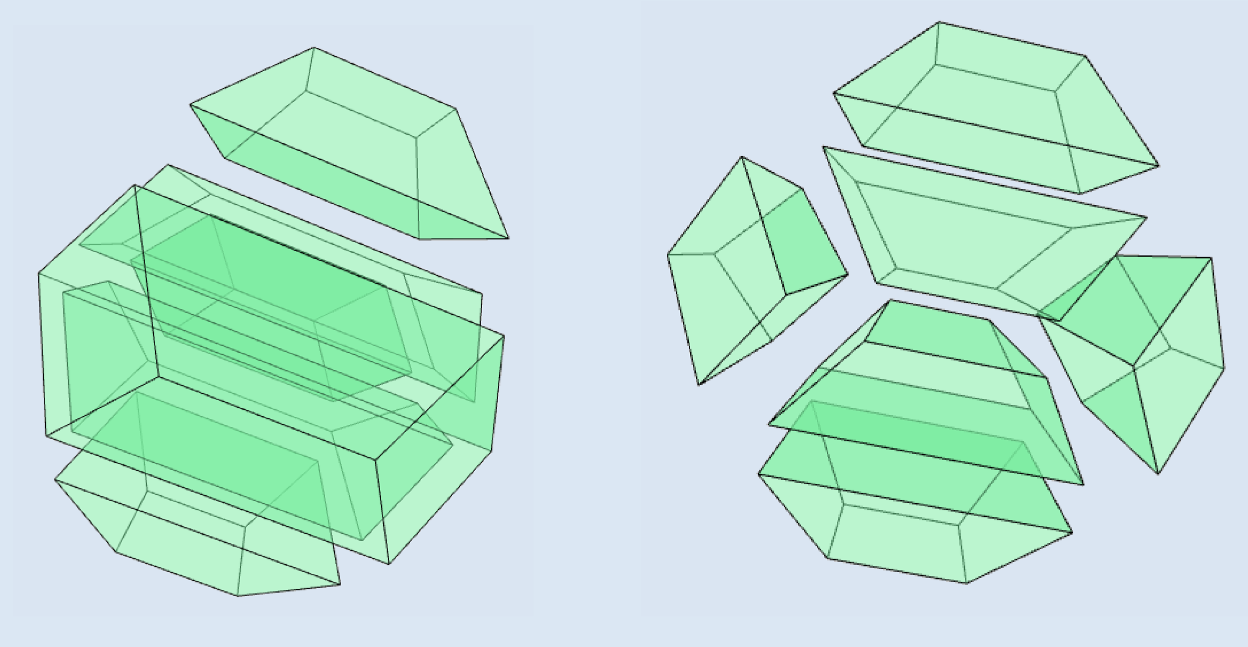

Figure 4: Two permutahedral subdivisions of Π_4 inducing the same subdivision on the 2-skeleton.

A **valuated matroid** is a function $\mathcal{B}(M) \to \mathbb{R}$ inducing a regular permutahedral subdivision on P_M. A **valueted flag matroid** $(\mu_1, \dots, \mu_n)$ is a sequence of valuated matroids such that the mixed subdivision of $P_{\mathcal{M}}$ is permutahedral [3]. Points in the $\mathrm{trop}(\mathrm{Fl}(n))$ are valueted flag matroids.

Main results

In [7] it is shown that a subdivision is matroidal, if it is matroidal in the 3-skeleton. We show a flag analogue:

Theorem 1 ([4])**.** *A function $w\colon \mathrm{Sym}(n) \to \mathbb{R}$ induces a permutahedral subdivision if and only if it induces a permutahedral subdivision of the 2-skeleton of Π_n. That is:*

(HEX) for every hexagon $abcdef$ in the 2-skeleton of Π_n, we have

(HXE) $w(a) + w(c) + w(e) = w(b) + w(d) + w(f)$,

(HXM) the maximum in $\max(w(a)+w(d), w(b)+w(e), w(c)+w(f))$ is attained twice;

(SQR) for every square face $abcd$ of Π_n, $w(a) + w(c) = w(b) + w(d)$.

Remark: We are using the min convention; (HXM) is degree -2 tropical equation.
In [1, 6] it is shown that the tropicalization of the positive Grassmannian corresponds to positroidal subdivisions. We show a flag analogue:

Theorem 2 ([4])**.** *Let $w : \mathrm{Sym}_n \to \mathbb{R}$. The following are equivalent*

1. *$w(\sigma) = \mu_1(I_1) + \cdots + \mu_n(I_n)$ where $I_1 \subset \cdots \subset I_n$ is the flag corresponding to σ and $(\mu_1, \dots, \mu_n)$ is a valuated flag matroid coming from the tropicalization of the positive flag variety.*
2. *The regular subdivision induced by w consists of Bruhat polytopes.*
3. *w satisfies (SQR), (HXE) and*

(HXM+) for every hexagon $abcdef$ of Π_n, where b is the lowest permutation, $w(b) + w(e) = \max(w(a) + w(d), w(c) + w(f))$.

The conditions above define the **positive flag Dressian**.

References

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