# Rowmotion on Fence Posets

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## Overview

- We initiate the study of rowmotion dynamics on antichains and on order-ideals of *fence posets*, including orbit structure and homomesy.
- One major tool is a representation of rowmotion orbits as 3-color tilings of a cylinder by certain tiles.
- We isolate a new phenomenon called *homometry*, where statistics must have the same average whenever orbit sizes are the same.
- For order-ideal rowmotion on fence posets, we show that we get the same homomesies and homometries independently of the order in which we toggle the elements.
- We also prove a general homomesy result for order rowmotion on any self-dual poset.

This poster is available digitally at: `www.matthewplante.com`

## Definitions

### Definition: Fence Poset

A *fence* is a poset with elements $F = \{x_1, x_2, \ldots, x_n\}$ and covers

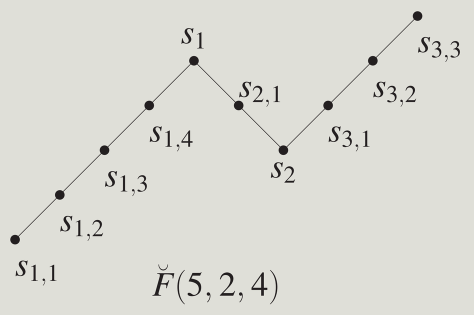
$$x_1 \lhd x_2 \lhd \ldots \lhd x_a \rhd x_{a+1} \rhd \ldots \rhd x_b \lhd x_{b+1} \lhd \cdots$$

where $a, b, \ldots$ are positive integers.

Fences have important connections with cluster algebras, $q$-analogues, unimodality, and Young diagrams [C20, MGO21, OR21]. The maximal chains of $F$ are called *segments*, where the $i$th segment is denoted by $S_i$. Elements on two segments are called *shared*. All other elements are *unshared*. If $F$ has $t$ segments, then we let $F = \check{F}(\alpha_1, \ldots, \alpha_t)$ where for all $i$

$$\alpha_i = (\#\text{ of unshared elements on segment } i) + 1.$$

In $F$, we let $\check{S}_i$ = the set of unshared elements of segment $S_i$; $s_{i,j}$ = the $j$th smallest element of $\check{S}_i$; and $s_i$ = the unique element of $S_i \cap S_{i+1}$.




### Definition: Rowmotion and transfer maps

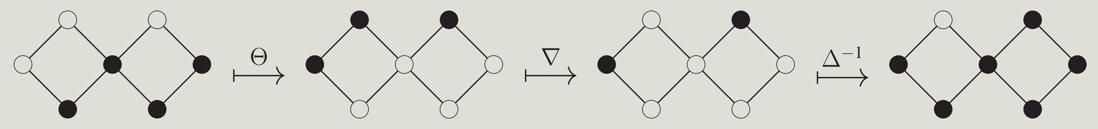
Let $P$ be a generic poset. We define natural bijections between the sets $\mathcal{J}(P)$ of all *order ideals* (aka downsets) of $P$, $\mathcal{F}(P)$ of all *order filters* (aka upsets) of $P$, and $\mathcal{A}(P)$ of all *antichains* of $P$.

- The map $\Theta : 2^P \to 2^P$ where $\Theta(S) = P \setminus S$ is the **complement** of $S$ (sending order ideals to filters and vice versa).
- The **up-transfer** $\Delta : \mathcal{J}(P) \to \mathcal{A}(P)$, where $\Delta(I)$ is the set of maximal elements of $I$. For an antichain $A \in \mathcal{A}(P)$, $\Delta^{-1}(A) = \{x \in P : x \le y \text{ for some } y \in A\}$.
- The **down-transfer** $\nabla : \mathcal{F}(P) \to \mathcal{A}(P)$, where $\nabla(F)$ is the set of minimal elements of $F$. For an antichain $A \in \mathcal{A}(P)$, $\nabla^{-1}(A) = \{x \in P : x \ge y \text{ for some } y \in A\}$.

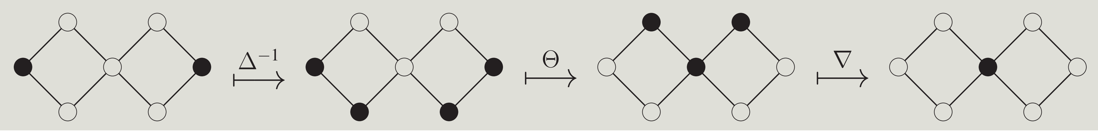
**Order ideal rowmotion** is the map $\widehat{\rho} : \mathcal{J}(P) \to \mathcal{J}(P)$ given by the composition $\widehat{\rho} = \Delta^{-1} \circ \nabla \circ \Theta$.
**Antichain rowmotion** is the map $\rho : \mathcal{A}(P) \to \mathcal{A}(P)$ given by the composition $\rho = \nabla \circ \Theta \circ \Delta^{-1}$.

### Example of Order Ideal and Antichain Rowmotion

In each step, the elements of the subset of the poset are given by the filled-in circles. First, an example of order ideal rowmotion.




Here is an example of antichain rowmotion. Notice the change in the number of antichain elements.




### Definition: Homomesy and Homometry

Let $G$ be a finite group acting on a finite set $S$. Let $\mathrm{st} : S \to \mathbb{Q}$ be a statistic. If $\mathcal{O} \subseteq S$, then we let

$$\mathrm{st}\,\mathcal{O} = \sum_{x \in \mathcal{O}} \mathrm{st}\,x.$$

Call st *homomesic* if $\mathrm{st}\,\mathcal{O}/\#\mathcal{O}$ is constant over all orbits $\mathcal{O}$, where the hash-tag means cardinality. In particular, st is *c-mesic* if, for all orbits $\mathcal{O}$,

$$\frac{\mathrm{st}\,\mathcal{O}}{\#\mathcal{O}} = c.$$

Call st *homometric* if for any two orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ we have

$$\#\mathcal{O}_1 = \#\mathcal{O}_2 \implies \mathrm{st}\,\mathcal{O}_1 = \mathrm{st}\,\mathcal{O}_2.$$

Note that homomesy implies homometry, but not conversely.

### Example homomesy from binary strings with rotation

$$\mathcal{B}_{n,k} := \{w_1w_2\ldots w_n \mid w_i \in \{0,1\} \text{ for all } i;\ w_1 + \cdots + w_n = k\}$$

with rotation $w_1w_2\ldots w_n \mapsto w_nw_1\ldots w_{n-1}$ that generates a cyclic group $C_n$ which acts on $\mathcal{B}_{n,k}$, and *inversion statistic*

$$\mathrm{inv}\, w_1w_2\ldots w_n = \#\{(i,j) \mid i < j \text{ and } w_i > w_j\}.$$

**Ex.** When $n = 4$ and $k = 2$ there are two orbits

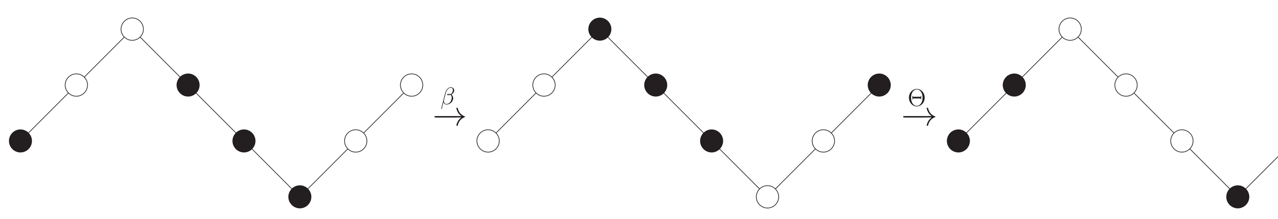
| $w$ | inv $w$ | $w$ | inv $w$ |
|---|---|---|---|
| 1100 | 4 | 1010 | 3 |
| 0110 | 2 | 0101 | 1 |
| 0011 | 0 | | |
| 1001 | 2 | | |
| average = 8/4 = 2 | | average = 4/2 = 2 | |

## Dual Posets

Let $P^*$ be the dual of poset $P$. Suppose $P$ is self dual so that $P \cong P^*$. Thus there exists an order-reversing bijection $\beta : P \to P$ which naturally extends to a bijection $\beta : 2^P \to 2^P$ that acts as a 180 degree rotation of the Hasse diagram. Define the *ideal complement* of $I \in \mathcal{J}(P)$ as

$$\overline{I} = \Theta \circ \beta(I)$$

where $\Theta(I) = P \setminus I$. Note that $\#I + \#\overline{I} = \#P$.




We denote the indicator function of $x$ in antichain $A$ by $\chi_x(A)$ and the indicator function of $x$ in order ideal $I$ by $\widehat{\chi}_x(I)$. Similarly, let $\chi(\mathcal{O})$ be the number of antichain elements in an orbit $\mathcal{O}$ of $\rho$, and $\widehat{\chi}(\mathcal{O})$ be the number of order ideal elements in an orbit $\mathcal{O}$ of $\widehat{\rho}$.

### Theorem [EPRS]: Homomesy for self-dual posets

Let $P$ be self-dual with $n = \#P$, and fix an order-reversing bijection $\beta : 2^P \to 2^P$. Let $I \in \mathcal{J}(P)$.

- If $I, \overline{I} \in \mathcal{O}$ for some orbit $\mathcal{O}$, then $\dfrac{\widehat{\chi}(\mathcal{O})}{\#\mathcal{O}} = \dfrac{n}{2}$.
- If $I \in \mathcal{O}$ and $\overline{I} \in \overline{\mathcal{O}}$ for some orbits $\mathcal{O}$ and $\overline{\mathcal{O}}$ with $\mathcal{O} \neq \overline{\mathcal{O}}$, then $\#\mathcal{O} = \#\overline{\mathcal{O}}$ and $\dfrac{\widehat{\chi}(\mathcal{O} \uplus \overline{\mathcal{O}})}{\#(\mathcal{O} \uplus \overline{\mathcal{O}})} = \dfrac{n}{2}$.

## Toggling

Cameron and Fon-der-Flaass showed that order ideal rowmotion can be realized as a composition of toggling involutions by "toggling once at each element of $P$ along any linear extension (from top to bottom)" [CF95]. Similarly, Joseph showed antichain rowmotion is a product of antichain toggles from bottom to top along any linear extension [J19]. Here are the definitions of order ideal toggling on the left and antichain toggling on the right.

$$\widehat{\tau}_i(I) = \begin{cases} I \setminus \{i\} & \text{if } i \in I \text{ and } I \setminus \{i\} \in \mathcal{J}(P) \\ I \cup \{i\} & \text{if } i \notin I \text{ and } I \cup \{i\} \in \mathcal{J}(P) \\ I & \text{otherwise.} \end{cases} \qquad \tau_i(I) = \begin{cases} I \setminus \{i\} & \text{if } i \in I \\ I \cup \{i\} & \text{if } i \notin I \text{ and } I \cup \{i\} \in \mathcal{A}(P) \\ I & \text{otherwise.} \end{cases}$$

### Theorem [EPRS]: Toggling order of order ideals of a fence

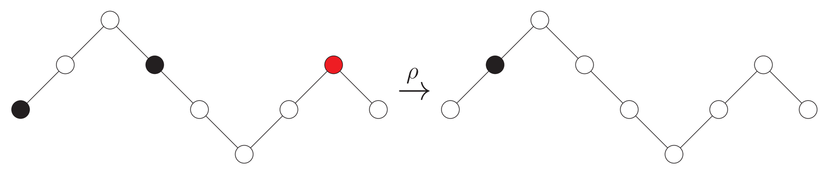
Let $F$ be a fence poset. Let $\psi$ be a product of the *order ideal* toggles $\widehat{\tau}_1, \ldots, \widehat{\tau}_n$ in any order. Any statistic which is a linear combination of the indicator functions $\widehat{\chi}_j$ is $c$-mesic under the action of $\widehat{\rho}$ if and only if it is $c$-mesic under the action of $\psi$.

### Conjecture [EPRS]: Toggling order of antichains of a fence

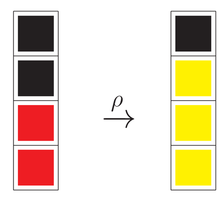
Let $F$ be a fence poset. Let $\psi$ be a product of the *antichain* toggles $\tau_1, \ldots, \tau_n$ in any order. Any statistic which is a linear combination of the indicator functions $\chi$ is $c$-mesic under the action of $\rho$ if and only if it is $c$-mesic under the action of $\psi$.

## Bijection between orbits of antichain rowmotion on fence posets and cylinder tilings

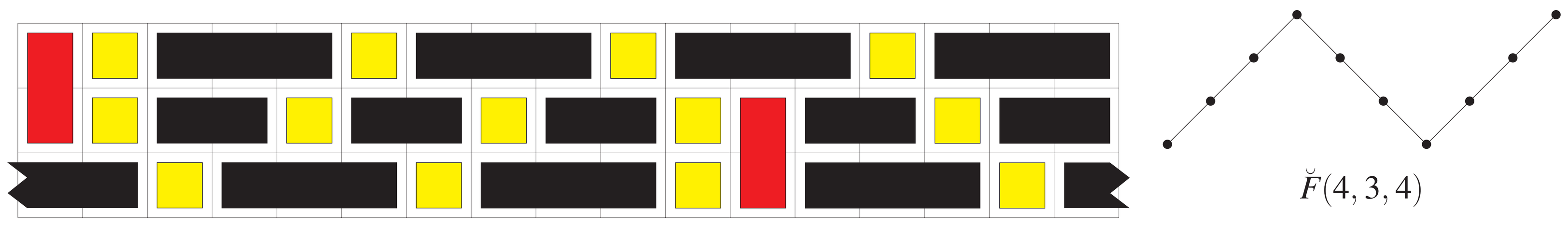
Here is an example of antichain rowmotion on $F = \check{F}(3,3,2,2)$:




Antichains are represented with a column of boxes, one for each segment. Shade a square black if the corresponding segment includes an unshared element, red for a shared element, yellow for no element. For the example on the right, represent an antichain $A \subset F$ using a column of 4 boxes, with the box in row $i$ from the top corresponding to the $i$th segment $S_i$ from the left.




We can model any orbit of $\rho$ by collecting these columns into a cylinder. Here is one orbit of size 17 for $\check{F}(4,3,4)$:




If $F = \check{F}(\alpha_1, \ldots, \alpha_t)$ with $\alpha = (\alpha_1, \ldots, \alpha_t)$ the tiling is composed of black $1 \times (\alpha_i - 1)$ tiles in row $i$, yellow $1 \times 1$ tiles, and red $2 \times 1$ tiles.

- If $\alpha_i > 1$ and the red tiles are ignored, then the black and yellow tiles alternate in row $i$.
- There is a red tile in a column covering rows $i$ and $i+1$ if and only if either the previous column contains two yellow tiles in those two rows when $i$ is even, or the next column contains two yellow tiles in those two rows when $i$ is odd.

## Orbit and homomesy results for general fences

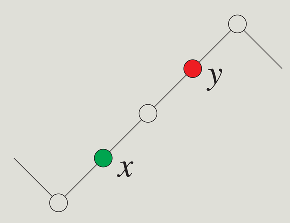
Let

- $b_i :=$ the number of black tiles in row $i$ of a tiling,
- $r_i :=$ the number of red tiles whose top box is in row $i$ of a tiling, and
- $\chi(\mathcal{O}) :=$ the number of antichain elements in orbit $\mathcal{O}$.

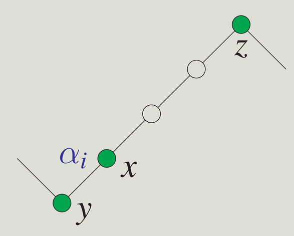
### Theorem [EPRS]: Cardinality statistic homomesies for fences

Given an orbit $\mathcal{O}$ in fence $\check{F}(\alpha)$ with corresponding $\alpha$-tiling,

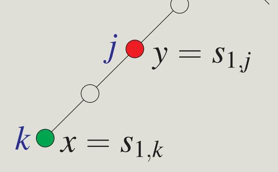
(a) If $x, y \in \check{S}_i$ for some $i$, then $\chi_x - \chi_y$ is 0-mesic.



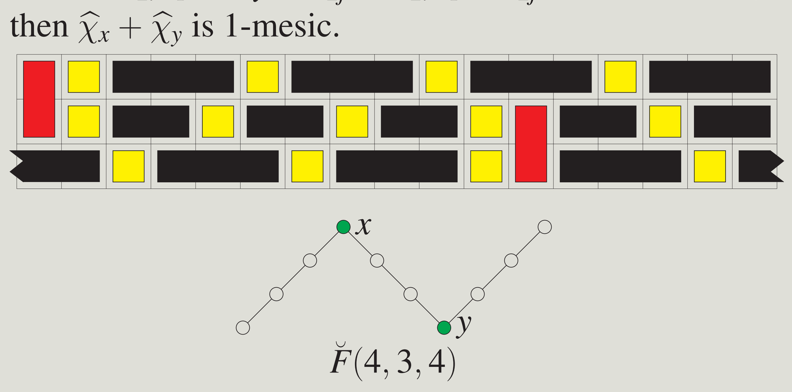

(b) If $x \in \check{S}_i$, $y = s_i$ and $z = s_{i-1}$, then $\alpha_i\chi_x + \chi_y + \chi_z$ is 1-mesic.




(c) If $x = s_{1,j}$ and $y = s_{1,k}$, then $k\widehat{\chi}_x - j\widehat{\chi}_y$ is $(k-j)$-mesic.



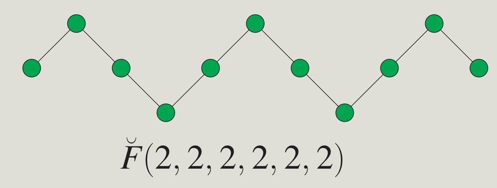

(d) Let $x = s_{2i-1}$ and $y = s_{2j}$. If $r_{2i-1} = r_{2j}$ for all orbits $\mathcal{O}$, then $\widehat{\chi}_x + \widehat{\chi}_y$ is 1-mesic.




(e) If $t$ is odd and all the $\alpha_i$ are even, then $\#\mathcal{O}$ is even for all orbits $\mathcal{O}$.

(f) If $\alpha_i = 2$ for all $i \in [t]$, then $\chi$ is $t/2$-mesic.

(g) $\chi(\mathcal{O}) = \displaystyle\sum_{i=1}^{s} (b_i\alpha_i - b_i + r_i)$




## Results for fences of the form $\check{F} = (a, b, a)$

### Theorem [EPRS]: Orbit and homomesy of $\check{F}(a, b, a)$

Consider $\check{F}(a,b,a)$ and define $g = \gcd(a,b)$, $\overline{a} = a/g$, $\overline{b} = b/g$, $\ell = \mathrm{lcm}(a,b)$. Since $\overline{a}, \overline{b}$ are relatively prime, there exists a smallest positive integer $m$ such that $m\overline{a} = q\overline{b} + 1$ for some positive integer $q$. Then the orbits of rowmotion on $\check{F}(a,b,a)$ can be partitioned by length into three sets $\mathcal{S}, \mathcal{M}, \mathcal{L}$, which we call *small*, *medium*, and *large*, having the following properties.

(a) We have $\#\mathcal{O} = \begin{cases} \ell & \text{if } \mathcal{O} \in \mathcal{S}, \\ a(2b - 2\overline{b} + m) + g & \text{if } \mathcal{O} \in \mathcal{M}, \\ a(2b - \overline{b} + m) + g & \text{if } \mathcal{O} \in \mathcal{L}. \end{cases}$

with $\#\mathcal{S} = \overline{a}(g-1)^2$, $\#\mathcal{M} = \dfrac{\overline{a}m - 1}{\overline{b}}$, $\#\mathcal{L} = \dfrac{\overline{a}(\overline{b} - m) + 1}{\overline{b}}$,

(b) For rowmotion on antichains, $\chi$ is homometric.

(c) For rowmotion on ideals, $\widehat{\chi}$ is $n/2$-mesic where $n = \#\check{F}(a,b,a) = 2a + b - 1$.

We have similar results charactertising orbit sizes and homomesies for the following set of fence poset classes: $\check{F}(a,b)$, $\check{F}(a,a,a,a)$, and $\check{F}(a,1,a,1,a)$.

## Palindromic Fences

### Proposition [EPRS]: Relationship between red and black tiles for palindromic fences

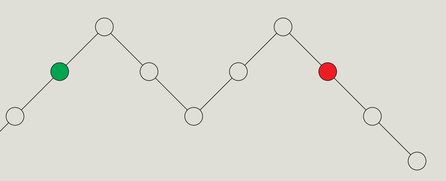
Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_t)$ be palindromic where $\alpha_i \ge 2$ for all $i \in [t]$, and let $F = \check{F}(\alpha)$. Then for any orbit $\mathcal{O}$ of $F$, the black tile sequence $b_1, b_2, \ldots, b_t$ is palindromic if and only if the red tile sequence $r_1, r_2, \ldots, r_{t-1}$ is palindromic.

### Proposition [EPRS]: Homomesies for palindromic fences

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_t)$ where $\alpha_i \ge 2$ for all $i \in [t]$. Also let $F = \check{F}(\alpha)$ and $n = \#F$. If $\alpha$ as well as the black and red tile sequences are all palindromic, then one has the following homomesies.

(a) For all $k \in [n]$ the statistic $\chi_k - \chi_{n-k+1}$ is 0-mesic.
(b) If $t$ is odd, then for all $k \in [n]$ the statistic $\widehat{\chi}_k + \widehat{\chi}_{n-k+1}$ is 1-mesic.

$\check{F}(4,2,2,4)$

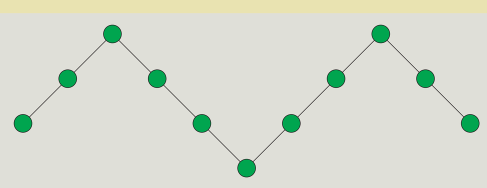


We conjecture that even more is true for constant $\alpha$.

### Conjecture [EPRS]: Homomesy and Homometry for $F(a, a, \ldots, a)$

Let $\alpha = (a^t)$ and let $F = \check{F}(\alpha)$.
(a) The statistic $\chi$ is homometric.
(b) If $t$ is odd, then the statistic $\widehat{\chi}$ is $n/2$-mesic where $n = \#F$.

$\check{F}(3,3,3,3)$



## Selected References (See abstract for more references and details)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