Deformation cone of hypergraphic polytopes

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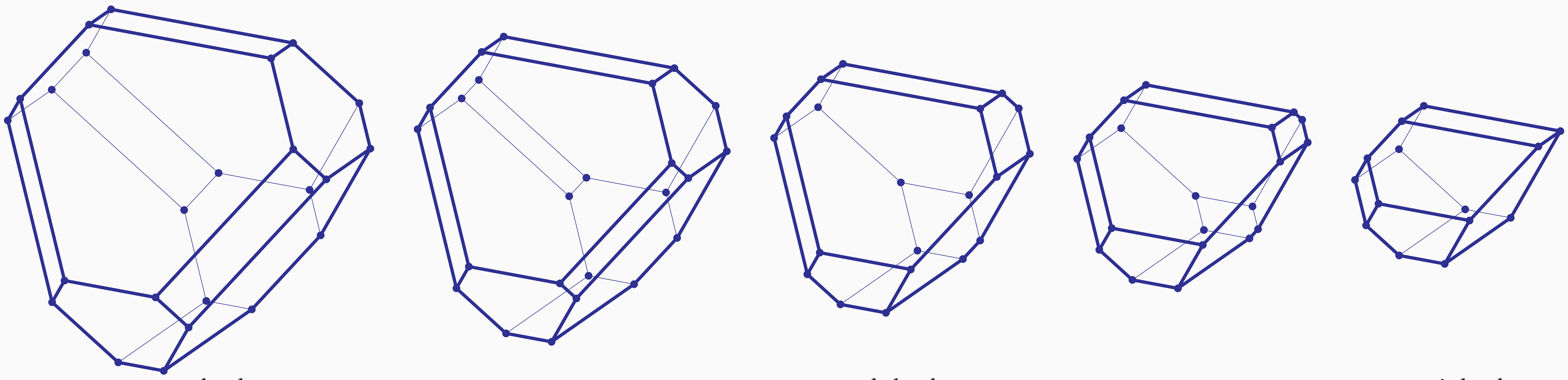
Deformation of polytopes

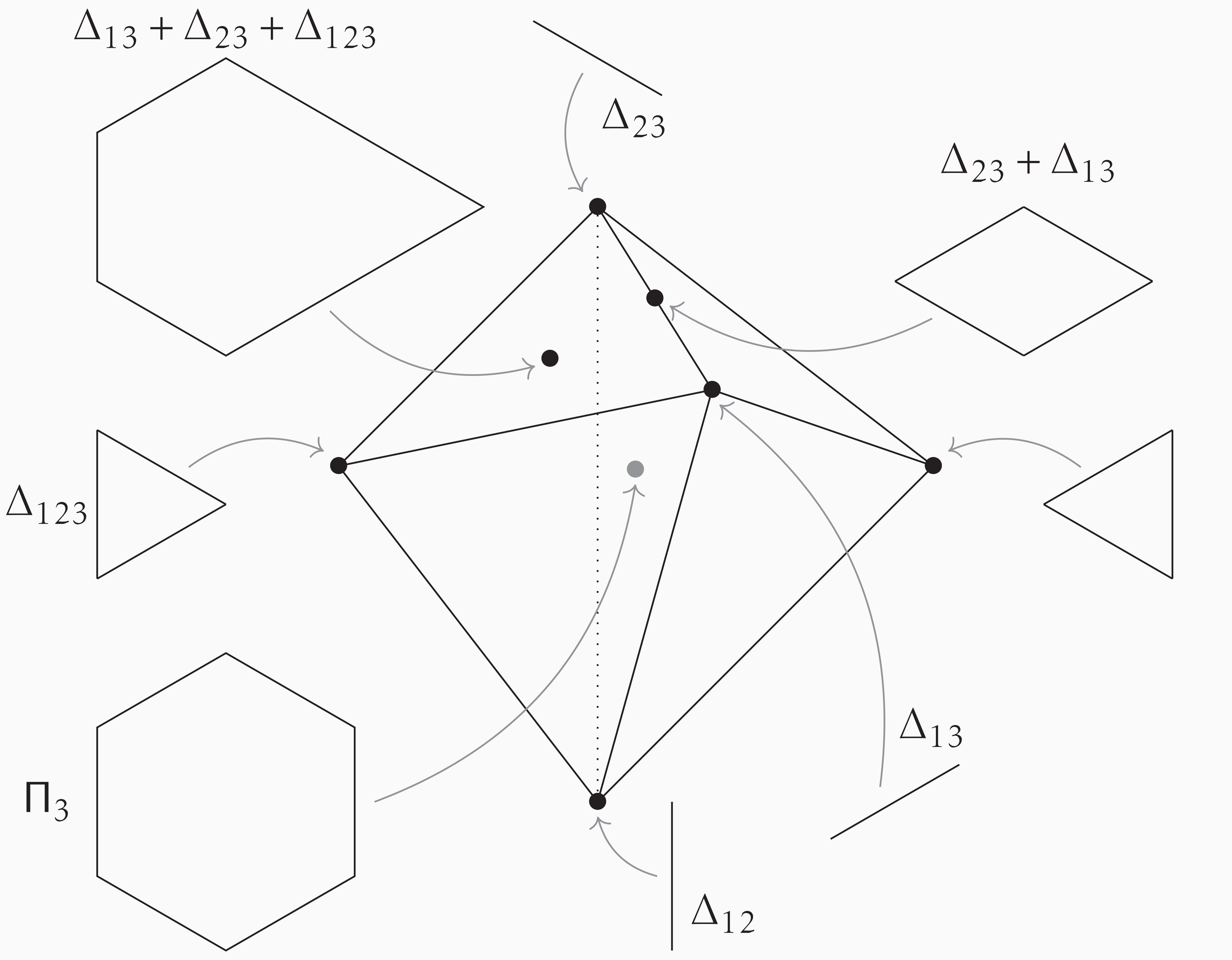
deformation cone $\mathbb{DC}(\mathsf{P})$ = set of all polytopes whose normal fans coarsen the normal fan of P.
PROP. $\mathbb{DC}(\mathsf{P})$ is a closed convex cone (under dilation and Minkowski sum) and contains a lineality subspace of dimension d (translations).
Standard permutohedron Π_n = $\mathrm{conv}\{(\sigma(i))_{i\in[\![1,n]\!]}\ ;\ \sigma\in\mathcal{S}_n\}$. Here below: 5 deformations of Π_4.



permutahedron — cyclohedron — associahedron

Deformed permutahedra and $\mathbb{DC}(\Pi_n)$

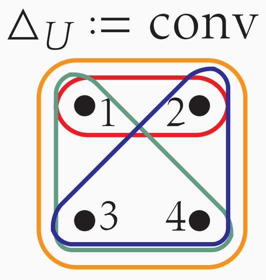
$\mathbb{DC}(\Pi_n)$ = deformed permutahedra = cone of submodular functions
$\dim\mathbb{DC}(\Pi_n) = 2^n - n - 1$; number of facets of $\mathbb{DC}(\Pi_n) = 2^{n-2}\binom{n}{2}$.
Remains to understand since the 70s: rays, faces...
Here below: $3D$ affine section of $\mathbb{DC}(\Pi_3)$ ($4D$-cone with 5 rays).

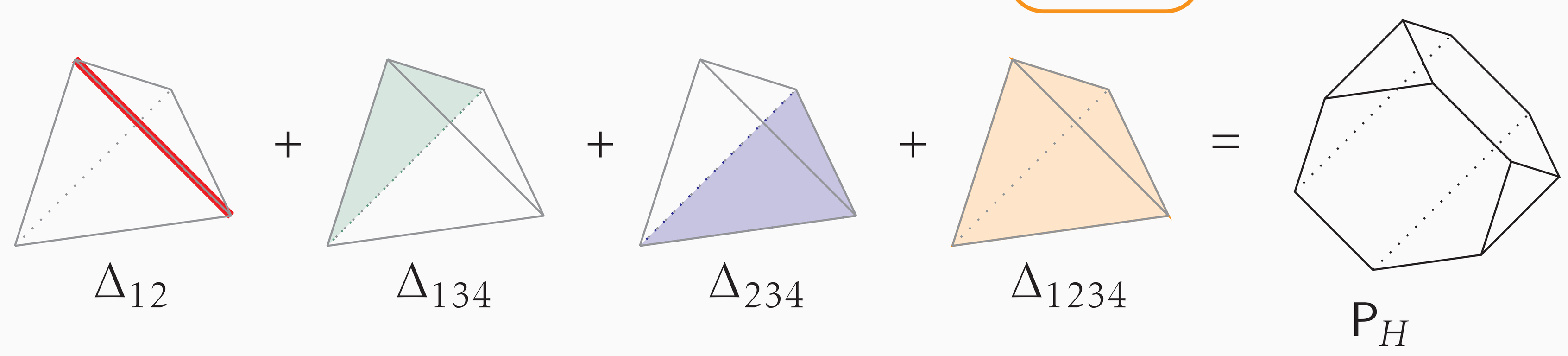


Hypergraphic polytopes P_H

Hypergraph on V = collection H of $U\subseteq V$ such that $|U|\geq 2$.
Hypergraphic polytope P_H = $\sum_{U\in H}\Delta_U$ with $\Delta_U := \mathrm{conv}\{e_u \mid u\in U\}$.

Here below: hypergraphic polytope for H:





PROP. Q deformation of P $\Rightarrow$ $\mathbb{DC}(\mathsf{Q})$ face of $\mathbb{DC}(\mathsf{P})$.
Hypergraphic fan coarsens braid fan (normal fan of Π_n) $\Rightarrow \mathsf{P}_H\in\mathbb{DC}(\Pi_n)$ and $\mathbb{DC}(\mathsf{P}_H)$ is a face of $\mathbb{DC}(\Pi_n)$. We study $\mathbb{DC}(\mathsf{P}_H)$.
Wall-crossing inequalities give a redundant description of $\mathbb{DC}(\mathsf{P}_H)$.
THM. The deformation cone $\mathbb{DC}(\mathsf{P}_H)$ is isomorphic to the set of polytopes $\left\{x\in\mathbb{R}^V \;\middle|\; \sum_{u\in U}x_u - \sum_{v\notin U}x_v \leq \boldsymbol{h}_U \text{ for all } U\subseteq V\right\}$ for all $\boldsymbol{h}$ in the cone of $\mathbb{R}^{2^V}$ defined by the following **redundant** description:

- $\boldsymbol{h}_\varnothing = -\boldsymbol{h}_V$,
- $\boldsymbol{h}_{S\cup\{u\}} + \boldsymbol{h}_{S\cup\{v\}} = \boldsymbol{h}_S + \boldsymbol{h}_{S\cup\{u,v\}}$ for each $S\subseteq V$ and each $\{u,v\}\subseteq V\smallsetminus S$ such that $U\notin H$ for any $\{u,v\}\subseteq U\subseteq V\smallsetminus S$,
- $\boldsymbol{h}_{S\cup\{u\}} + \boldsymbol{h}_{S\cup\{v\}} \geq \boldsymbol{h}_S + \boldsymbol{h}_{S\cup\{u,v\}}$ for each $\{u,v\}\subseteq U\in H$ and $S\subseteq V\smallsetminus U$.

Dimension of $\mathbb{DC}(\mathsf{P}_H)$

$K\subseteq V$ induced clique = $\forall u,v\in K, \exists U\in H,\ \{u,v\}\subseteq U\subseteq K$.

THM. Span $(\mathbb{DC}(\mathsf{P}_H))$ **independent** eqns:

- $\boldsymbol{h}_\varnothing = -\boldsymbol{h}_V$; $\boldsymbol{h}_{S\cup\{u\}} + \boldsymbol{h}_{S\cup\{v\}} = \boldsymbol{h}_S + \boldsymbol{h}_{S\cup\{u,v\}}$ for $\varnothing\neq S\subseteq V$ with $V\backslash S$ not an induced clique, $U\notin H$ for any $\{u,v\}\subseteq U\subseteq V\backslash S$.

CORO. The simplices Δ_K for the induced cliques $K\neq\varnothing$ of H form a linear basis of the vector space spanned by $\mathbb{DC}(\mathsf{P}_H)$.
$\dim\mathbb{DC}(\mathsf{P}_H)$ = number induced cliques $K\neq\varnothing$.

For some classes of hypergraph, we have an irredundant description of $\mathbb{DC}(\mathsf{P}_H)$: Graphical zonotopes and Nestohedra.

Want more details?

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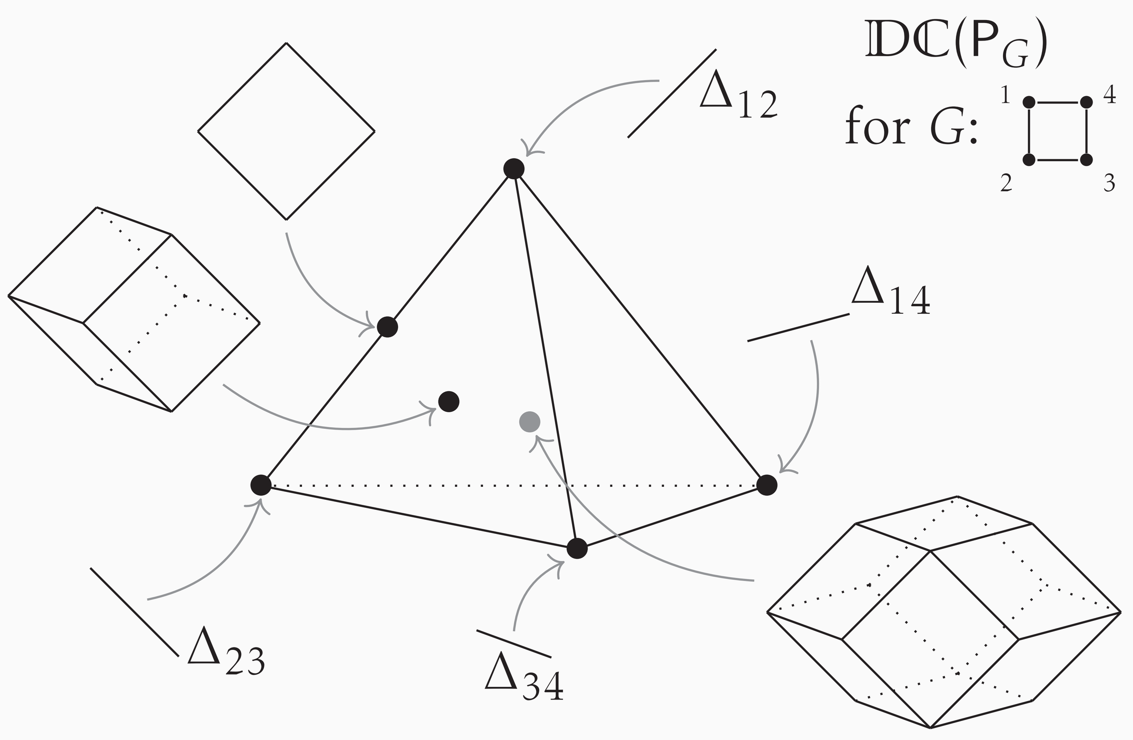
Graphical zonotopes

Graphical zonotope = P_G when $H = G$ a graph ($\forall U,\ |U| = 2$)



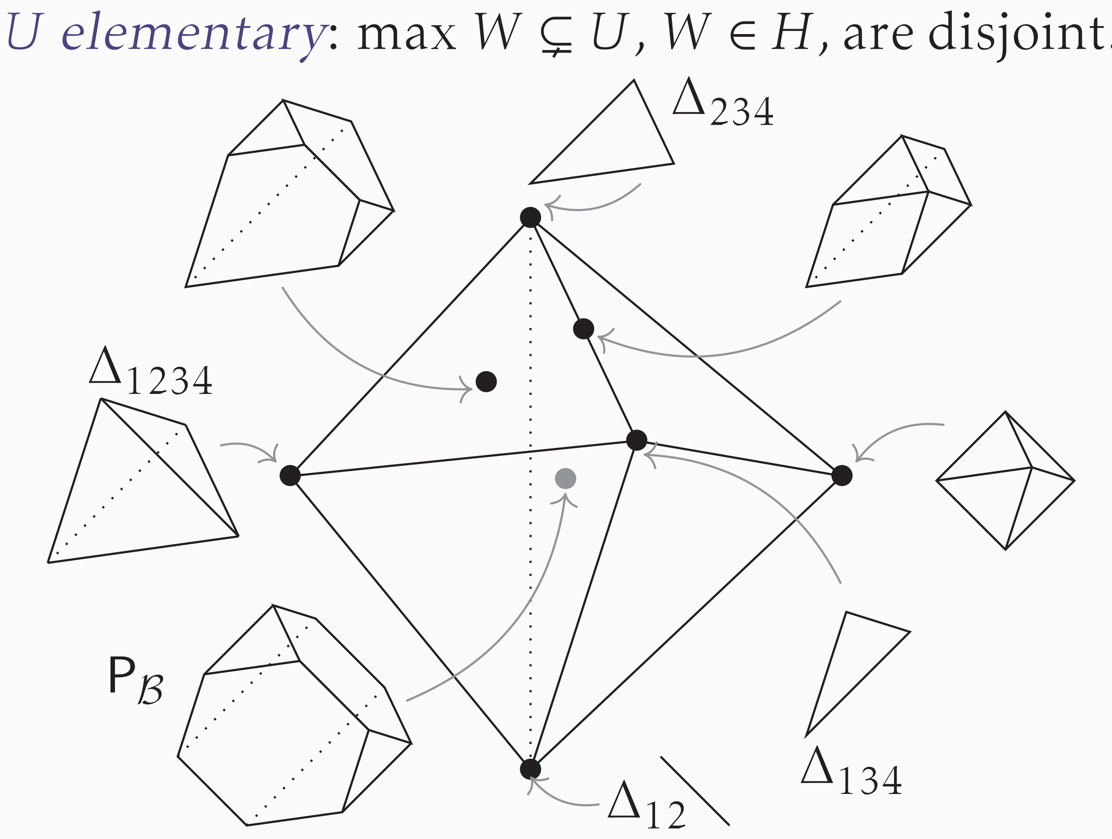
THM. Irredundant description $\mathbb{DC}(\mathsf{P}_G)$:

- $\boldsymbol{h}_\varnothing = -\boldsymbol{h}_V$,
- $\boldsymbol{h}_{S\smallsetminus\{u\}} + \boldsymbol{h}_{S\smallsetminus\{v\}} = \boldsymbol{h}_S + \boldsymbol{h}_{S\smallsetminus\{u,v\}}$ for each $\varnothing\neq S\subseteq V$ and any $\{u,v\}\in\binom{S}{2}\smallsetminus E$,
- $\boldsymbol{h}_{S\cup\{u\}} + \boldsymbol{h}_{S\cup\{v\}} \geq \boldsymbol{h}_S + \boldsymbol{h}_{S\cup\{u,v\}}$ for each $\{u,v\}\in E$ and $S\subseteq N(u)\cap N(v)$.

CORO. $\mathbb{DC}(\mathsf{P}_G)$ simplicial $\Leftrightarrow$ G has no triangle (clique of size 3).

Nestohedra

Nestohedron = $\mathsf{P}_{\mathcal{B}}$ when $H = \mathcal{B}$ a building set ($\forall U_1,U_2\in H,\ U_1\cap U_2\neq\varnothing \Rightarrow U_1\cup U_2\in H$)
U elementary: max $W\subsetneq U$, $W\in H$, are disjoint.



THM. Irredundant description $\mathbb{DC}(\mathsf{P}_{\mathcal{B}})$:

- $\sum_{K\in\overline{\kappa}(\mathcal{B})}\boldsymbol{h}_K = 0$ ($\overline{\kappa}(\mathcal{B})$: max $U\in H$ and $\varnothing$),
- $\sum_{B\in\mu(P)}\boldsymbol{h}_B \geq \boldsymbol{h}_P$ for elementary $P\in H$,
- $\boldsymbol{h}_A + \boldsymbol{h}_B + \sum_{K\in\kappa(P\smallsetminus(A\cup B))}\boldsymbol{h}_K \geq \boldsymbol{h}_P + \sum_{K\in\kappa(A\cap B)}\boldsymbol{h}_K$
  P not elementary, $A\neq B$ maximal in P.

CORO. $\mathbb{DC}(\mathsf{P}_{\mathcal{B}})$ simplicial $\Leftrightarrow$ all U with ≥ 3 distinct maximal subblocks are elementary.