# Homomorphism complexes, reconfiguration, and homotopy for directed graphs

Anton Dochtermann and **Anurag Singh**

Department of Mathematics, Texas State University and IIT Bhilai

## Introduction

For any pair of directed graphs $G$ and $H$ the polyhedral complex $\overrightarrow{\mathrm{Hom}}(G,H)$ parametrizes the directed graph homomorphisms $f : G \to H$. This construction can be seen as a special case of the poset structure on the set of multihomomorphisms in more general categories, as introduced by Kozlov, Matsushita, and others. Hom complexes of directed graphs have applications in the study of chains in graded posets and cellular resolutions of monomial ideals.

## Homomorphism complexes for directed graphs

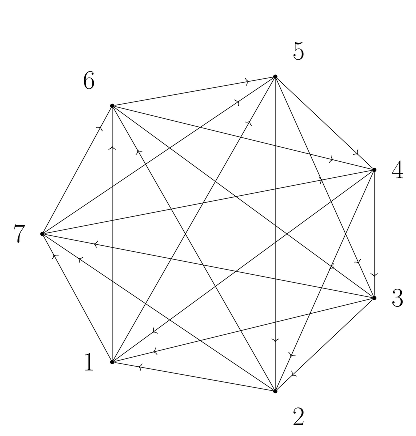
Let $G$ and $H$ are directed graphs.

- A **multihomomorphism** is a map $\alpha : V(G) \to 2^{V(H)}\backslash\{\emptyset\}$ such that if $(v,w) \in E(G)$ we have $\alpha(v) \times \alpha(w) \subseteq E(H)$.
- The **directed Hom complex** from $G$ to $H$, denoted by $\overrightarrow{\mathrm{Hom}}(G,H)$, is the polyhedral complex with cells given by all multihomomorphisms $\alpha : V(G) \to 2^{V(H)}\backslash\{\emptyset\}$. In particular, the vertex set of $\overrightarrow{\mathrm{Hom}}(G,H)$ is given by the set of all directed graph homomorphisms $f : G \to H$.

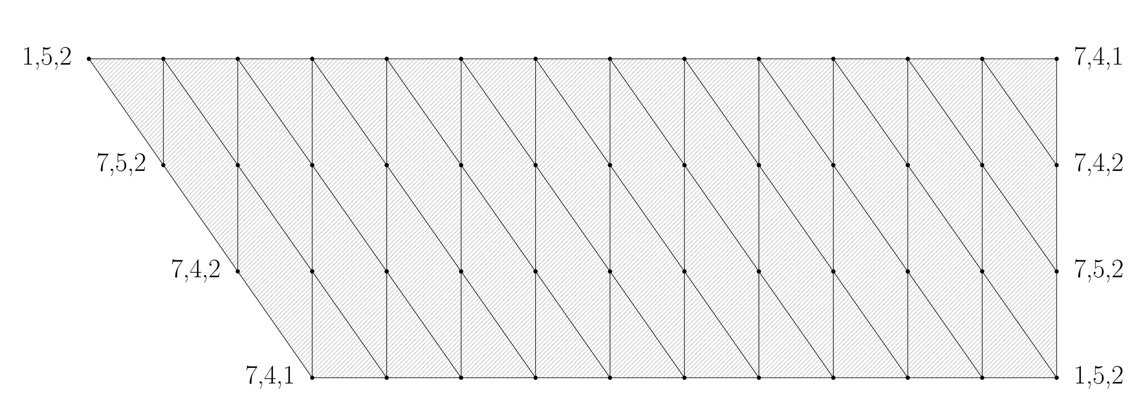
## Examples

Let $\overrightarrow{L}_n$, $\overrightarrow{C}_n$ and $\overrightarrow{K_n}$ denote the directed path graph $1 \to 2 \to \cdots \to n$, directed cycle graph $1 \to 2 \to \cdots \to n \to 1$, and the transitive $n$-tournament, respectively. Then,

- $\overrightarrow{\mathrm{Hom}}(\overrightarrow{L}_r, \overrightarrow{L}_s)$ is a disjoint union of $s-r+1$ points if $s \geq r$, and is empty otherwise.
- $\overrightarrow{\mathrm{Hom}}(\overrightarrow{C}_r, \overrightarrow{C}_s)$ is a disjoint union of $s$ points if $s$ divides $r$, and is empty otherwise.
- $\overrightarrow{\mathrm{Hom}}(\overrightarrow{K}_{n-1}, \overrightarrow{K_n})$ is a path on $n$ vertices.
- The tournament $T_7$ and its complex of morphisms from the 3-cycle $\overrightarrow{C}_3$.



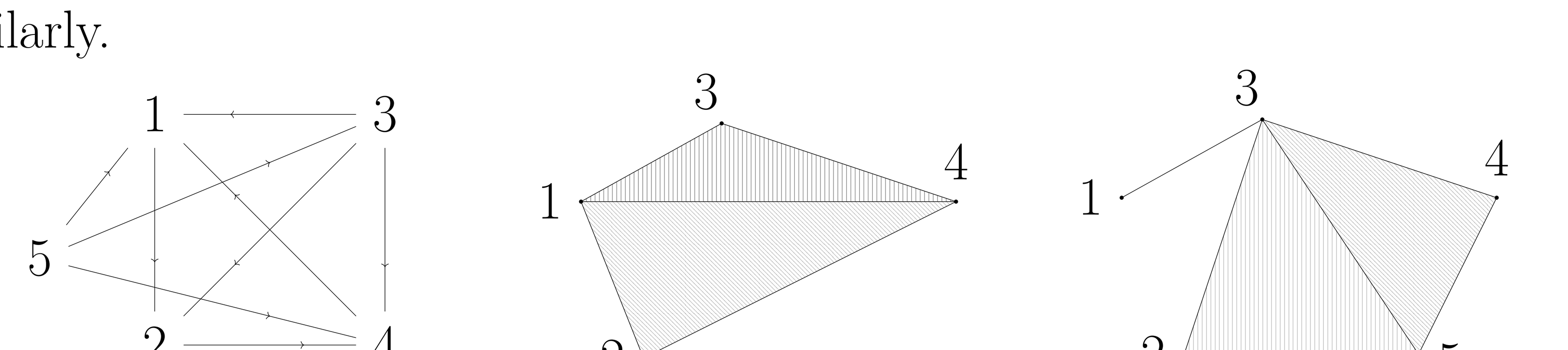

$T_7$




$\overrightarrow{\mathrm{Hom}}(\overrightarrow{C}_3, T_7) \cong$ Möbius strip

## Directed neighborhood complex

- The **out-neighborhood complex** $\overrightarrow{\mathcal{N}}(G)$ is the simplicial complex on vertex set $\{v \in V(G) : \mathrm{indeg}(v) > 0\}$, with facets given by the out neighborhoods $\overrightarrow{N}_G(v)$ for all $v \in V(G)$. The **in-neighborhood complex** $\overleftarrow{\mathcal{N}}(G)$ is defined similarly.




(a) $G$ (b) $\overrightarrow{\mathcal{N}}(G)$ (c) $\overleftarrow{\mathcal{N}}(G)$

Let $\overrightarrow{K}_{m,n}$ is the graph with vertex set $[m] \cup [n]$ and with all directed edges $\{(i,j) : i \in [m], j \in [n]\}$. Our main results here are the following.

**Theorem 1.** Let $G$ be a simple directed graph. Then,

1. $\overrightarrow{\mathcal{N}}(G) \simeq \overleftarrow{\mathcal{N}}(G) \simeq \overrightarrow{\mathrm{Hom}}(K_2, G)$.
2. If $G$ does not contain a copy of $\overrightarrow{K}_{m,n}$ (for any $m+n=d$) then $\overrightarrow{\mathcal{N}}(G)$ admits a strong deformation retract onto a complex of dimension at most $d-3$.
3. If $|V(G)| \leq 2n+2$ then $\overrightarrow{\mathcal{N}}(G)$ is $n$-Leray. Furthermore, this bound is tight in the sense that there exists a simple digraph $T_m$ on $m = 2n+3$ vertices with $\overrightarrow{\mathcal{N}}(T_m) \simeq \mathbb{S}^n$.

## Structural results

The directed $\overrightarrow{\mathrm{Hom}}$ complexes have various structural properties that parallel those of homomorphism complexes in the undirected setting. In particular,

**Theorem 2.** For digraphs $A$, $B$, and $C$, we have homotopy equivalences

- $\overrightarrow{\mathrm{Hom}}(A, B \times C) \simeq \overrightarrow{\mathrm{Hom}}(A,B) \times \overrightarrow{\mathrm{Hom}}(A,C)$;
- $\overrightarrow{\mathrm{Hom}}(A \times B, C) \simeq \overrightarrow{\mathrm{Hom}}(A, C^B)$.

If $v, w \in V(G)$ are vertices of a digraph $G$ with the property that $\overrightarrow{N}_G(v) \subseteq \overrightarrow{N}_G(w)$ and $\overleftarrow{N}_G(v) \subseteq \overleftarrow{N}_G(w)$ then we have a homomorphism $G \to G\backslash\{v\}$ given by $v \mapsto w$ (and $u \mapsto u$ for all $u \neq v$) called a **(directed) folding**.

**Theorem 3.** If $G \to G\backslash\{v\}$ is a directed folding then for any digraph $H$, $\overrightarrow{\mathrm{Hom}}(H,G) \simeq \overrightarrow{\mathrm{Hom}}(H, G\backslash\{v\})$ and $\overrightarrow{\mathrm{Hom}}(G\backslash\{v\}, H) \simeq \overrightarrow{\mathrm{Hom}}(G,H)$.

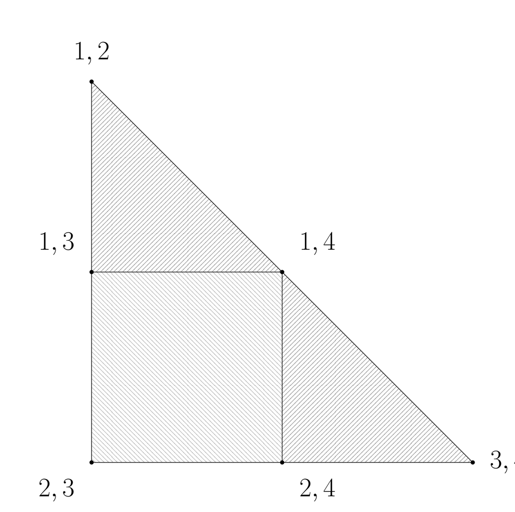
## Reconfiguration into tournaments

The connectivity of $\overrightarrow{\mathrm{Hom}}(G, T_n)$ is a natural place to study **reconfiguration** questions as a digraph analogue of the well-studied question of mixings of (undirected) graph colorings. Our main result in this setting is the following.

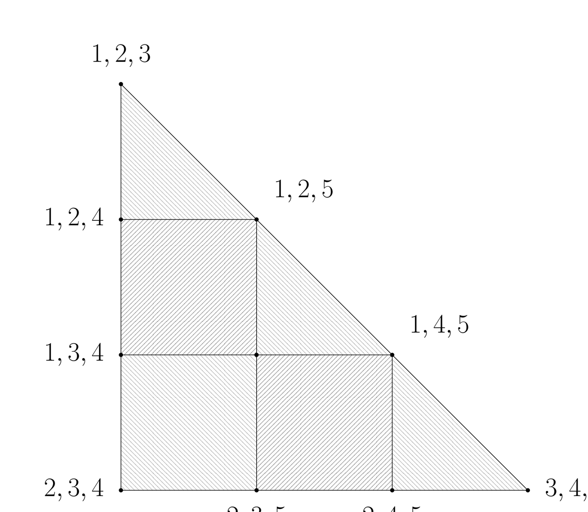
**Theorem 4.** For any digraph $G$ the complex $\overrightarrow{\mathrm{Hom}}(G, \overrightarrow{K_n})$ is either empty or contractible. Furthermore, if $\overrightarrow{\mathrm{Hom}}(G, \overrightarrow{K_n})$ is nonempty then the diameter of its 1-skeleton satisfies

$$diam((\overrightarrow{\mathrm{Hom}}(G, \overrightarrow{K_n}))^{(1)}) \leq |V(G)|.$$

For $G = \overrightarrow{K_m}$, the complexes $\overrightarrow{\mathrm{Hom}}(\overrightarrow{K_m}, \overrightarrow{K_n})$ can be recovered as certain mixed subdivisions of a dilated simplex $m\Delta^{n-m}$.




$\overrightarrow{\mathrm{Hom}}(\overrightarrow{K}_2, \overrightarrow{K}_4)$




$\overrightarrow{\mathrm{Hom}}(\overrightarrow{K}_3, \overrightarrow{K}_5)$

## Discrete homotopy for directed graphs

Suppose $f, g : G \to H$ are homomorphisms of directed graphs.

- $f$ and $g$ are **bihomotopic**, denoted $f \overset{\leftrightarrow}{\simeq} g$, if there exists a path from $f$ to $g$ in the complex $\overrightarrow{\mathrm{Hom}}(G,H)$.

**Theorem 5.** Bihomotopy of digraphs satisfy the following properties.

- We have $f \overset{\leftrightarrow}{\simeq} g$ if and only if there exists a bidirected path from $f$ to $g$ in $H^G$;
- Directed foldings $G \to G - v$ preserve bihomotopy type;
- $G \overset{\leftrightarrow}{\simeq} \mathbf{1}$ if and only if $\overrightarrow{\mathrm{Hom}}(T,G)$ is connected for any digraph $T$.

Exponential graph leads to other notions: $f$ and $g$ are

- **dihomotopic** if there exists a directed path from $f$ to $g$ in the graph $(H^G)^o$,
- **line-homotopic** if there exists a path from $f$ to $g$ in the underlying undirected graph of $(H^G)^o$.

## References

[1] E. Babson and D. N. Kozlov. Complexes of graph homomorphisms. *Israel J. Math.*, 152:285–312, 2006.

[2] A. Dochtermann. Hom complexes and homotopy theory in the category of graphs. *European J. Combin.*, 30(2):490–509, 2009.

[3] A. Dochtermann and A. Singh. Homomorphism complexes, reconfiguration, and homotopy for directed graphs. *arXiv: 2108.10948*, 2021.

[4] T. Matsushita. Morphism complexes of sets with relations. *Osaka J. Math.*, 53(1):267–283, 2016.