Row-strict dual immaculate functions and 0-Hecke modules

34th FPSAC July 2022
shsund@comcast.net

Elizabeth Niese Sheila Sundaram Stephanie van Willigenburg Julianne Vega Shiyun Wang

Indian Institute of Science
Bangalore, India

Our Results

- We introduce a new basis of quasisymmetric functions, the **row-strict dual immaculate functions**, with a cyclic, indecomposable 0-Hecke algebra module.
- We uncover the remarkable properties of the **immaculate Hecke poset** induced by the 0-Hecke action on standard immaculate tableaux, revealing other submodules and quotient modules, cyclic and indecomposable.
- We complete the combinatorial and representation-theoretic picture by showing that the generating functions of *all* the possible variations of tableaux occur as characteristics of 0-Hecke modules, all captured in the **immaculate Hecke poset**.

Quasisymmetric functions, 0-Hecke algebra

A composition α of n is a sequence of positive integers summing to n. Write $\alpha \vDash n$.

Ex The diagram of $\alpha = (3,2,4) \vDash 9$ is $\begin{array}{|c|c|c|c|}\hline \ & \ & \ & \ \\ \hline \ & \ \\ \cline{1-2} \ & \ & \ \\ \cline{1-3}\end{array}$.

Compositions α of n

- map to subsets of $[n-1] = \{1, 2, \ldots, n-1\}$.
- are partially ordered by refinement: β refines α if each part of α is a sum of **consecutive** parts of β.

The monomial quasisymmetric function indexed by α:

$$M_\alpha = \sum_{\substack{(i_1, i_2, \ldots, i_k) \\ i_1 < i_2 < \ldots < i_k}} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}.$$

The fundamental quasisymmetric function indexed by α:

$$F_\alpha := \sum_{\beta \text{ refines } \alpha} M_\beta.$$

Defn $\{M_\alpha : \alpha \vDash n\}$ and $\{F_\alpha : \alpha \vDash n\}$ are bases for the degree-n homogeneous component QSym_n of the algebra of quasisymmetric functions, the **monomial** basis and the **fundamental** basis.

Defn The 0-Hecke algebra $H_n(0)$ is a deformation of the group algebra of the symmetric group, of dimension $n!$

Thm(Norton1979) $H_n(0)$ admits exactly 2^{n-1} simple modules L_α, one for each $\alpha \vDash n$, all one-dimensional.

For finite-dimensional $H_n(0)$-modules M, there is an analogue of the Frobenius characteristic:

$$M \mapsto \mathrm{ch}(M) = \sum_{\alpha \in \mathcal{C}(M)} F_\alpha \in \mathrm{QSym},$$

where $\mathcal{C}(M)$ is a subset of compositions associated to M. Ex $\mathrm{ch}(L_\alpha) = F_\alpha$, the fundamental quasiysmmetric.

Standard Immaculate Tableaux

Defn (Berg-Bergeron-Saliola-Serrano-Zabrocki2014) A standard immaculate tableau (SIT) of shape $\alpha \vDash n$ has n distinct entries taken from $[n]$, such that

1. The leftmost column increases bottom to top.
2. All rows increase, left to right.

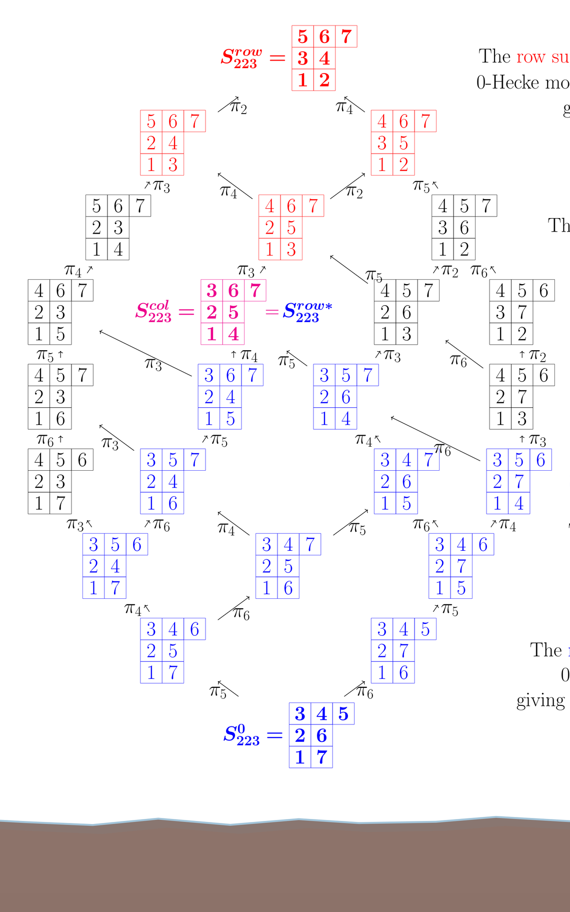
Ex $\alpha = (3,2,4)$, $T = \begin{array}{|c|c|c|c|}\hline 4 & 5 & 8 & 9 \\ \hline 3 & 7 \\ \cline{1-2} 1 & 2 & 6 \\ \cline{1-3}\end{array}$.

Thm(NSvWVW2022) Define a partial order $S \prec_{\mathcal{RS}^*_\alpha} T \iff \boldsymbol{T = s_i(S)}$ on $\mathrm{SIT}(\alpha)$; $s_i(S)$ switches i and $i+1$ in S.

The immaculate Hecke poset $\mathcal{PRS}^*_\alpha$ has rank $\binom{|\alpha|}{2} + \binom{\ell(\alpha)}{3} - \sum_{i=1}^{\ell(\alpha)} \binom{\alpha_i + (i-1)}{2}$, with bottom element S^0_α and top element S^{row}_α.

For any $T \in \mathrm{SIT}(\alpha)$, there are saturated chains from S^0_α to T, and from T to S^{row}_α.

The Immaculate Hecke Poset α=223



The row superstandard tableau S^{row}_α generates the indecomposable 0-Hecke module for the $\mathfrak{S}^*$-action on $\mathrm{SIT}(\alpha)$, **by reversing arrows**, giving the **dual immaculate basis** for QSym_n.

The column superstandard tableau S^{row}_α generates an indecomposable 0-Hecke module for the $\mathcal{RS}^*$-action on $\mathrm{SET}(\alpha)$, and a quotient module for the $\mathfrak{S}^*$-action **by reversing arrows**. These give the extended Schur functions $\mathcal{RE}_\alpha$ and $\mathcal{E}_\alpha$ **bases**.

$\mathrm{SIT}^*(\alpha)$ has first column consisting of the smallest integers; $S^{row*}_{223} = S^{col}_{223}$ generates a submodule for the $\mathfrak{S}^*$-action. The quotient $\langle S^{row}_\alpha \rangle / \langle S^{row*}_\alpha \rangle$ is cyclic and indecomposable. **(By reversing arrows)**.

The minimal tableau S^0_α generates the indecomposable 0-Hecke module for the $\mathcal{RS}^*$-action on $\mathrm{SIT}(\alpha)$. giving the **row-strict dual immaculate basis** for QSym_n.

Berg-Bergeron-Saliola-Serrano-Zabrocki-2014
Dual Immaculate Basis

Defn An immaculate tableau of shape $\alpha \vDash n$ satisfies:

1. The leftmost column increases strictly, bottom to top.
2. All rows increase weakly, left to right.

The dual immaculate function $\mathfrak{S}^*_\alpha$ indexed by $\alpha \vDash n$ is the generating function for these tableaux.

Thm $\{\mathfrak{S}^*_\alpha\}_{\alpha \vDash n}$ is a basis of QSym_n.

$$\mathfrak{S}^*_{12} : \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 1 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 1 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} + \cdots$$

Defn The $\mathfrak{S}^*$-descent set $\mathrm{Des}_{\mathfrak{S}^*}(S)$ of a standard immaculate tableau S is

$$\mathrm{Des}_{\mathfrak{S}^*}(S) := \{i : i+1 \text{ appears strictly above } i \text{ in } S\}.$$

Thm The set $\{\mathfrak{S}^*_\alpha\}_{\alpha \vDash n}$ is a basis for QSym_n.

The dual immaculate function expands positively in the fundamental basis as $\mathfrak{S}^*_\alpha = \sum_S F_{\mathrm{comp}(\mathrm{Des}_{\mathfrak{S}^*}(S))}$, sum over all SIT S of shape α. For $\alpha = (1,3)$, the unique SIT $\begin{array}{|c|c|c|}\hline 2 & 3 & 4 \\ \hline 1 \\ \cline{1-1}\end{array}$ has $\mathfrak{S}^*$-descent set $\{1\}$; $\mathfrak{S}^*_{(1,3)} = F_{\mathrm{comp}(\{1\})} = F_{(1,3)}$.

Row-strict Dual Immaculate Basis

Defn(NSvWVW 2022) A row-strict immaculate tableau of shape $\alpha \vDash n$ satisfies:

1. The leftmost column increases weakly, bottom to top.
2. The rows increase strictly, left to right.

The row-strict dual immaculate function $\mathcal{RS}^*_\alpha$ indexed by $\alpha \vDash n$ is the generating function for these tableaux.

Thm $\{\mathcal{RS}^*_\alpha\}_{\alpha \vDash n}$ a basis of QSym_n.

$$\mathcal{RS}^*_{12} : \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 1 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 1 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} + \cdots$$

Defn The $\mathcal{RS}^*$-descent set $\mathrm{Des}_{\mathcal{RS}^*}(S)$ of a SIT S is

$$\mathrm{Des}_{\mathcal{RS}^*}(S) := \{i : i+1 \text{ appears weakly below } i \text{ in } S\}.$$

Thm The row-strict dual immaculate function expands positively in the fundamental basis as $\mathcal{RS}^*_\alpha = \sum_S F_{\mathrm{comp}(\mathrm{Des}_{\mathcal{RS}^*}(S))}$, sum over all SIT S of shape α.

For $\alpha = (1,3)$, the unique SIT $\begin{array}{|c|c|c|}\hline 2 & 3 & 4 \\ \hline 1 \\ \cline{1-1}\end{array}$ has $\mathcal{RS}^*$-descent set $\{2,3\}$; $\mathcal{RS}^*_{(1,3)} = F_{\mathrm{comp}\{2,3\}} = F_{(2,1,1)}$.

Defn There is an involution ψ on QSym: $\psi(F_\alpha) = F_{\alpha^c}$, where $\mathrm{set}(\alpha^c)$ is the complement in $[n-1]$ of $\mathrm{set}(\alpha)$.

Thm $\mathcal{RS}^*_\alpha = \psi(\mathfrak{S}^*_\alpha)$.

0-Hecke modules for $\mathfrak{S}^*_\alpha$ and $\mathcal{RS}^*_\alpha$

Thm(BBSSZ2015) There is an indecomposable cyclic 0-Hecke algebra module $\mathcal{W}_\alpha = \langle S^{row}_\alpha \rangle$, defined via the $\mathfrak{S}^*$-descent set, whose characteristic is the dual immaculate function $\mathfrak{S}^*_\alpha$: $\mathrm{ch}(\mathcal{W}_\alpha) = \mathfrak{S}^*_\alpha$.

Thm(NSvWVW2022) There is an indecomposable cyclic 0-Hecke algebra module $\mathcal{V}_\alpha = \langle S^0_\alpha \rangle$, defined via the $\mathcal{RS}^*$-descent set, whose characteristic is the row-strict dual immaculate function $\mathcal{RS}^*_\alpha$: $\mathrm{ch}(\mathcal{V}_\alpha) = \mathcal{RS}^*_\alpha$.

Row-strict extended Schur functions

Let $\mathrm{SET}(\alpha)$ be the subset of $\mathrm{SIT}(\alpha)$ with ALL columns increasing. Let S^{col}_α be the column superstandard tableau.

Thm $\mathrm{SET}(\alpha)$ is the interval $[S^{col}_\alpha, S^{row}_\alpha]$ of the Hecke poset; it is a basis for an **indecomposable, cyclic**

- submodule $\mathcal{Z}_\alpha$ of $\mathcal{V}_\alpha = \langle S^0_\alpha \rangle$, with characteristic $\mathcal{RE}_\alpha$.
- quotient module $\langle S^{\mathrm{row}}_{223} \rangle / \langle SIT(\alpha) \setminus \mathrm{SET}(\alpha) \rangle$ of $\mathcal{W}_\alpha = \langle S^{\mathrm{row}}_{223} \rangle$, with characteristic $\mathcal{E}_\alpha$.

$\{\mathcal{RE}_\alpha\}_{\alpha \vDash n}$(resp.$\{\mathcal{E}_\alpha\}_{\alpha \vDash n}$) are **bases** for QSym_n; when α is a partition λ, $\mathcal{RE}_\alpha = s_{\lambda'}$ (resp. $\mathcal{E}_\alpha = s_\lambda$) (Schur function).

Thm The row-strict extended Schur function $\mathcal{RE}_\alpha$ is the generating function for tableaux of shape α with all rows strictly increasing, and ALL columns weakly increasing.

Thm (Campbell-Feldman-Light-Shuldiner-Xu 2014) The extended Schur function $\mathcal{E}_\alpha$ is the generating function for tableaux with all rows weakly increasing, ALL columns strictly increasing.

$\mathcal{E}_\alpha$ also in Assaf-Searles (2019); Searles (2020) obtains (differently) a 0-Hecke module.

More 0-Hecke modules

$\mathrm{SIT}^*(\alpha)$ is the set of tableaux in $\mathrm{SIT}(\alpha)$ with first column consisting of the smallest integers; $S^{\mathrm{row}*}_\alpha \in \mathrm{SIT}^*(\alpha)$ has its remaining cells filled consecutively along rows, bottom to top, left to right.

$$S^{\mathrm{row}*}_{223} = S^{col}_{223};\ S^{\mathrm{row}*}_{332} = \begin{array}{|c|c|c|}\hline 3 & 8 \\ \cline{1-2} 2 & 6 & 7 \\ \hline 1 & 4 & 5 \\ \hline\end{array} \neq S^{col}_{332}.$$

$\langle S^{\mathrm{row}*}_{223} \rangle$ is an invariant submodule of the $\mathfrak{S}^*$-action, with basis $\mathrm{SIT}^*(223)$. The quotient $\langle S^{\mathrm{row}}_{223} \rangle / \langle S^{\mathrm{row}*}_{223} \rangle$ is cyclic and indecomposable.

More descent sets

Defn For $T \in \mathrm{SIT}(\alpha)$ define:

$\mathrm{Des}_{\mathcal{A}^*}(T) := \{i : i+1 \text{ is strictly below } i \text{ in } T\}$;

$\mathrm{Des}_{\bar{\mathcal{A}}^*}(T) := \{i : i+1 \text{ is weakly above } i \text{ in } T\}$.

Thm There is a cyclic $H_n(0)$-module $\mathcal{A}^*_\alpha$, generated by the least element S^0_α of the poset $\mathcal{PRS}^*(\alpha)$, with quasisymmetric characteristic

$$\mathrm{ch}(\mathcal{A}^*_\alpha) = \sum_{T \in \mathrm{SIT}(\alpha)} F_{\mathrm{comp}(\mathrm{Des}_{\mathcal{A}^*}(T))}.$$

$\mathcal{A}^*_\alpha$ is the generating function for all tableaux of shape α with first column and all rows weakly increasing.

Thm There is a cyclic $H_n(0)$-module $\bar{\mathcal{A}}^*_\alpha$, generated by the top element S^{row}_α of the poset $\mathcal{PRS}^*(\alpha)$, with quasisymmetric characteristic

$$\mathrm{ch}(\bar{\mathcal{A}}^*_\alpha) = \sum_{T \in \mathrm{SIT}(\alpha)} F_{\mathrm{comp}(\mathrm{Des}_{\bar{\mathcal{A}}^*}(T))}.$$

$\bar{\mathcal{A}}^*_\alpha$ is the generating function for all tableaux of shape α with first column and all rows strictly increasing.

Note that, as is the case with $\mathfrak{S}^*_\alpha$ and $\mathcal{RS}^*_\alpha$, the two characteristics are related by the involution ψ : $\psi(\mathrm{ch}(\bar{\mathcal{A}}^*_\alpha)) = \mathrm{ch}(\mathcal{A}^*_\alpha)$.

Partial order via $\mathcal{A}^*$- & $\bar{\mathcal{A}}^*$-actions

Thm All four actions are captured in the same Hecke poset on $\mathrm{SIT}(\alpha)$.

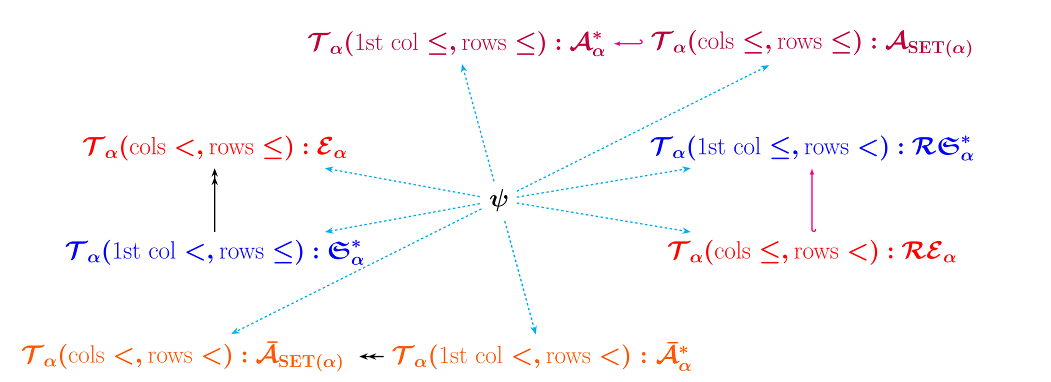
Order ideals give 0-Hecke-submodules;
order filters give 0-Hecke-quotient modules.

All descent sets and tableaux

The unified combinatorial picture for the semistandard tableaux:

Tableaux $\mathrm{SIT}(\alpha)$	$\mathfrak{S}^*_\alpha$ Dual immaculate BBSSZ 2015	$\mathcal{RS}^*_\alpha$ **Row-strict dual imm.** NSvWVW 2022 **this paper**	$\mathcal{A}^*_\alpha$ **ch** $(\mathcal{A}^*_\alpha)$ NSvWVW 2022 **this paper**	$\bar{\mathcal{A}}^*_\alpha$ **ch** $(\bar{\mathcal{A}}^*_\alpha)$ NSvWVW 2022 **this paper**
1st Col bottom to top	strict ↗	weak ↗	weak ↗	strict ↗
Rows left to right	weak ↗	strict ↗	weak ↗	strict ↗
Descents for standard tableaux	i such that $i+1$ strictly above i	i such that **$i+1$ weakly below i**	i such that **$i+1$ strictly below i**	i such that **$i+1$ weakly above i**
$(2,2)$	$\begin{array}{\|c\|c\|}\hline 3 & 4 \\ \hline 1 & \mathbf{2} \\ \hline\end{array}, \begin{array}{\|c\|c\|}\hline 2 & 4 \\ \hline \mathbf{1} & \mathbf{3} \\ \hline\end{array}, \begin{array}{\|c\|c\|}\hline 2 & 3 \\ \hline \mathbf{1} & 4 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline \mathbf{3} & 4 \\ \hline \mathbf{1} & 2 \\ \hline\end{array}, \begin{array}{\|c\|c\|}\hline \mathbf{2} & 4 \\ \hline 1 & 3 \\ \hline\end{array}, \begin{array}{\|c\|c\|}\hline \mathbf{2} & \mathbf{3} \\ \hline 1 & 4 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 3 & 4 \\ \hline 1 & 2 \\ \hline\end{array}, \begin{array}{\|c\|c\|}\hline \mathbf{2} & 4 \\ \hline 1 & 3 \\ \hline\end{array}, \begin{array}{\|c\|c\|}\hline 2 & \mathbf{3} \\ \hline 1 & 4 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline \mathbf{3} & 4 \\ \hline \mathbf{1} & \mathbf{2} \\ \hline\end{array}, \begin{array}{\|c\|c\|}\hline 2 & 4 \\ \hline \mathbf{1} & \mathbf{3} \\ \hline\end{array}, \begin{array}{\|c\|c\|}\hline \mathbf{2} & 3 \\ \hline \mathbf{1} & 4 \\ \hline\end{array}$
	$F_{(2,2)} + F_{(1,2,1)} + F_{(1,3)}$	$F_{(1,2,1)} + F_{(2,2)} + F_{(2,1,1)}$	$F_{(4)} + F_{(2,2)} + F_{(3,1)}$	$F_{(1^4)} + F_{(1,2,1)} + F_{(1,1,2)}$
Tableaux $\mathrm{SET}(\alpha)$	$\mathcal{E}_\alpha$ CFLSX '14, AS '19, S '20	$\mathcal{RE}_\alpha$ NSvWVW 2022	$\mathcal{A}_{\mathrm{SET}(\alpha)}$ NSvWVW 2022	$\bar{\mathcal{A}}_{\mathrm{SET}(\alpha)}$ NSvWVW 2022
ALL Cols bottom to top	strict ↗	weak ↗	weak ↗	strict ↗
Rows left to right	weak ↗	strict ↗	weak ↗	strict ↗
$(2,2)$	$F_{(2,2)} + F_{(1,2,1)}$	$F_{(1,2,1)} + F_{(2,2)}$	$F_{(4)} + F_{(2,2)}$	$F_{(1^4)} + F_{(1,2,1)}$
Basis for QSym?	Yes	Yes	No	No
Kostka matrix	upper triangular, 1's on the diagonal	?	?	upper triangular, but 0's on the diagonal

The complete picture



The eight flavours of tableaux, their 0-Hecke modules, with characteristics related in pairs by the involution ψ, from the four descent sets. The double arrow-head indicates a quotient module, and the hooked arrow indicates a submodule.

E. Niese, S. Sundaram, S. van Willigenburg, J. Vega, S. Wang: *0-Hecke modules for row-strict dual immaculate functions*, arXiv:2202.00708