Elena Tielker
Bielefeld
etielker@math.uni-bielefeld.de

# WEIGHTED EHRHART SERIES AND A TYPE-B ANALOGUE OF A FORMULA OF MACMAHON

arXiv 2203.1577

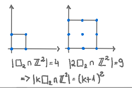
## ABSTRACT

We show a relationship between permutation statistics and lattice point enumeration in polytopes. Using this connection, properties of the generalised Eulerian numbers (palindromicity, unimodality) correspond to certain properties of the corresponding polytope (Gorenstein, anti-blocking). In our paper (arXiv: 2203.1577) we also present results on generalising this connection to coloured multiset permutations.

## $S_n$ AN EASY EXAMPLE

For $n=2$, the Eulerian polynomial $d_{S_2}(t) := \sum_{\pi\in S_2} t^{\mathrm{des}(\pi)} = 1+t$ is the numerator of the Ehrhart series of the unit square $\square_2$:

$$\frac{d_{S_2}(t)}{(1-t)^3} = \sum_{k\geq 0}(k+1)^2t^k = \sum_{k\geq 0}|k\square_2\cap\mathbb{Z}^2|\,t^k =: \mathrm{Ehr}_{\square_2}(t)$$




## PERMUTATION STATISTICS

For $\eta=(\eta_1,\dots,\eta_r)$, a composition of $n\in\mathbb{N}$, we denote by $B_\eta$ the set of signed multiset permutations, i.e. $\pi\in B_\eta$ is a pair $\pi=(w,\varepsilon)$, where $w=w_1\dots w_n$ is a multiset permutation and $\varepsilon$ a sign vector which attaches to every $w_i$ a positive or negative sign. We define a new major index and descent statistic on these permutations.

AIM Study the descent polynomial $d_{B_\eta}(t)$ and the joint distribution of major index and descent $C_{B_\eta}(q,t)$.

## WEIGHTED EHRHART SERIES

For certain families of polytopes in $\mathbb{R}^n$ (products of simplices and cross polytopes) and a so-called weight function $\bar\mu_n$, which we specify in our paper, we define

$$\mathrm{Ehr}_{\mathcal{P},\bar\mu_n}(q,t) := \sum_{k\geq 0}\sum_{x\in k\mathcal{P}\cap\mathbb{Z}^n} q^{\bar\mu_n(x)}t^k \in \mathbb{Q}(q,t),$$

the weighted Ehrhart series of $\mathcal{P}$.

AIM Associate certain permutations $X$ to polytopes $\mathcal{P}$ s.t. $\boxed{\mathrm{Ehr}_{\mathcal{P},\bar\mu_n}(q,t) = \frac{C_X(q,t)}{\prod_{i=0}^n(1-q^it)}}$

## TYPE A

### $S_\eta$ Multiset permutations & simplices [MacMahon's formula of type A]

The joint distribution of major index and descent on $S_\eta$ is the numerator of the weighted Ehrhart series of $\prod_{i=1}^r\Delta_{\eta_i}$ wrt. $\bar\mu_n$, i.e.

$$\frac{C_{S_\eta}(q,t)}{\prod_{i=0}^n(1-q^it)} = \sum_{k\geq 0}\prod_{i=1}^r\binom{k+\eta_i}{\eta_i}_q t^k = \mathrm{Ehr}_{\prod_{i=1}^r\Delta_{\eta_i},\bar\mu_n}(q,t).$$

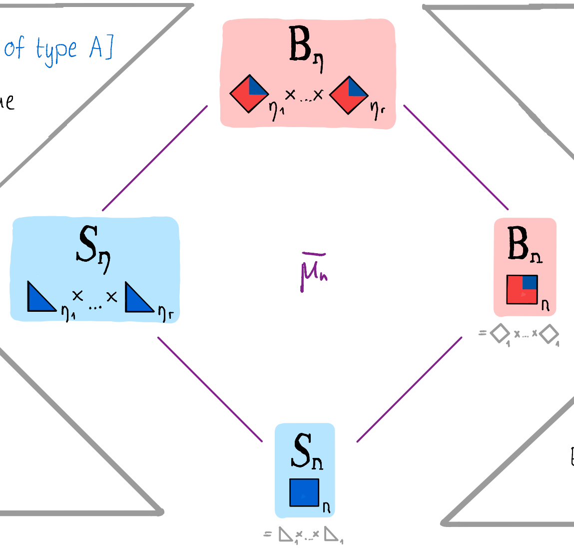
a formula of MacMahon

$\Downarrow$ $\eta=(1,\dots,1)$

### $S_n$ Permutations & the unit cube

Carlitz polynomial is the numerator of the weighted Ehrhart series of $\square_n$ wrt. $\bar\mu_n$, i.e.

$$\frac{C_{S_n}(q,t)}{\prod_{i=0}^n(1-q^it)} = \sum_{k\geq 0}\binom{k+1}{1}_q^n t^k = \mathrm{Ehr}_{\square_n,\bar\mu_n}(q,t).$$




## TYPE B

### $B_\eta$ Signed multiset permutations & cross polytopes [MacMahon's formula of type B]

The joint distribution of major index and descent on $B_\eta$ is the numerator of the weighted Ehrhart series of $\prod_{i=1}^r\Diamond_{\eta_i}$ wrt. $\bar\mu_n$, i.e.

$$\frac{C_{B_\eta}(q,t)}{\prod_{i=0}^n(1-q^it)} = \sum_{k\geq 0}\prod_{i=1}^r\sum_{j=0}^{\eta_i} q^{\frac{j(j-1)}{2}}\binom{\eta_i}{j}_q\binom{k-j+\eta_i}{\eta_i}_q t^k = \mathrm{Ehr}_{\prod_{i=1}^r\Diamond_{\eta_i},\bar\mu_n}(q,t).$$

$\Downarrow$ $\eta=(1,\dots,1)$

### Signed permutations & the cube centred in 0 $B_n$

The 'type-B Carlitz polynomial is the numerator of the weighted Ehrhart series of $\boxdot_n$ wrt. $\bar\mu_n$:

$$\frac{C_{B_n}(q,t)}{\prod_{i=0}^n(1-q^it)} = \sum_{k\geq 0}\left(\binom{k+1}{1}_q\binom{k}{1}_q\right)^n t^k = \mathrm{Ehr}_{\boxdot_n,\bar\mu_n}(q,t).$$

## PROPERTIES OF THE GEN. EULERIAN POLYNOMIALS

In the $q=1$-case, this connection to polytopes can be used to study palindromicity & unimodality of the descent polynomials, the (generalised) Eulerian polynomials of types A & B.

The generalised Eulerian polynomial of type B is

- palindromic for all $\eta$ since $\Diamond_{\eta_i}$ and therefore $\prod\Diamond_{\eta_i}$ is Gorenstein
- unimodular for all $\eta$ since $\Diamond_{\eta_i}$ and therefore $\prod\Diamond_{\eta_i}$ is anti-blocking