Highest weight crystals for Schur Q-functions

Eric Marberg, Tong Kam Hung*

Hong Kong University of Science and Technology
FPSAC 2022 Poster Presentation

July 18-22, 2022

Symmetric polynomials

Three classical families of symmetric polynomials indexed by partitions:

- Schur polynomials $s_\lambda(x_1,\dots,x_n) := \sum_{T\in \mathsf{SSYT}_n(\lambda)} x^T$
- Schur P-polynomials $P_\mu(x_1,\dots,x_n) := \sum_{T\in \mathsf{ShSSYT}_n(\mu)} x^T$
- Schur Q-polynomials $Q_\mu(x_1,\dots,x_n) := \sum_{T\in \mathsf{ShSSYT}^+_n(\mu)} x^T = 2^{\ell(\mu)} P_\mu$

Here $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0)$ can be any integer partition.
But $\mu = (\mu_1 > \mu_2 > \cdots > \mu_l > 0)$ must be a strict partition.

These polynomials are generating functions for semistandard (shifted) tableaux with all entries at most n. They appear in representation theory of classical groups, as cohomology classes of Schubert varieties, etc.

Each family has positive multiplicative structure constants. For example, the Littlewood-Richardson rule $s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda,\mu} s_\nu$ has each $c^\nu_{\lambda,\mu} \in \mathbb{N}$.

Abstract $\mathfrak{gl}_n$-crystals

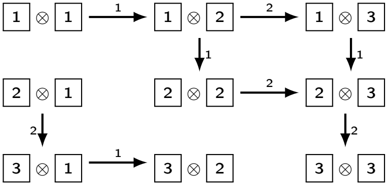
- A $\mathfrak{gl}_n$-crystal is a directed acyclic graph $\mathcal{B}$ with edges labeled by $i \in [n-1]$ and a weight map $\mathrm{wt} : \mathcal{B} \to \mathbb{Z}^n$, satisfying certain axioms
- For example, the standard $\mathfrak{gl}_n$-crystal $\mathbb{B}_n$ is

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3} \xrightarrow{3} \cdots \xrightarrow{n-1} \boxed{n}$$

  with weight map $\mathrm{wt}(\boxed{i}) = \mathbf{e}_i = (0, \dots, 0, 1, 0, \dots) \in \mathbb{Z}^n$.
- Two ways of making another $\mathfrak{gl}_n$-crystal from $\mathfrak{gl}_n$-crystals $\mathcal{B}$ and $\mathcal{C}$:

$$\mathcal{B} \otimes \mathcal{C} = \{b \otimes c : b \in \mathcal{B}, c \in \mathcal{C}\} \quad \text{and} \quad \mathcal{B} \oplus \mathcal{C} = \mathcal{B} \sqcup \mathcal{C}.$$

  For example, the tensor product $\mathbb{B}_3 \otimes \mathbb{B}_3$ is



with $\mathrm{wt}(\boxed{i} \otimes \boxed{j}) = \mathbf{e}_i + \mathbf{e}_j$

Normal $\mathfrak{gl}_n$-crystals

- A $\mathfrak{gl}_n$-crystal $\mathcal{B}$ has a character $\mathrm{ch}(\mathcal{B}) = \sum_{b \in B} x^{\mathrm{wt}(b)} \in \mathbb{Z}[x_1, \dots, x_n]$:

$$\mathrm{ch}\left(\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{n-1} \boxed{n} \right) = x_1 + x_2 + \cdots + x_n.$$

  It holds that $\mathrm{ch}(\mathcal{B} \oplus \mathcal{C}) = \mathrm{ch}(\mathcal{B}) + \mathrm{ch}(\mathcal{C})$ and $\mathrm{ch}(\mathcal{B} \otimes \mathcal{C}) = \mathrm{ch}(\mathcal{B})\mathrm{ch}(\mathcal{C})$.

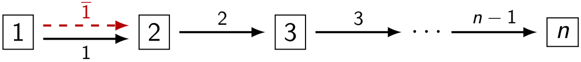
- A finite $\mathfrak{gl}_n$-crystal is normal if each of its connected components is isomorphic to a connected component of $\mathbb{B}_n^{\otimes m} = \mathbb{B}_n \otimes \cdots \otimes \mathbb{B}_n$ for some m.

Theorem (Kashiwara; Lusztig, 1990–1995)

Suppose $\mathcal{B}$ and $\mathcal{C}$ are normal $\mathfrak{gl}_n$-crystals. Then:

- $\mathcal{B} \cong \mathcal{C}$ *if and only if* $\mathrm{ch}(\mathcal{B}) = \mathrm{ch}(\mathcal{C})$.
- $\mathcal{B}$ *is connected iff* $\mathrm{ch}(\mathcal{B}) = s_\lambda(x_1, \dots, x_n)$ *for a partition* $\lambda \in \mathbb{N}^n$.
- *Every* $s_\lambda(x_1, \dots, x_n)$ *for* $\lambda \in \mathbb{N}^n$ *occurs as* $\mathrm{ch}(\mathcal{B})$ *for some normal* $\mathcal{B}$.
- *The character of any normal $\mathfrak{gl}_n$-crystal is Schur positive.*

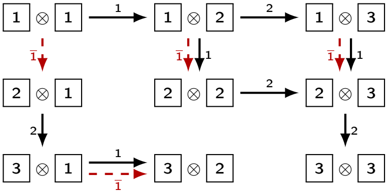
- A $\mathfrak{q}_n$-crystal is a $\mathfrak{gl}_n$-crystal with certain extra arrows labeled by $i = \overline{1}$.
- The standard $\mathfrak{q}_n$-crystal $\mathbb{B}_n$ is



with weight map $\mathrm{wt}(\boxed{i}) = \mathbf{e}_i = (0, \dots, 0, 1, 0, \dots) \in \mathbb{Z}^n$.

- There are $\mathfrak{q}_n$-versions of $\oplus$ and $\otimes$: $\oplus$ is again disjoint union, $\otimes$ more involved

For example, the tensor product $\mathbb{B}_3 \otimes \mathbb{B}_3$ is



with $\mathrm{wt}(\boxed{i} \otimes \boxed{j}) = \mathbf{e}_i + \mathbf{e}_j$

Normal $\mathfrak{q}_n$-crystals

- The character of a $\mathfrak{q}_n$-crystal is its character as a $\mathfrak{gl}_n$-crystal.
- A finite $\mathfrak{q}_n$-crystal is normal if each of its connected components is isomorphic to a connected component of $\mathbb{B}_n^{\otimes m} = \mathbb{B}_n \otimes \cdots \otimes \mathbb{B}_n$ for some m.

Theorem (Grantcharov, Jung, Kang, Kashiwara, Kim, 2012)

Suppose $\mathcal{B}$ and $\mathcal{C}$ are normal $\mathfrak{q}_n$-crystals. Then:

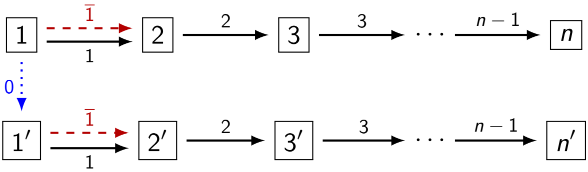
- *$\mathcal{B} \cong \mathcal{C}$ if and only if $\mathrm{ch}(\mathcal{B}) = \mathrm{ch}(\mathcal{C})$.*
- *$\mathcal{B}$ is connected iff $\mathrm{ch}(\mathcal{B}) = P_\mu(x_1, \ldots, x_n)$ for a strict partition $\mu \in \mathbb{N}^n$.*
- *Every $P_\mu(x_1, \ldots, x_n)$ for $\mu \in \mathbb{N}^n$ occurs as $\mathrm{ch}(\mathcal{B})$ for some normal $\mathcal{B}$.*
- *The character of any normal $\mathfrak{q}_n$-crystal is Schur P-positive.*

Normal $\mathfrak{q}_n$-crystals $\rightsquigarrow$ explanations of Schur P-positivity

Key question: does there exist a category of crystals that can be used to demonstrate the stronger property of Schur Q-positivity? **Answer:** Yes!

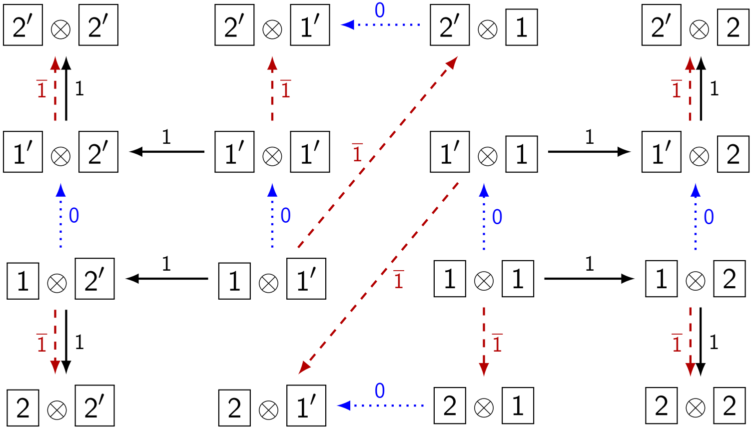
Abstract $\mathfrak{q}_n^+$-crystals

- An extended q_n-crystal or $\mathfrak{q}_n^+$-crystal is $\mathfrak{q}_n$-crystal with extra arrows labeled by $i = 0$, satisfying certain conditions.
- If we retain only the 0-arrows, a $\mathfrak{q}_n^+$-crystal becomes a $\mathfrak{gl}_{1|1}$-crystal.
- The standard $\mathfrak{q}_n^+$-crystal $\mathbb{B}_n^+$ is



  with weight map $\mathrm{wt}(\boxed{i}) = \mathrm{wt}(\boxed{i'}) = \mathbf{e}_i = (0, \dots, 0, 1, 0, \dots) \in \mathbb{Z}^n$.
- The $\mathfrak{q}_n^+$-version of $\oplus$ is again disjoint union, but the $\mathfrak{q}_n^+$-tensor product $\otimes$ is more complicated than the $\mathfrak{q}_n$-version.

The tensor product $\mathbb{B}_2^+ \otimes \mathbb{B}_2^+$ is



with $\mathrm{wt}(\boxed{i} \otimes \boxed{j}) = \mathrm{wt}(\boxed{i}) + \mathrm{wt}(\boxed{j})$

Normal $\mathfrak{q}_n^+$-crystals

- The character of a $\mathfrak{q}_n$-crystal is its character as a $\mathfrak{q}_n$-crystal:

$$\mathrm{ch}\left(\mathbb{B}_n^+\right) = 2x_1 + 2x_2 + \cdots + 2x_n.$$

- A finite $\mathfrak{q}_n^+$-crystal is normal if each of its connected components is isomorphic to a connected component of $(\mathbb{B}_n^+)^{\otimes m}$ for some m.

Theorem (Marberg, Tong, 2021)

Suppose $\mathcal{B}$ and $\mathcal{C}$ are normal $\mathfrak{q}_n^+$-crystals. Then:

- $\mathcal{B} \cong \mathcal{C}$ *if and only if* $\mathrm{ch}(\mathcal{B}) = \mathrm{ch}(\mathcal{C})$.
- $\mathcal{B}$ *is connected iff* $\mathrm{ch}(\mathcal{B}) = Q_\mu(x_1, \ldots, x_n)$ *for a strict partition* $\mu \in \mathbb{N}^n$.
- *Every* $Q_\mu(x_1, \ldots, x_n)$ *for* $\mu \in \mathbb{N}^n$ *occurs as* $\mathrm{ch}(\mathcal{B})$ *for some normal* $\mathcal{B}$.
- *The character of any normal $\mathfrak{q}_n^+$-crystal is Schur Q-positive.*

Weyl group action on $\mathfrak{q}_n^+$ crystals

- Let $\mathcal{B}$ be a $\mathfrak{q}_n^+$-crystal. Choose $i \in \{0\} \cup [n-1]$.

  If we retain only i-arrows, then $\mathcal{B}$ is a disjoint union of paths.

  Define $\sigma_i : \mathcal{B} \to \mathcal{B}$ to reverse the order of the elements in each path.

- Let W_n^{BC} denote the type BC_n Coxeter group with simple generators $t_0 := (-1, 1)$ and $t_i := (i, i+1)(-i, -i-1)$ for $i \in [n-1]$.

Theorem (Kashiwara, 1990s)

If $\mathcal{B}$ is a normal $\mathfrak{gl}_n$-crystal, then there is a unique group action of the symmetric group S_n on $\mathcal{B}$ in which the transposition $(i, i+1)$ acts as σ_i.

Theorem (Marberg, Tong 2021)

If $\mathcal{B}$ is a normal $\mathfrak{q}_n^+$-crystal, then there exists a unique action of W_n^{BC} on $\mathcal{B}$ in which the generators t_0 and t_i for $i \in [n-1]$ act as σ_0 and σ_i.

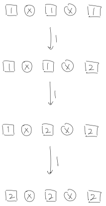
There does not seem to be an analogous result for normal $\mathfrak{q}_n$-crystals.

Subcrystals of $\mathbb{B}_2 \otimes \mathbb{B}_2 \otimes \mathbb{B}_2$ with highest weight $(3, 0)$

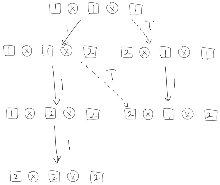
The highest weight of a connected normal $\mathfrak{gl}_n$-, $\mathfrak{q}_n$, or $\mathfrak{q}_n^+$-crystal is the partition $\lambda \in \mathbb{N}^n$ such that the character is s_λ, P_λ, or Q_λ, respectively.

Key fact: $\exists$ exactly one isomorphism class of such crystals for each λ.

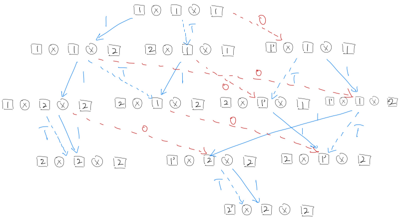
However, for a given λ the $\mathfrak{gl}_n$-, $\mathfrak{q}_n$-, and $\mathfrak{q}_n^+$-crystals look quite different.



type $\mathfrak{gl}_2$

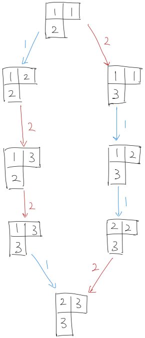


type $\mathfrak{q}_2$

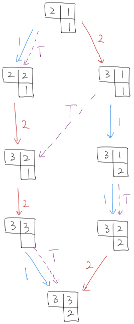


type $\mathfrak{q}_2^+$

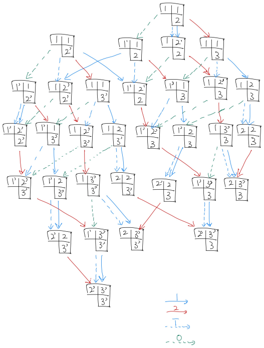
Normal crystals of tableaux with highest weight $(2, 1, 0)$



type $\mathfrak{gl}_3$



type $\mathfrak{q}_3$



type $\mathfrak{q}_3^+$