Chromatic Quasisymmetric Class Functions of linearized combinatorial Hopf monoids

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Quasisymmetric Class Functions

A quasisymmetric class function is a function $\Psi : \mathfrak{G} \to Qsym$ that is constant on conjugacy classes.

Given an integer composition $\alpha \models n$ with k parts, we define the **monomial quasisymmetric function**

$$M_\alpha = \sum_{i_1 < i_2 < \ldots < i_k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}$$

Given a quasisymmetric class function Ψ of degree n, we write $\Psi(\mathfrak{g}) = \sum_{\alpha \models n} \psi_\alpha(\mathfrak{g}) M_\alpha$.

Then ψ_α are class functions, and we can write $\Psi = \sum_{\alpha \models n} \psi_\alpha M_\alpha$.

Coloring Mixed Graphs

Given a finite set N, an **acyclic mixed graph** is a triple $(N, U, \vec{D})$ with a *undirected edge set* U, a *directed edge set* $\vec{D}$, and no directed cycles.

An **automorphism** of G is a bijection $\mathfrak{g} : G \to G$ that preserves edges (of both types). Let $\mathfrak{G}$ be a group of automorphisms of G.

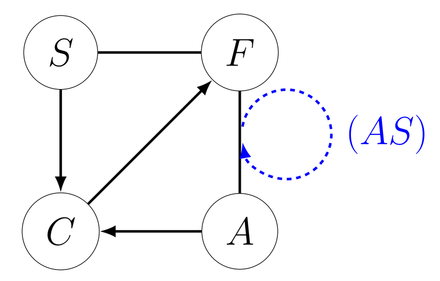

Figure 1. an acyclic mixed graph.

A (weak / strong) **coloring** of G is a function $f : N \to \mathbb{N}$ subject to:

1. For every $uv \in U$, we have $f(u) \neq f(v)$.
2. For every $(u, v) \in D$, we have $f(u) \leq f(v)$.
3. For every $(u, v) \in D$, we have $f(u) < f(v)$.

Let $\mathcal{F}(G)$ ($\overline{\mathcal{F}}(G)$) denote the set of weak/strong colorings.

Chromatic Quasisymmetric Class Function

If $\mathfrak{G} \curvearrowright G$, then $\mathfrak{G} \curvearrowright \mathcal{F}(G)$ and $\mathfrak{G} \curvearrowright \overline{\mathcal{F}}(G)$ via $\mathfrak{g}f = f \circ \mathfrak{g}^{-1}$, where $f \in \mathcal{F}(G)$ and $\mathfrak{g} \in \mathfrak{G}$.

Given commuting indeterminates $x_1, x_2, \ldots$ and $\mathfrak{g} \in \mathfrak{G}$, we define

$$X(G, \mathfrak{G}, \mathbf{x}; \mathfrak{g}) = \sum_{f \in \mathcal{F}(G) : \mathfrak{g}f = f} \prod_{v \in N} x_{f(v)}.$$

and

$$\overline{X}(G, \mathfrak{G}, \mathbf{x}; \mathfrak{g}) = \sum_{f \in \overline{\mathcal{F}}(G) : \mathfrak{g}f = f} \prod_{v \in N} x_{f(v)}.$$

Thus $X(G, \mathfrak{G}, \mathbf{x})$ is the weak chromatic quasisymmetric class function of $(G, \mathfrak{G})$, and $\overline{X}(G, \mathfrak{G}, \mathbf{x})$ is the strong chromatic quasisymmetric class function.

The orbital chromatic polynomials $\chi(G, \mathfrak{G}, x)$ and $\overline{\chi}(G, \mathfrak{G}, x)$ count the number of orbits of proper colorings with largest color at most x.

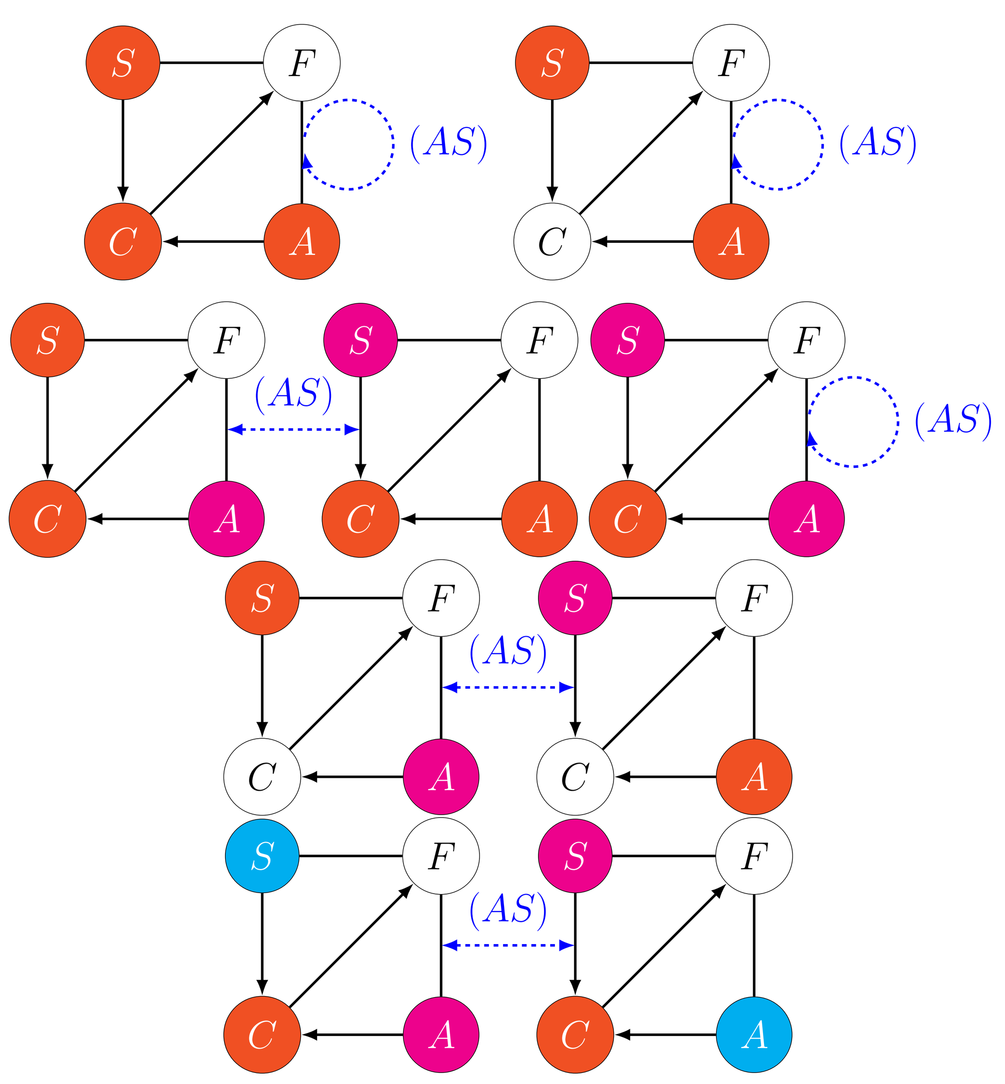

Figure 2. various colorings, where white < orange < magenta < cyan

There are two irreducible characters for $\mathbb{Z}/2\mathbb{Z}$. We let $\det$ denote the nontrivial character, and ρ denote the regular character. For our example mixed graph, we have

$$\begin{aligned} X(G, \mathfrak{G}, \mathbf{x}) &= M_{1,3} + M_{2,2} + \rho M_{1,2,1} + \rho M_{2,1,1} + M_{1,1,2} + \rho M_{1,1,1,1} \\ &= F_{1,3} + F_{2,2} + \det F_{2,1} + \det F_{2,1,1} - F_{1,1,2} - \det F_{1,1,1,1} \end{aligned}$$

$$\chi(G, \mathfrak{G}, x) = 2\binom{x}{2} + 3\binom{x}{3} + \binom{x}{4}$$

$$\overline{X}(G, \mathfrak{G}, \mathbf{x}) = M_{1,1,2} + \rho M_{1,1,1,1}$$

$$\overline{\chi}(G, \mathfrak{G}, x) = \binom{x}{3} + \binom{x}{4}$$

Given two group characters χ and ψ, we write $\chi \leq_{\mathfrak{G}} \psi$ if $\psi - \chi$ is a group character. Given a quasisymmetric class function Ψ, we let $[M_\alpha]\Psi$ be the coefficient of M_α in Ψ, which is a class function.

Theorem

We have $X(G, \mathfrak{G}, \mathbf{x})$ and $\overline{X}(G, \mathfrak{G}, \mathbf{x})$ are quasisymmetric class functions. Moreover, for any $\alpha \models |N|$, then $[M_\alpha]X(G, \mathfrak{G}, \mathbf{x})$ and $[M_\alpha]\overline{X}(G, \mathfrak{G}, \mathbf{x})$ are permutation characters.

For α a coarsening of β, we have $[M_\alpha]X(G, \mathfrak{G}, \mathbf{x}) \leq_{\mathfrak{G}} [M_\beta]X(G, \mathfrak{G}, \mathbf{x})$ and $[M_\alpha]\overline{X}(G, \mathfrak{G}, \mathbf{x}) \leq_{\mathfrak{G}} [M_\beta]\overline{X}(G, \mathfrak{G}, \mathbf{x})$.

Similarly, if we write $\chi(G, \mathfrak{G}, x) = \sum_{i=0}^{|N|} f_i \binom{x}{i}$, then we have:

1. For $i \leq j \leq |N| + 1 - i$, we have $f_i \leq f_j$.
2. For all $1 \leq i \leq j$, we have $\binom{|N|-1}{j-1} f_i \leq \binom{|N|-1}{i-1} f_j$.

If we write $\overline{\chi}(G, \mathfrak{G}, x) = \sum_{i=0}^{|N|} f_i \binom{x}{i}$, we obtain the same inequalities. For ordinary graphs, with trivial group action, the last inequality is new.

We say that $X(G, \mathfrak{G}, \mathbf{x})$ is M-increasing and $\chi(G, \mathfrak{G}, x)$ is strongly flawless.

Are there similar results for:

- P-partitions of a poset or double poset?
- generic functions on matroids, or generalized permutohedra?
- generalized colorings of a graph where every connected component of every monochromatic subgraph has a Hamilton path?

Idea: work with linearized combinatorial Hopf monoids in species.

1. The Hopf algebra / monoid structure allows us to define 'colorings',
2. The species structure allows us to define group actions.

Species and group actions

- Set species = an endofunctor $\mathbf{F} : Set \to Set$ on the category of finite sets with bijections. Given a finite set N, we obtain a set $\mathbf{F}_N$ such that $\mathfrak{S}_N \curvearrowright \mathbf{F}_N$.
- Linear species = a functor $\mathcal{F} : Set \to Vec$ to the category of finite dimensional vector spaces over a field $\mathbb{K}$ and linear transformations.
- Given $\mathbf{F}$, the linearization $\mathbb{K}\mathbf{F}$ is defined by letting $(\mathbb{K}\mathbf{F})_N$ be the vector space with basis $\mathbf{F}_N$.

Examples:

- If we let $\mathbf{E}_N = \{1\}$ for every finite set N, then we obtain the **exponential species**.
- If we let $\mathbf{MG}_N$ denote the set of acyclic mixed graphs on N, we obtain a species.
- We can also consider the subspecies of graphs $\mathbf{G}$ or posets $\mathbf{P}$ of $\mathbf{MG}$.

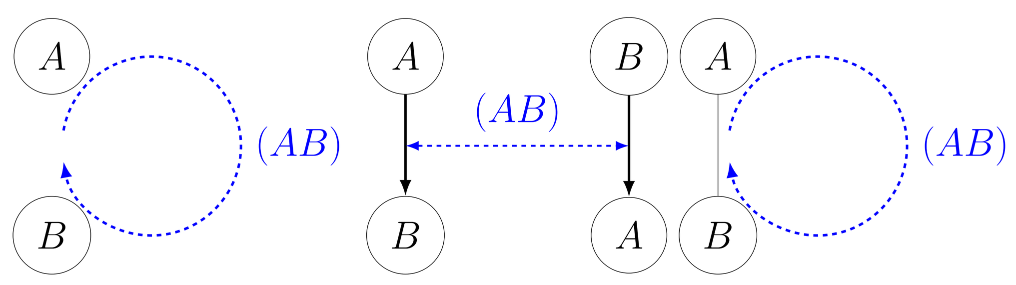

Figure 3. Acyclic mixed graphs

Given $\mathbf{f} \in \mathbf{F}_N$, and $\mathfrak{g} \in \mathfrak{S}_N$, we say $\mathfrak{g}$ is an automorphism of $\mathbf{f}$ if $\mathfrak{g}\mathbf{f} = \mathbf{f}$.

Linearized combinatorial Hopf monoids

A connected Hopf monoid is a linear species $\mathcal{H}$ with maps

1. $\mu_{S,T} : \mathcal{H}_S \otimes \mathcal{H}_T \to \mathcal{H}_{S \sqcup T}$
2. $\Delta_{S,T} : \mathcal{H}_{S \sqcup T} \to \mathcal{H}_S \otimes \mathcal{H}_T$

for every pair of disjoint sets S, T, subject to axioms:

1. **Associativity**: For all disjoint A, B, C, we have $\mu_{A, B \sqcup C} \circ (1_A \otimes \mu_{B,C}) = \mu_{A \sqcup B, C} \circ (\mu_{A,B} \otimes 1_C)$.
2. **Coassociativity**: For all disjoint A, B, C, we have $(1_A \otimes \Delta_{B,C}) \circ \Delta_{A, B \sqcup C} = (\Delta_{A,B} \otimes 1_C) \circ \Delta_{A \sqcup B, C}$.
3. **Compatibility**: For all disjoint sets A, B, C, D, we have $\Delta_{A \sqcup C, B \sqcup D} \circ \mu_{A \sqcup B, C \sqcup D} = (\mu_{A,C} \otimes \mu_{B,D}) \circ (1_A \otimes \beta_{B,C} \otimes 1_D) \circ (\Delta_{A,B} \otimes \Delta_{C,D})$ where $\beta_{B,C}(x \otimes y) = y \otimes x$.
4. **Connectedness**: $\dim \mathcal{H}_\emptyset = 1$.

A linearized Hopf monoid is a set species $\mathbf{H}$, such that $\mathbb{K}(\mathbf{H})$ is a Hopf monoid, and furthermore, for every pair of disjoint finite sets M and N, we have the following:

1. For every $\mathbf{x} \in \mathbf{H}_M, \mathbf{y} \in \mathbf{H}_N$, we have $\mu(\mathbf{x} \otimes \mathbf{y}) \in \mathbf{H}_{M \sqcup N}$.
2. For every $\mathbf{h} \in \mathbf{H}_{M \sqcup N}$, if $\Delta_{M,N}(\mathbf{h}) \neq 0$ then there exists $\mathbf{h} \setminus N \in \mathbf{H}_M$ and $\mathbf{h}/M \in \mathbf{H}_N$ such that $\Delta_{M,N}(\mathbf{h}) = \mathbf{h} \setminus N \otimes \mathbf{h}/M$.

For $\mathbf{MG}$, the product is disjoint union of mixed graphs. Given disjoint sets M, N, and a mixed graph $\mathbf{g} \in \mathbf{MG}_{M \cup N}$, we define

$$\Delta_{M,N}(\mathbf{g}) = \begin{cases} 0 & \text{if there exists } m \in M, n \in N, (n, m) \in \mathbf{g} \\ \mathbf{g}|_M \otimes \mathbf{g}|_N & \text{otherwise} \end{cases}$$

where $\mathbf{g}|_S$ is the induced subgraph on S. Thus $\mathbf{MG}$ is a linearized Hopf monoid. It contains graphs $\mathbf{G}$ and posets $\mathbf{P}$ as Hopf submonoids.

Chromatic Quasisymmetric Class Function

A **Hopf monoid character** is a natural transformation $\varphi : \mathbb{K}(\mathbf{H}) \to \mathbb{K}(\mathbf{E})$ such that, for all disjoint finite sets M and N, and all $x \in \mathbf{H}_M$ and all $y \in \mathbf{H}_N$, we have $\varphi_M(x) \cdot \varphi_N(y) = \varphi_{M \sqcup N}(x \cdot y)$.

It is **linearized** if $\varphi(\mathbf{h}) \in \{0, 1\}$ for all $\mathbf{h} \in \mathbf{H}_N$ and all N.

For $\mathbf{MG}$, we define

$$\varphi(\mathbf{g}) = \begin{cases} 0 & \text{there exists } uv \in \mathbf{g} \\ 1 & \text{otherwise} \end{cases}$$

and

$$\overline{\varphi}(\mathbf{g}) = \begin{cases} 1 & \mathbf{g} \text{ has no directed or undirected edges} \\ 0 & \text{otherwise} \end{cases}$$

Given $\mathbf{H}$, $\mathbf{h} \in \mathbf{H}_N$, $i \in \mathbb{N}$, and $f : N \to \mathbb{N}$, we define monochromatic subobjects

$$\mathbf{h}_{f,i} := \mathbf{h} \setminus f^{-1}(\mathbb{N} \setminus [i]) / f^{-1}([i-1]).$$

Given φ, a φ-proper coloring is a function $f : N \to \mathbb{N}$ such that $\varphi(\mathbf{h}_{f,i}) = 1$ for all $i \in \mathbb{N}$. Let $\mathcal{F}_\varphi(\mathbf{h})$ denote the set of φ-proper colorings. If $\mathfrak{G} \curvearrowright \mathbf{h}$ as automorphisms, then $\mathfrak{G} \curvearrowright \mathcal{F}_\varphi(\mathbf{h})$ via $\mathfrak{g}f = f \circ \mathfrak{g}^{-1}$, where $f \in \mathcal{F}_\varphi(\mathbf{h})$ and $\mathfrak{g} \in \mathfrak{G}$.

Given commuting indeterminates $x_1, x_2, \ldots$ and $\mathfrak{g} \in \mathfrak{G}$, we define

$$\Psi_{\mathbf{H},\varphi}(\mathbf{h}, \mathfrak{G}, \mathbf{x}; \mathfrak{g}) = \sum_{f \in \mathcal{F}_\varphi(\mathbf{h}) : \mathfrak{g}f = f} \prod_{v \in N} x_{f(v)}.$$

Thus $\Psi_{\mathbf{H},\varphi}(\mathbf{h}, \mathfrak{G}, \mathbf{x})$ is the φ-chromatic quasisymmetric class function of $(\mathbf{h}, \mathfrak{G})$.

The orbital chromatic polynomial $\Psi_{\mathbf{H},\varphi}(\mathbf{h}, \mathfrak{G}, x)$ counts the number of orbits of proper colorings with largest color at most x.

Given $G \in \mathbf{MG}_N$, and $\mathfrak{G} \curvearrowright G$ as automorphisms, We have

1. $\Psi_{\mathbf{MG},\varphi}(G, \mathfrak{G}, \mathbf{x}) = X(G, \mathfrak{G}, \mathbf{x})$.
2. $\Psi_{\mathbf{MG},\overline{\varphi}}(G, \mathfrak{G}, \mathbf{x}) = \overline{X}(G, \mathfrak{G}, \mathbf{x})$.

Theorem

Let $\mathbf{H}$ be a linearized Hopf monoid and φ be a linearized Hopf monoid character. Then the following are equivalent:

1. For every finite set N, every $\mathbf{h} \in \mathbf{H}_N$, and every group $\mathfrak{G} \curvearrowright \mathbf{h}$ as automorphisms, we have $\Psi_{\mathbf{H},\varphi}(\mathbf{h}, \mathfrak{G}, \mathbf{x})$ is M-increasing.
2. For every finite set N, every $\mathbf{h} \in \mathbf{H}_N$, and every $k > 0$ with $k < |N|$, if $\varphi(\mathbf{h}) = 1$, then there exists $S \subset N$ with $|S| = k$ and $\varphi(\mathbf{h} \setminus S) = \varphi(\mathbf{h}/(N \setminus S)) = 1$.

Moreover, if $\Psi_{\mathbf{H},\varphi}(\mathbf{h}, \mathfrak{G}, \mathbf{x})$ is M-increasing, then:

1. $\Psi_{\mathbf{H},\varphi}(\mathbf{h}, \mathfrak{G}, x)$ is strongly flawless.
2. If we write $\Psi_{\mathbf{H},\varphi}(\mathbf{h}, \mathfrak{G}, x) = \sum_{i=0}^{|N|} f_i \binom{x}{i}$, then for all $i \leq j$, we have $\binom{|N|-1}{j-1} f_i \leq \binom{|N|-1}{i-1} f_j$.

Wanted: A condition on $\mathbf{H}$ and φ that ensures $\Psi_{\mathbf{H},\varphi}(\mathbf{h}, \mathfrak{G}, \mathbf{x})$ is F-effective for all $\mathbf{h}$. Here, F-effective means that $[\mathcal{F}_\alpha]\Psi_{\mathbf{H},\varphi}(\mathbf{h}, \mathfrak{G}, \mathbf{x})$ is a character for all α.

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