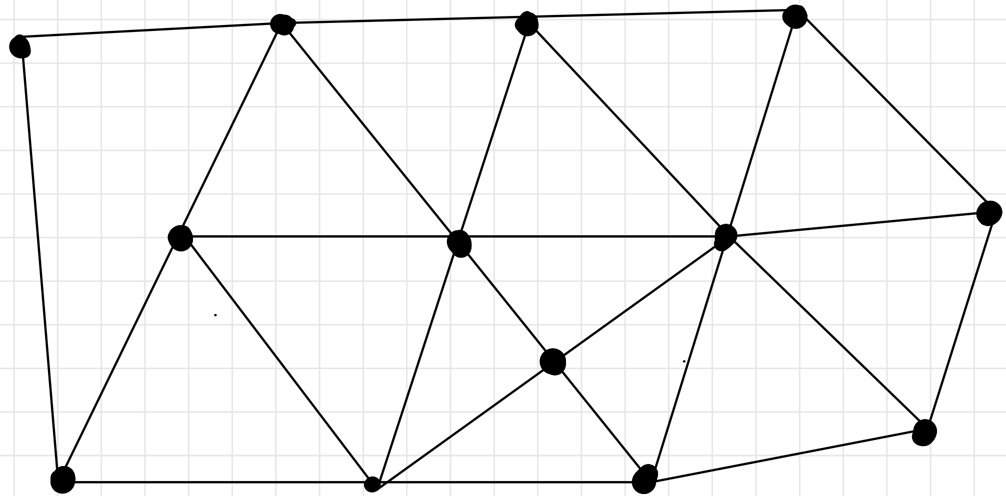
Even subgraphs and Loop O(1) as factors of IID

with Gourab Ray + Yinon Spinka

FPSAC 2022

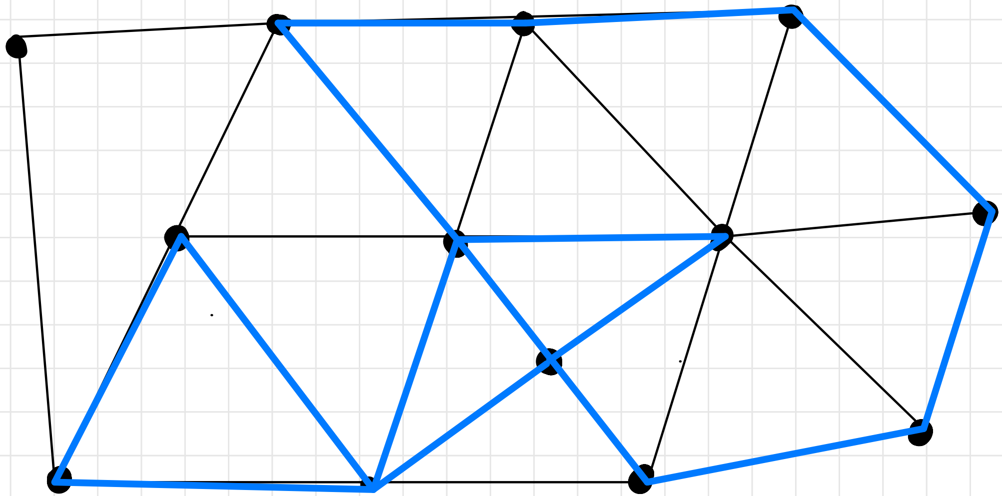
Bangalore

Random even subgraphs



G

Random even subgraphs



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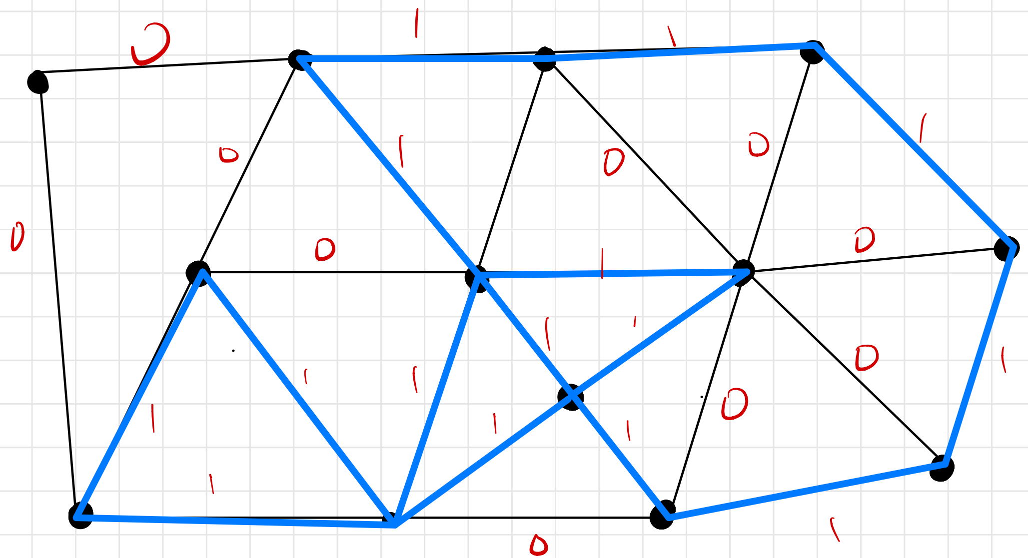
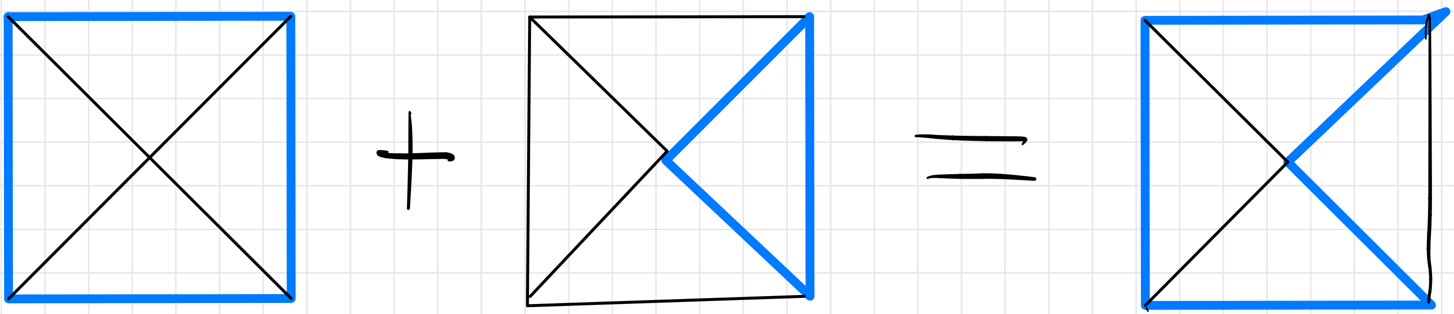


subgraphs of G $\Longleftrightarrow$ $\mathbb{Z}_2^E$

Random even subgraphs



Even subgraphs are a subspace

$$\mathcal{E} \subset \mathbb{Z}_2^E$$

(also called the cycle space)

Random even subgraphs

One way to pick a uniform $H \subset \mathcal{E}$:

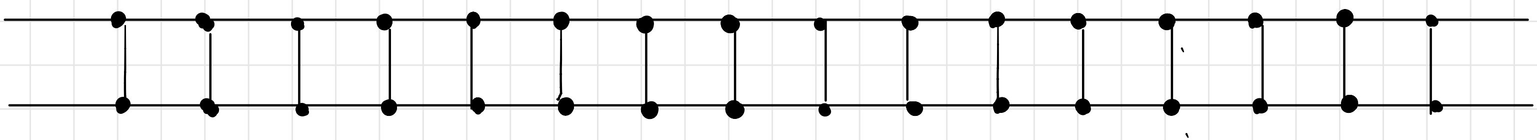
- fix a basis B for $\mathcal{E}$
- take independent uniform $\varepsilon_i \in \{0,1\}$
- Take $H = \sum \varepsilon_i b_i$

Random even subgraphs

What if G is infinite?

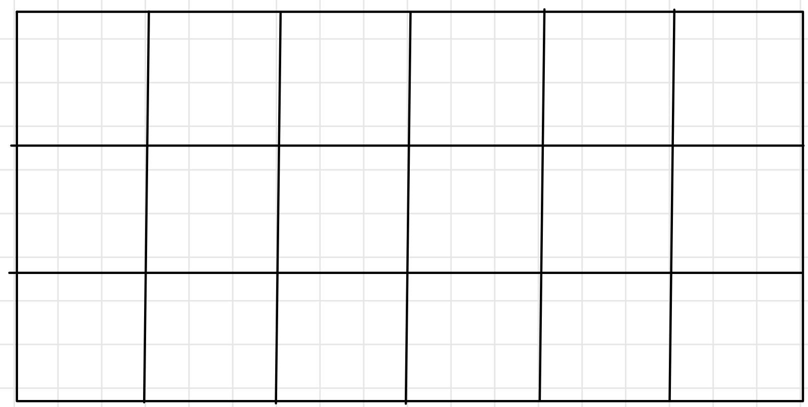
Random even subgraphs

What if G is infinite?



Random even subgraphs

What if G is infinite?



Random even subgraphs

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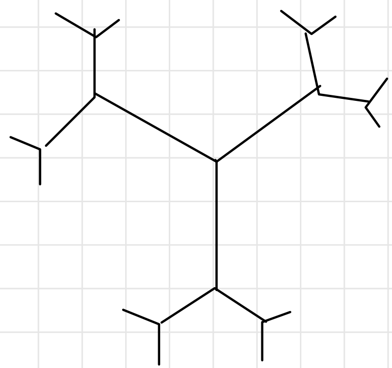


Random even subgraphs

Free limit

- Take $G_n \subset G$ finite, $G_n \nearrow G$
- Let $H_n \subset G_n$ be a unif. even subgraph
- Take the limit (in dist.) of H_n.

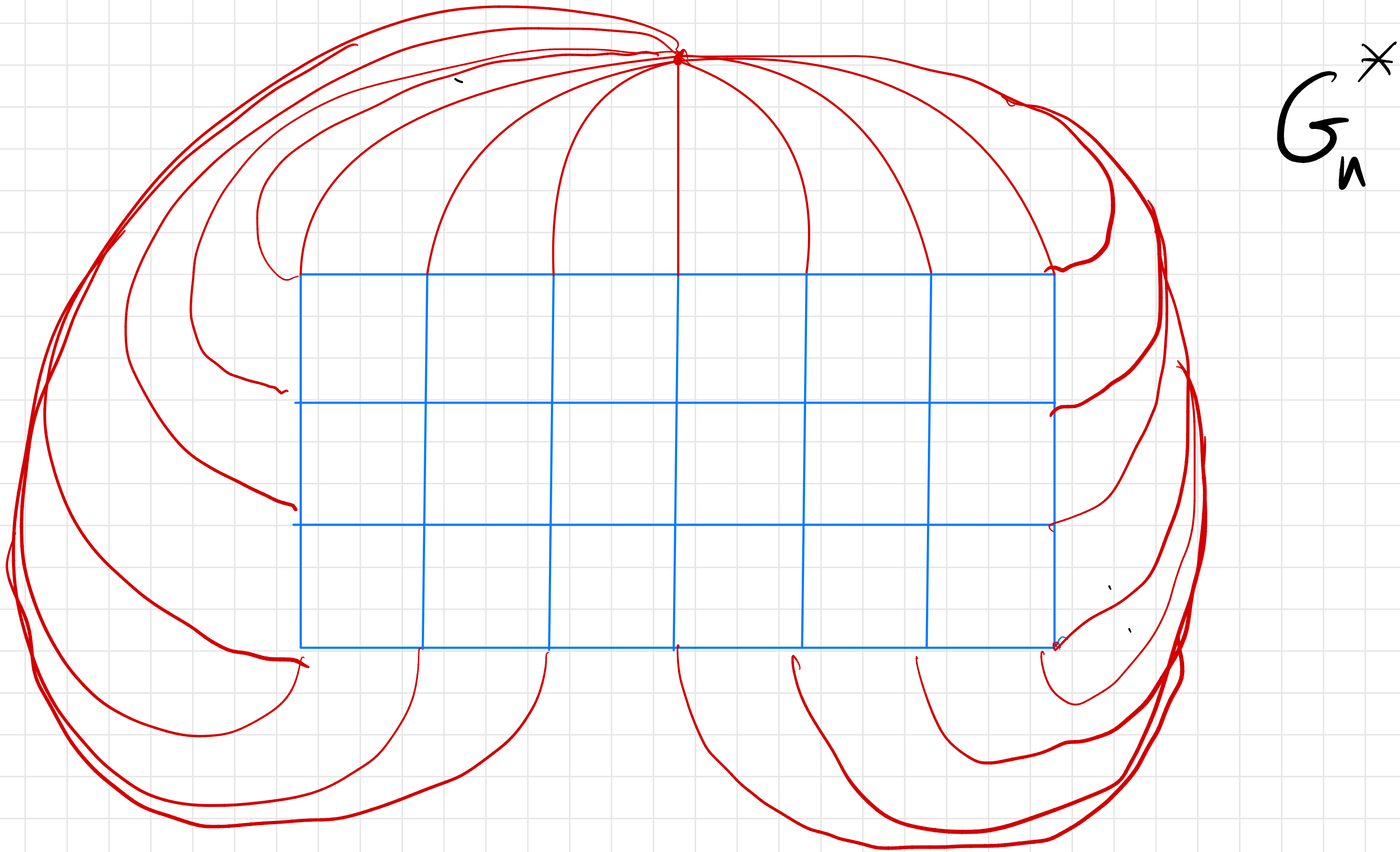
> **Thm** the limit exists, does not depend on the choice of exhaustion.

Random even subgraphs

Wired limit

- Take $G_n \subset G$ finite, $G_n \nearrow G$
- Let G_n^* be G with all of $G \setminus G_n$ contracted to a single vertex.

Random even subgraphs



Random even subgraphs

Wired limit

- Take $G_n \subset G$ finite, $G_n \nearrow G$
- Let G_n^* be G with all of $G \setminus G_n$ contracted to a single vertex.
- $H_n \subset G_n^*$ unif. even subgraph.
- take distrib. limit.

Random even subgraphs

Thm the free and wired even subgraph limits exist, do not depend on the choice of exhaustion.

Note The limits may be equal or not. It is not always obvious which.

Generating even subgraphs

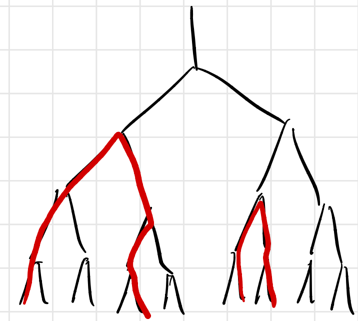
$\mathcal{E}^f$ = free even subgraph space, spanned by cycles.

$\mathcal{E}^w$ = wired even subgraph space, includes also doubly infinite paths

Generating even subgraphs

e.g. $G =$ reg. tree: $\qquad \mathcal{E}^f = \{\emptyset\}$

$\mathcal{E}^w$ includes paths:



Generating even subgraphs

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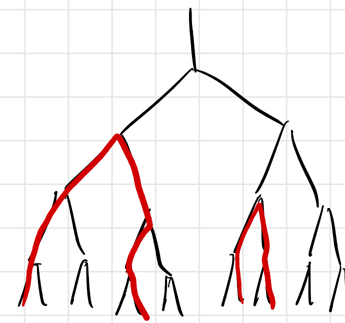


e.g. $G = \mathbb{Z}^2$:

Is $\mathcal{E}^f = \mathcal{E}^w$?

Factors of iid

Consider processes $(X_n)_{n\in\mathbb{Z}}$, $(Y_n)_{n\in\mathbb{Z}}$.

X is a factor of Y if there is a func. φ s.t. $X=\varphi(Y)$ and φ is translation equivariant:

$$\varphi(\text{shift}(Y)) = \text{shift}(\varphi(Y))$$

Factors of iid

If (Y_n) are iid and X is a factor of Y we say X is a factor of iid (FIID).

Factors of iid

examples:

$$X_n = \begin{cases} 1 & Y_n > Y_{n+1} \\ 0 & Y_n \leq Y_{n+1} \end{cases}$$

$$X_n = \max(Y_{n-1}, Y_n, Y_{n+1})$$

$$X_n = \begin{cases} 1 & Y_{n+m} \leq Y_n + |m| \quad \forall m \\ 0 & \text{if not.} \end{cases}$$

Factors of iid

Finitary factor: reveal terms one by one; at some finite time X_0 is determined.

$$X_n = \max(Y_{n-1}, Y_n, Y_{n+1})$$

$$X_n = \begin{cases} 1 & Y_{n+m} \leq Y_n + |m| \quad \forall m \\ 0 & \text{if not.} \end{cases}$$

(not finitary) ✓

Factors of iid

Major problem: understanding which processes are factors of others, and what properties can the factor maps possess.

Factors of iid

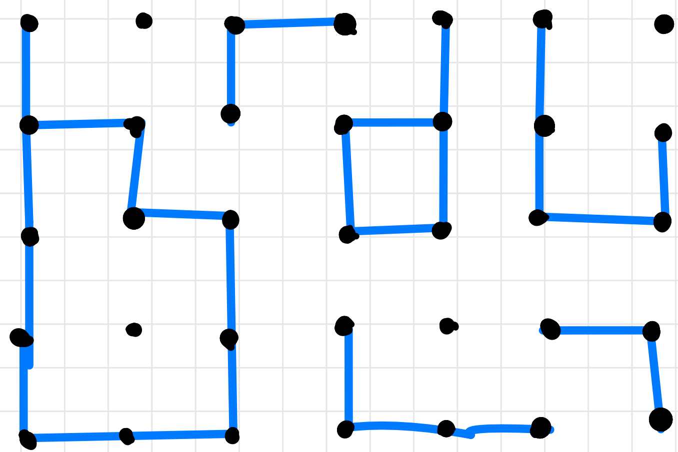
Factors are of interest on any graph G: $(X_v)_{v\in G}$ is a proc.

$$X = \varphi(Y) \text{ with } \varphi \text{ equivariant.}$$

<u>note</u>: X, Y can live on edges or vertices or both.

Factors of iid

e.g. Let $G = \mathbb{Z}^2$. Let $H \subset \mathbb{Z}^2$ include each edge independently with prob. p.



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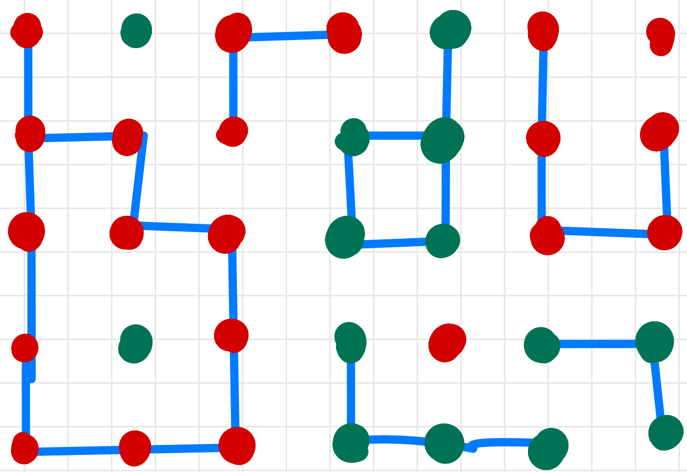
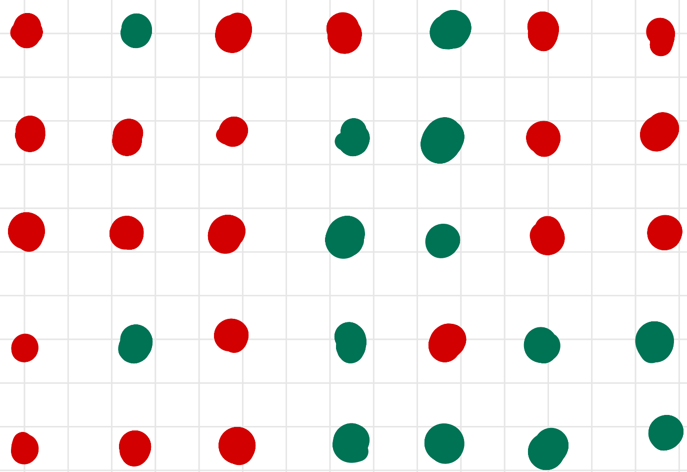


colour each cluster $\bullet$ / $\bullet$ by a coin toss.

Factors of iid

e.g. Let $G = \mathbb{Z}^2$. Let $H \subset \mathbb{Z}^2$ include each edge independently with prob. p.

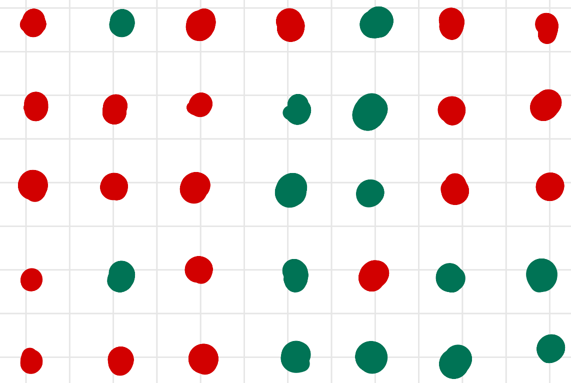


- colour each cluster red / green
- forget the edges.

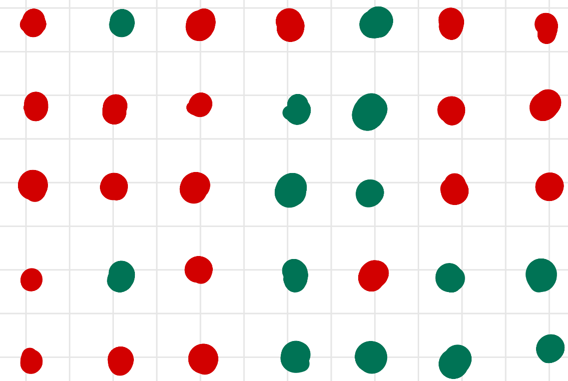
Factors of iid

X_v = colour of v.

Is X a factor of iid?



Factors of iid



X_v = colour of v.

Is X a factor of iid?

If $p < p_c$ all clusters are finite $\Rightarrow$ YES

If $p > p_c$ there is an ∞ cluster $\Rightarrow$ NO.

Factors of iid

Qn: what about the same model on other graphs?

If there are no ∞ clusters: YES

unique ∞ cluster: NO

∞ many ∞ clusters: unclear.

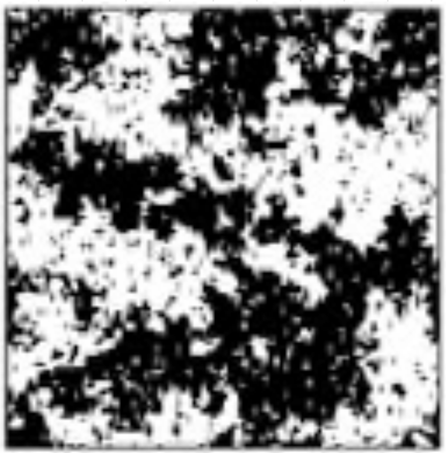
(open in general)

The Ising model (magnetism, 1920's)

configuration $\sigma \in \{\pm 1\}^V$

distrib. $p(\sigma) \propto e^{\beta \sum_{x \sim y} \sigma_x \sigma_y}$

$\beta > \beta_c$



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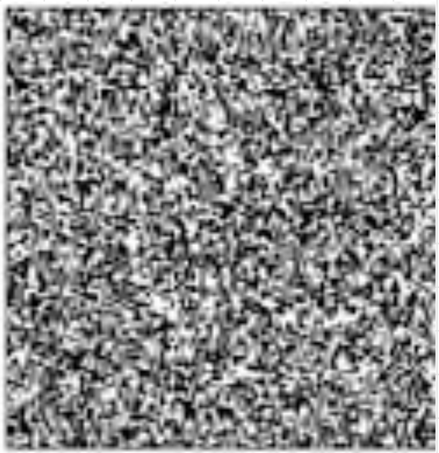


$\beta < \beta_c$

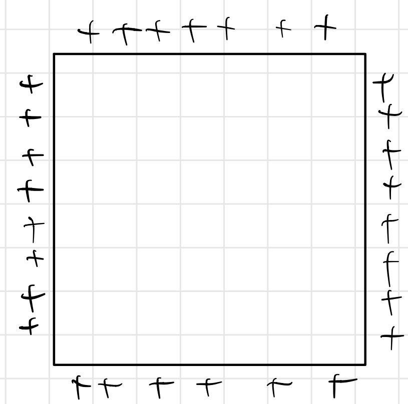
The Ising model

configuration $\sigma \in \{\pm 1\}^V$

distrib. $p(\sigma) \propto e^{\beta \sum_{x \sim y} \sigma_x \sigma_y}$

On an infinite graph we can take free or wired limits μ^f, μ^+, μ^-.

μ_+ : limit with + boundary cond.



The Ising model

$\boxed{\text{\underline{Thm} (Ornstein-Weiss ; Adams) } \mu^+ \text{ is FIID for } \forall\beta \text{ (On any amenable graph).}}$

Harel-Spinka : This holds for "monotone" models.

The Ising model

Thm (Ornstein–Weiss ; Adams) μ^+ is FIID for $\forall \beta$ (on any amenable graph).

Thm (van den Berg – Steif) On $\mathbb{Z}^d$, μ^+ is FFIID if and only if $\mu^+ = \mu^-$ (i.e. $\beta \leq \beta_c$)

Key obstacle : Large deviation probabilities :

$\mu^+(\text{more} - \text{in } [n]^d) \geq \exp(-cn^{d-1})$

but for FFIID it must be $\leq \exp(-cn^d)$.

The Ising model: Beyond $\mathbb{Z}^d$.

On the regular tree $\mathbb{T}_d$:

- μ^+ is a $\begin{cases} \text{FIID} & \forall \beta \\ \text{FFIID} & \text{if } \mu^+ = \mu^- \\ \text{FFIID} & \text{for } \beta > \beta_0 \quad \text{(Harel-Ray-Spinka)} \end{cases}$

- μ^f is $\begin{cases} \text{a FFIID} & \text{if } \tanh \beta < \frac{1}{d-1} \quad \text{(uniqueness)} \\ \text{not FFIID} & \text{if } \tanh \beta > \frac{1}{\sqrt{d-1}} \quad \text{(reconstruction)} \\ \text{a FIID} & \text{if } \tanh \beta \leq c(d) > \frac{1}{d-1} \quad \text{(Nam-Sly-Zhang)} \end{cases}$

The Ising model: Ising gradient

Let $\omega_{xy} = \begin{cases} 1 & \sigma_x \neq \sigma_y \\ 0 & \sigma_x = \sigma_y \end{cases}$.

On $\mathbb{Z}^d$, (ω_e) is FFIID for all β (Ray-Spinka) requires new methods since it is not a monotone model.

A.-Ray-Spinka: also on planar lattices.

FK-Ising and Edwards-Sokal

On a finite graph, FK ising meas. on $\{0,1\}^E$

$$P(\omega) \propto p^{\text{open}(\omega)} (1-p)^{\text{closed}(\omega)} \, 2^{\text{clusters}(\omega)}$$

The free and wired limits are ϕ^f, ϕ^w

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The free and wired limits are ϕ^f, ϕ^w

Haggstrom-Jonasson-Lyons : ϕ^f, ϕ^w are FIID

Harel-Spinka : If $\phi^f = \phi^w$ then FFIID

if $\phi^w \neq \phi^f$ then ϕ^w not FFIID (extra cond.)

FK-Ising and Edwards-Sokal

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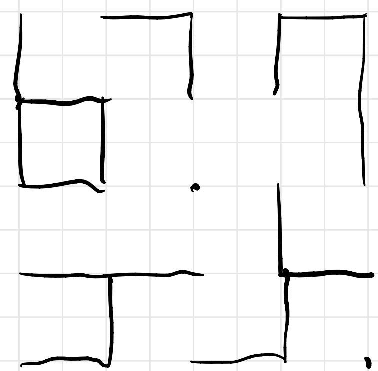
Edwards-Sokal: Assign each cluster in ω a sign to get Ising.



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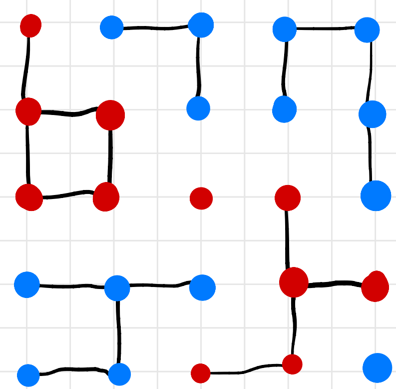

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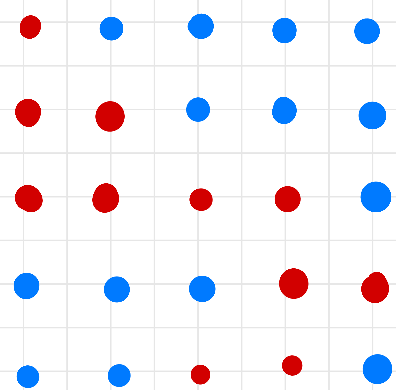


The Loop O(1) model

Meas. on config. $\eta \in \{0,1\}^{E \cup V}$

Parameters $x, y \geq 0$ $\quad x = \tanh \beta \quad y = \tanh h$

On finite graph:

$$P(\eta) \propto x^{\sum \eta(e)} y^{\sum \eta(v)} \mathbb{1}_{\eta \text{ even}}$$

Thm: Free and wired limits exist: P^f, P^w.

Special case: $x=1, y=0$: unif. even subgraph.

The Loop O(1) model

<u>Thm</u> (A.-Ray-Spinka) for $x, y \in [0,1]$, P^ω is FIID unless ($x=1$ and $y=0$ and G is 2-ended)

The Loop O(1) model

Thm (A.-Ray-Spinka) for $x,y \in [0,1]$, P^{ω} is FIID unless ($x=1$ and $y=0$ and G is 2-ended)

Thm (A-R-S) P^{f} is FIID in many cases:

- $y>0$
- G amenable or planar
- ϕ^{f} has $<\infty$ geodesic cycles through e

Conj: P^{f} is always FIID

The Loop O(1) model

Aizenman – Duminil-Copin – Sidoravicius :

Loop O(1) $\Longleftrightarrow$ FK Ising $\qquad p = \frac{2x}{1+x} \quad q = \frac{2y}{1+y}$

Given a Loop O(1) config., add each edge(rect) with prob. x (y) to get FK Ising.

Given FK Ising, take a uniform even subgraph

$\Rightarrow$ Loop O(1) can be a factor.

Generating even subgraphs

Want $H = \sum \varepsilon_i b_i$ for some spanning set of $\mathcal{E}^f$ or $\mathcal{E}^w$.

* This is only defined if $\{b_i\}$ is locally finite.
* For a factor, want $\{b_i\}$ to be chosen "invariatly".

e.g. If G is planar, the faces span $\mathcal{E}^f$.

Recall: ⊛ A ray is an ∞ simple path.

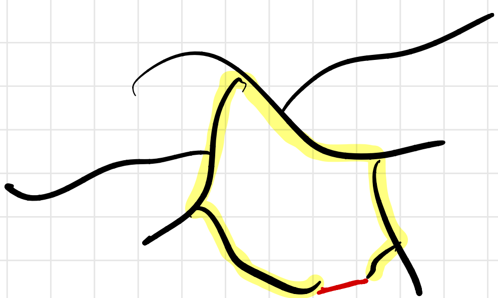
⊛ 2 rays are equivalent if $\forall$ finite sets S they are eventually in same component of $G \setminus S$.

⊛ An end is an equivalence class of rays.

e.g. $\mathbb{Z}$, $\mathbb{Z}^2$, Trees

A tree $T \subset G$ is end-faithful if ends of T are in bijection with ends of G.

Claim: If T is an end faithful spanning tree then it gives a locally finite spanning set for $\mathcal{E}^f$.



$e \notin T \iff C_e$ cycle

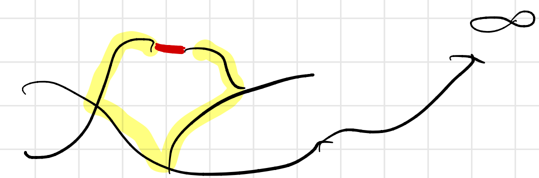
Timar: G amenable, unimodular, one-ended then G has a one-ended sp. tree as a FIID.

Benjamini-Lyons-Peres-Schramm: 2 ended

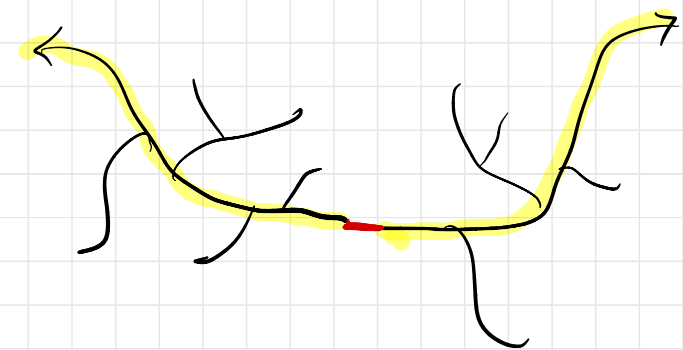
Qn: what if G has ∞ ends?

Generating $\mathcal{E}^w$

Let F be a spanning forest with 1-ended trees.



For $e \notin F$ define C_e extend e to a cycle or bi-infinite path in F.



The WUSF is such a forest, and is a FIID.

THANK YOU