Acting by Separable Permutations on the Kazhdan-Lusztig Basis

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Context

$\mathfrak{S}_n$ acts on $\mathcal{S}^\lambda$ equipped with the *Kazhdan-Lusztig basis* $\mathbb{KL}_\lambda = \{C_T \mid T \in \mathsf{SYT}(\lambda)\}$.

Theorem (Berenstein-Zelevinksy, Stembridge, Mathas, '90s)

If $\lambda \vdash n$ is arbitrary:

$$w_0 \cdot C_T = \pm C_{\mathrm{ev}(T)}$$

Theorem (Rhoades, 2010)

If $\lambda \vdash n$ is **rectangular***:*

$$c \cdot C_T = \pm C_{\mathrm{pr}(T)}$$

Non-rectangular Case

$$\mathbb{KL} = \left(\begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} \end{array}, \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} \end{array}, \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2} \end{array}, \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2} \end{array}, \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} \end{array} \right)$$

$$[c]_{\mathbb{KL}} = \begin{bmatrix} 0 & 0 & 0 & - & 0 \\ 0 & 0 & - & 0 & 0 \\ 0 & 0 & 0 & 0 & - \\ + & 0 & - & 0 & - \\ 0 & + & 0 & - & 0 \end{bmatrix} \qquad [\mathrm{pr}]_{\mathbb{KL}} = \begin{bmatrix} 0 & 0 & 0 & + & 0 \\ 0 & 0 & + & 0 & 0 \\ 0 & 0 & 0 & 0 & + \\ + & 0 & 0 & 0 & 0 \\ 0 & + & 0 & 0 & 0 \end{bmatrix}$$

Non-rectangular Case

$$\mathbb{KL} = \left(\begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} \end{array}, \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} \end{array}, \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2} \end{array}, \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2} \end{array}, \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} \end{array} \right)$$

$$[c]_{\mathbb{KL}} = \begin{bmatrix} 0 & 0 & 0 & - & 0 \\ 0 & 0 & - & 0 & 0 \\ 0 & 0 & 0 & 0 & - \\ + & 0 & - & 0 & - \\ 0 & + & 0 & - & 0 \end{bmatrix} \qquad [\mathrm{pr}]_{\mathbb{KL}} = \begin{bmatrix} 0 & 0 & 0 & + & 0 \\ 0 & 0 & + & 0 & 0 \\ 0 & 0 & 0 & 0 & + \\ + & 0 & 0 & 0 & 0 \\ 0 & + & 0 & 0 & 0 \end{bmatrix}$$

$$[43512]_{\mathbb{KL}} = \begin{bmatrix} 0 & 0 & + & - & + \\ 0 & 0 & 0 & 0 & + \\ 0 & 0 & 0 & - & + \\ 0 & - & 0 & 0 & + \\ + & - & 0 & 0 & 0 \end{bmatrix} \qquad [??]_{\mathbb{KL}} = \begin{bmatrix} 0 & 0 & + & 0 & 0 \\ 0 & 0 & 0 & 0 & + \\ 0 & 0 & 0 & + & 0 \\ 0 & + & 0 & 0 & 0 \\ + & 0 & 0 & 0 & 0 \end{bmatrix}$$

Main Result

Theorem (G., Yacobi)

Fix $\lambda \vdash n$ **arbitrary**.

$$c \cdot C_T = \mathrm{pr}(\pm C_T + \mathrm{l.o.t})$$

where $C_R < C_T$ *when* $\mathrm{sh}(\partial(R)) \prec \mathrm{sh}(\partial(T))$

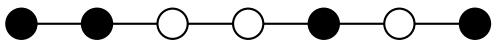
Theorem (G., Yacobi)

Fix $\lambda \vdash n$ **arbitrary**. *Let* $w \in \mathfrak{S}_n$ *be any separable permutation.*

$$w \cdot C_T = \varphi_w(\pm C_T + \mathrm{l.o.t.})$$

for a unique bijection φ_w *and some order* $<_w$ *on* $\mathbb{KL}_\lambda$.

Separable Permutations

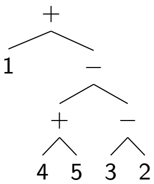
$J =$ 

w_J longest element of $\mathfrak{S}_J$

The *separable permutations* (OEIS A006318) are equivalent to the following:

~~2413~~

~~3142~~



$w = w_{J_1} \cdots w_{J_k}$

for chains
$J_1 \supset \cdots \supset J_k$

$$\sum_{n \geq 0} \mathrm{sep}(n) x^n = \frac{1 - x - \sqrt{1 - 6x + x^2}}{2}$$

The long cycle

Set $\mathfrak{S}_J := \langle s_1, \ldots, s_{n-2} \rangle \cong \mathfrak{S}_{n-1}$. Then $c = w_0 w_J$.

$$\mathbb{KL} = \left(\begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}, \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}, \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array}, \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}, \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array} \right)$$

$$[w_J]_{\mathbb{KL}} = \begin{bmatrix} + & 0 & - & 0 & - \\ 0 & + & 0 & - & 0 \\ 0 & 0 & 0 & 0 & - \\ 0 & 0 & 0 & - & 0 \\ 0 & 0 & - & 0 & 0 \end{bmatrix}$$

In fact, $\langle \mathbb{KL} \rangle \cong \mathcal{S}^{(2,2)}$ and $\langle \mathbb{KL} \rangle = \langle \mathbb{KL} \rangle / \langle \mathbb{KL} \rangle \cong \mathcal{S}^{(3,1)}$ as $\mathfrak{S}_{n-1}$ representations.

The long cycle

Set $\mathfrak{S}_J := \langle s_1, \ldots, s_{n-2} \rangle \cong \mathfrak{S}_{n-1}$. Then $c = w_0 w_J$.

$$\mathbb{KL} = \left({\color{red}\begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} \end{array}}, \; {\color{red}\begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} \end{array}}, \; {\color{blue}\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2} \end{array}}, \; {\color{blue}\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2} \end{array}}, \; {\color{blue}\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} \end{array}} \right)$$

$$[w_J]_{\mathbb{KL}} = \begin{bmatrix} {\color{red}+} & {\color{red}0} & - & 0 & - \\ {\color{red}0} & {\color{red}+} & 0 & - & 0 \\ 0 & 0 & {\color{blue}0} & {\color{blue}0} & {\color{blue}-} \\ 0 & 0 & {\color{blue}0} & {\color{blue}-} & {\color{blue}0} \\ 0 & 0 & {\color{blue}-} & {\color{blue}0} & {\color{blue}0} \end{bmatrix}$$

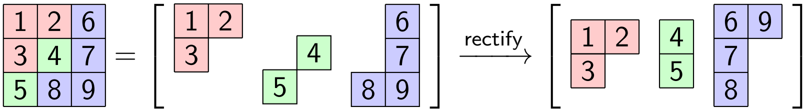
In fact, $\langle {\color{red}\mathbb{KL}} \rangle \cong \mathcal{S}^{(2,2)}$ and $\langle {\color{blue}\mathbb{KL}} \rangle = \langle \mathbb{KL} \rangle / \langle {\color{red}\mathbb{KL}} \rangle \cong \mathcal{S}^{(3,1)}$ as $\mathfrak{S}_{n-1}$ representations.

$$\begin{aligned} w_J \cdot C_T &= \mathrm{ev}_J(\pm C_T + \mathrm{l.o.t.}) \\ \Longrightarrow \; c \cdot C_T &= \mathrm{ev} \circ \mathrm{ev}_J(\pm C_T + \mathrm{l.o.t.}) \end{aligned}$$

and $\mathrm{ev} \circ \mathrm{ev}_J$ turns out to be pr!

Action of $\mathfrak{S}_J$

Suppose $\mathfrak{S}_J \cong \mathfrak{S}_3 \times \mathfrak{S}_2 \times \mathfrak{S}_4$. Write $T = [T_1 \;\; T_2 \;\; T_3]$



Theorem (G., Yacobi)

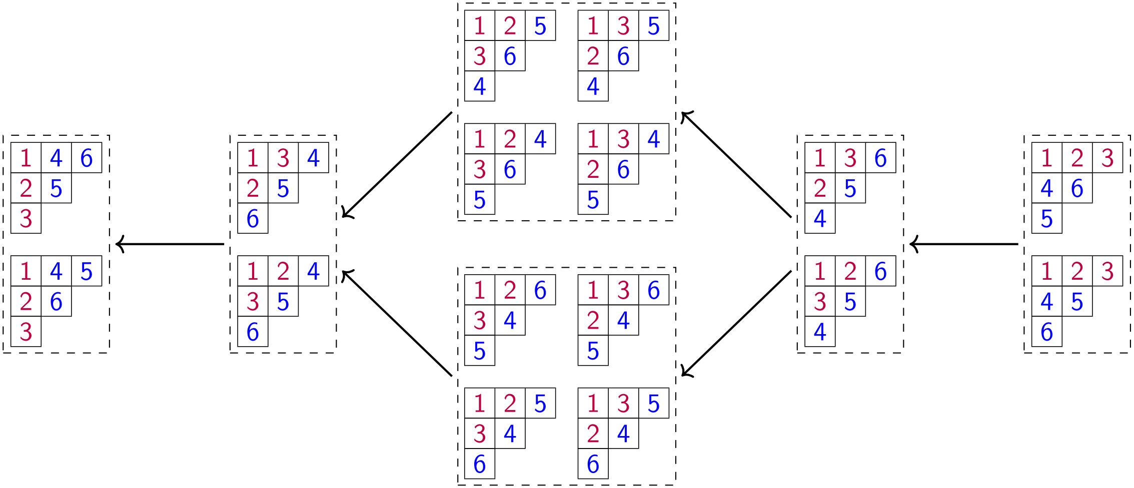
Suppose C_R appears in $\mathfrak{S}_J \cdot C_T$. Then

- $\mathrm{sh}(R_k) \preceq \mathrm{sh}(T_k)$ *for all* k
- *If all equalities,* $\mathrm{sh}(\mathrm{rect}(R_k)) \preceq \mathrm{sh}(\mathrm{rect}(T_k))$ *for all* k
- *If all equalities,* $R_k \approx T_k$ *are dual equivalent for all* k

This order gives a filtration $0 \subset \mathcal{S}_0^\lambda \subset \cdots \subset \mathcal{S}_r^\lambda = \mathcal{S}^\lambda$ *over* $\mathbb{KL}_\lambda$ *which is Jordan-Hölder for* $\mathfrak{S}_J$.

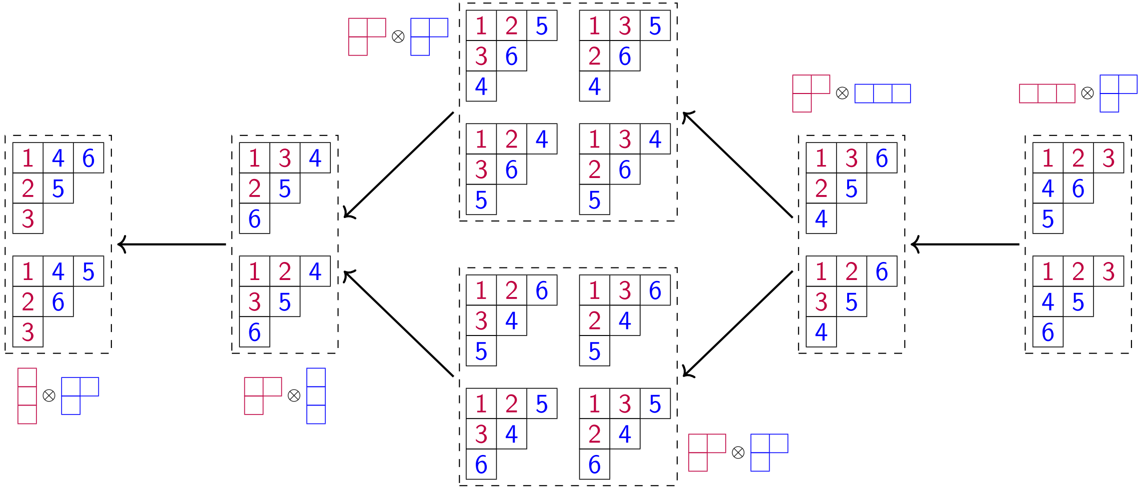
Example

$\lambda = (3, 2, 1)$ and $\mathfrak{S}_J \cong \mathfrak{S}_3 \times \mathfrak{S}_3$



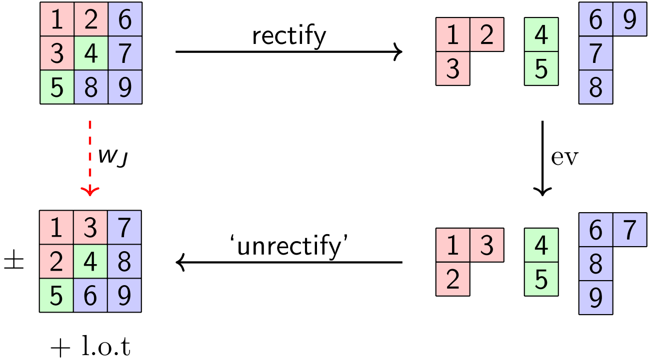
Example

$\lambda = (3, 2, 1)$ and $\mathfrak{S}_J \cong \mathfrak{S}_3 \times \mathfrak{S}_3$



Action of w_J

Irreps of $\mathfrak{S}_J$ are $\mathcal{S}^{\mu_1} \otimes \cdots \otimes \mathcal{S}^{\mu_m}$, and $w_J \cdot (C_{T_1} \otimes \cdots \otimes C_{T_m}) = \pm C_{\mathrm{ev}(T_1)} \otimes \cdots \otimes C_{\mathrm{ev}(T_m)}$



Bonus: Useful Trick

$$s_j \cdot C_T = \begin{cases} -C_T & j \in D(T) \\ C_T + \sum \overline{\mu}(R,T) C_R & j \notin D(T) \end{cases}$$

Fix $Q =$

1	4	7	9
2	5	8	
3	6		

Proposition (G., Yacobi)

Take $w_R \xrightarrow{\text{RSK}} (R, Q)$ and $w_T \xrightarrow{\text{RSK}} (T, Q)$ so $\overline{\mu}(R,T) := \overline{\mu}(w_R, w_T)$

- $w_R = \text{col}(R)$ and $w_T = \text{col}(T)$
- $w_R \leq w_T \implies R \preceq T$

Let $R = [R_1 \ \cdots \ R_m]$ and $T = [T_1 \ \cdots \ T_m]$ be the $\mathfrak{S}_J$ decompositions

- $\text{sh}(R_k) = \text{sh}(T_k)$ for all $k \implies w_R \in \mathfrak{S}_J w_T$
- If $\{i \mid R_i \neq T_i\} = \{k\}$, then $\overline{\mu}(R,T) = \overline{\mu}(\text{col}(R_k), \text{col}(T_k))$