Generalized quantum Yang-Baxter moves and their application to Schubert calculus

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July 21, 2022, FPSAC2022

(Based on arXiv:2105.02546)

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The quantum alcove model: introduced by Lenart-Lubovsky (2015)

- the quantum K-theory of flag manifolds
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- $B^{p,1}$: a column-shape Kirillov-Reshetikhin crystal (a combinatorial model for a certain finite-dimensional representation of a quantum affine algebra)

Fact (Lenart-Naito-Sagaki-Schilling-Shimozono (2017))

In arbitrary untwisted affine type, there exists a crystal isomorphism

$$\underbrace{\mathcal{A}(\Gamma)}_{\text{objects of the quantum alcove model}} \xrightarrow{\sim} B^{p_1,1} \otimes B^{p_2,1} \otimes \cdots \otimes B^{p_k,1} \ (\textit{only "dual Demazure arrows"}),$$

where Γ is a suitable sequence of roots, called a λ-chain.

The combinatorial R-matrix

- $(p_1, p_2, \ldots, p_k) \in \mathbb{Z}_{\geq 0}^k$
- $(p'_1, p'_2, \ldots, p'_k)$: a permutation of $(p_1, p_2, \ldots, p_k)$

Fact

There exists a crystal isomorphism

$$B^{p_1,1} \otimes B^{p_2,1} \otimes \cdots \otimes B^{p_k,1} \xrightarrow{\sim} B^{p'_1,1} \otimes B^{p'_2,1} \otimes \cdots \otimes B^{p'_k,1},$$

called a combinatorial R-matrix (realized as jeu de taquin on Young tableaux in type A).

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- λ: dominant integral weight
- Γ, Γ': two "reduced" (shortest) λ-chains

Theorem (Lenart-Lubovsky (2018))

There exists a crystal isomorphism $\mathcal{A}(\Gamma) \xrightarrow{\sim} \mathcal{A}(\Gamma')$, which is realized combinatorially by a sequence of quantum Yang-Baxter moves.

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Theorem (Lenart-Lubovsky (2018))

There exists a crystal isomorphism $\mathcal{A}(\Gamma) \xrightarrow{\sim} \mathcal{A}(\Gamma')$, which is realized combinatorially by a sequence of quantum Yang-Baxter moves.

$\rightarrow$ $\mathcal{A}(\Gamma)$ does not depend on the choice of Γ

Combinatorial R-matrix vs. QYB moves

$$\begin{tikzcd}
\mathcal{A}(\Gamma) \arrow[r, "\text{quantum Yang–Baxter moves}"] \arrow[d, "\simeq"'] & \mathcal{A}(\Gamma') \arrow[d, "\simeq"] \\
B^{p_1,1} \otimes \cdots \otimes B^{p_k,1} \arrow[r, "\substack{\text{combinatorial } R\text{-matrix} \\ \text{(jeu de taquin)}}"] & B^{p'_1,1} \otimes \cdots \otimes B^{p'_k,1}
\end{tikzcd}$$

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Conclusion

The quantum Yang-Baxter moves provide a realization (in the quantum alcove model) of the combinatorial R-matrix, which works uniformly for all untwisted affine root systems.

The generalization of the QYB move (1/3)

$\mathcal{A}(w, \Gamma)$: objects of the quantum alcove model (admissible subsets) generalized by Lenart-Naito-Sagaki (2020) for

- w (generalized from $w = e$ before): an element of the Weyl group
- Γ: a λ-chain (λ: an arbitrary integral weight)

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Applications (Lenart-Naito-Sagaki (2020))

- The Chevalley multiplication formula in the K-group of semi-infinite flag manifolds
- — in the quantum K-group of flag manifolds
- Character identities of level-zero Demazure modules over quantum affine algebras

The generalization of the QYB move (2/3)

Question

Is there a generalization of the quantum Yang-Baxter moves $\mathcal{A}(w, \Gamma) \to \mathcal{A}(w, \Gamma')$?

$\to \mathcal{A}(w, \Gamma)$ is independent of the choice of Γ

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Problem

In general, $|\mathcal{A}(w,\Gamma)| \neq |\mathcal{A}(w,\Gamma')|$. Hence there does not exist any bijection $\mathcal{A}(w,\Gamma) \to \Gamma(w,\Gamma')$.

$\to$ We need a new approach to generalize QYB moves.

The generalization of the QYB move (3/3)

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Definition (Fisher-Konvalinka (2020))

A sijection ("signed bijection") $S \Rightarrow T$ between signed sets S and T is a triple $(\iota_S, \iota_T, \varphi)$ consisting of

- $\varphi : S_0 \to T_0$: a sign-preserving bijection ($S_0 \subset S$, $T_0 \subset T$)
- ι_S (resp., ι_T): a sign-reversing involution on $S \setminus S_0$ (resp., $T \setminus T_0$)

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Theorem (KLN (2021))

For λ-chains Γ, Γ' such that Γ' is obtained from Γ by a "simple deformation procedure", there exists a sijection $\mathcal{A}(w,\Gamma) \Rightarrow \mathcal{A}(w,\Gamma')$ which preserves the related statistics end, down, wt, *and* height.

Settings

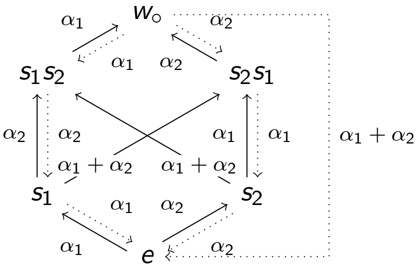
- $\mathfrak{g}$: a simple Lie algebra over $\mathbb{C}$
- Δ: the root system of $\mathfrak{g}$
- Δ^+: the set of positive roots
- P: the weight lattice
- P^+: the set of dominant integral weights
- $Q^\vee$: the coroot lattice
- W: the Weyl group
- $\ell : W \to \mathbb{Z}_{\geq 0}$: the length function

The quantum Bruhat graph

Definition (Brenti-Fomin-Postnikov (1999))

The quantum Bruhat graph $\mathrm{QBG}(W)$ is the labeled directed graph:

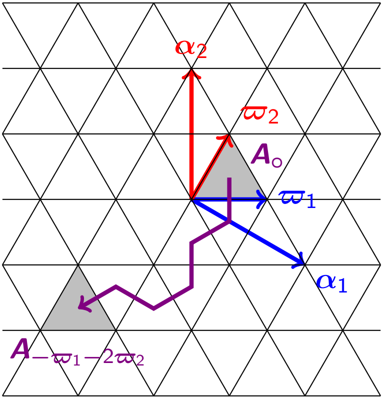
- Vertex set: W
- Label set: Δ^+
- Edge: $x \xrightarrow{\alpha} y$ $(x, y \in W,\ \alpha \in \Delta^+)$ $\Leftrightarrow$ $y = xs_\alpha$, and
  (Bruhat edge) $\ell(y) = \ell(x) + 1$, or
  (Quantum edge) $\ell(y) = \ell(x) - 2\langle \rho, \alpha^\vee \rangle + 1$ $(\rho := (1/2)\sum_{\alpha \in \Delta^+} \alpha)$.



Chains of roots

- $A_\circ := \{\nu \mid 0 < \langle \nu, \alpha^\vee \rangle < 1 \text{ for all } \alpha \in \Delta^+\}$: the fundamental alcove
- $\lambda \in P$

(reduced) λ-chain: a sequence $\Gamma = (\beta_1, \dots, \beta_r)$ of roots associated to a (shortest) path from $A_\circ$ to $A_{-\lambda} := A_\circ - \lambda$



[Type A_2]
$(\alpha_2, \alpha_1+\alpha_2, \alpha_2, \alpha_1+\alpha_2, \alpha_1, \alpha_1+\alpha_2)$
$(\varpi_1 + 2\varpi_2)$-chain

Admissible subsets (1/2)

Admissible subsets: main objects in the quantum alcove model

- $w \in W$
- $\lambda \in P$
- $\Gamma = (\beta_1, \dots, \beta_r)$: a λ-chain

Definition (Lenart-Lubovsky (2015), Lenart-Naito-Sagaki (2020))

A subset $A = \{i_1 < i_2 < \cdots < i_s\} \subset \{1, \dots, r\}$ is said to be *w-admissible* if

$$w = w_0 \xrightarrow{|\beta_{i_1}|} w_1 \xrightarrow{|\beta_{i_2}|} \cdots \xrightarrow{|\beta_{i_s}|} w_s \ (=: \mathrm{end}(A))$$

is a directed path in $\mathrm{QBG}(W)$. Set

$$\mathrm{down}(A) := \sum_{\substack{1 \le k \le s \\ w_{k-1} \to w_k \text{ is a quantum edge}}} |\beta_k|^\vee,$$

$$n(A) := |\{j \in A \mid \beta_j \in -\Delta^+\}|.$$

Admissible subsets (2/2)

Definition (Lenart-Lubovsky (2015), Lenart-Naito-Sagaki (2020))

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Remark

We can also define statistics $\mathrm{wt}(A) \in P$ and $\mathrm{height}(A) \in \mathbb{Z}$.

$\mathcal{A}(w, \Gamma) := \{A \subset \{1, \ldots, r\} \mid A \text{ is } w\text{-admissible}\}$ with sign $A \mapsto (-1)^{n(A)}$

Yang-Baxter transformation

- $\lambda \in P$
- $\Gamma = (\beta_1, \ldots, \beta_r)$: a λ-chain

Definition (e.g., Lenart-Postnikov (2007))

A Yang-Baxter transformation (YB): a procedure to obtain a new λ-chain

(1) Take a segment $(\beta_{t+1}, \ldots, \beta_{t+q})$ of Γ s.t.

- $\langle \beta_{t+1}, \beta_{t+q}^{\vee} \rangle \leq 0$,
- $(\beta_{t+1}, \ldots, \beta_{t+q}) = (\alpha, s_\alpha(\beta), s_\alpha s_\beta(\alpha), \ldots, s_\beta(\alpha), \beta)$ for some α, β.

(2) Reverse $(\beta_{t+1}, \ldots, \beta_{t+q})$ in Γ:

$$\Gamma' := (\beta_1, \ldots, \beta_t, \beta_{t+q}, \ldots, \beta_{t+1}, \beta_{t+q+1}, \ldots, \beta_r).$$

$\rightarrow \Gamma'$: λ-chain

Deletion

- $\lambda \in P$
- $\Gamma = (\beta_1, \ldots, \beta_r)$: a λ-chain

Definition (e.g., Lenart-Postnikov (2007))

A deletion (D): a procedure to obtain a new λ-chain

(1) Take a segment $(\beta_{t+1}, \beta_{t+2})$ in Γ s.t. $\beta_{t+2} = -\beta_{t+1}$.

(2) Delete the segment $(\beta_{t+1}, \beta_{t+2})$ in Γ:

$$\Gamma' := (\beta_1, \ldots, \beta_t, \beta_{t+3}, \ldots, \beta_r).$$

$\to \Gamma'$: λ-chain

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Fact (e.g., Lenart-Naito-Sagaki, Lenart-Postnikov)

From any λ-chain, we can obtain any reduced λ-chain by repeated application of (YB) and (D).

Quantum Yang-Baxter move

Theorem (Lenart-Lubovsky (2018))

Let $\lambda \in P^+$, and take reduced λ-chains Γ_1, Γ_2 s.t. $\Gamma_1 \xrightarrow{\text{(YB)}} \Gamma_2$. There exists a bijection $Y : \mathcal{A}(e, \Gamma_1) \to \mathcal{A}(e, \Gamma_2)$ s.t.

- $\mathrm{end}(Y(A)) = \mathrm{end}(A)$,
- $\mathrm{down}(Y(A)) = \mathrm{down}(A)$,
- $\mathrm{wt}(Y(A)) = \mathrm{wt}(A)$, *and*
- $\mathrm{height}(Y(A)) = \mathrm{height}(A)$.

This Y is called a quantum Yang-Baxter (QYB) move.

- A QYB move is a structure-preserving bijection.
  $\to$ $\mathcal{A}(e, \Gamma)$ does not depend on the choice of Γ.

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- A QYB move is a structure-preserving bijection.
  $\to$ $\mathcal{A}(e, \Gamma)$ does not depend on the choice of Γ.
- It is, in fact, an affine crystal isomorphism.
- It is a root system generalization of jeu de taquin in type A.

Generalization of QYB moves (1/2)

Theorem (KLN (2021))

Let $\lambda \in P$ and $w \in W$. Take λ-chains Γ_1, Γ_2 s.t.

- $\Gamma_1 \xrightarrow{\text{(YB)}} \Gamma_2$ *or*
- $\Gamma_1 \xrightarrow{\text{(D)}} \Gamma_2$ *in which a segment $(\beta, -\beta)$ in Γ_1, with β not a simple root, is deleted.*

There exist explicit subsets $\mathcal{A}_0(w, \Gamma_1) \subset \mathcal{A}(w, \Gamma_1)$ and $\mathcal{A}_0(w, \Gamma_2) \subset \mathcal{A}(w, \Gamma_2)$ s.t.

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(1) *there exists a bijection $Y : \mathcal{A}_0(w, \Gamma_1) \to \mathcal{A}_0(w, \Gamma_2)$ which preserves sign $(-1)^{n(A)}$ and which preserves* $\mathrm{end}(\cdot)$, $\mathrm{down}(\cdot)$, $\mathrm{wt}(\cdot)$, *and* $\mathrm{height}(\cdot)$,

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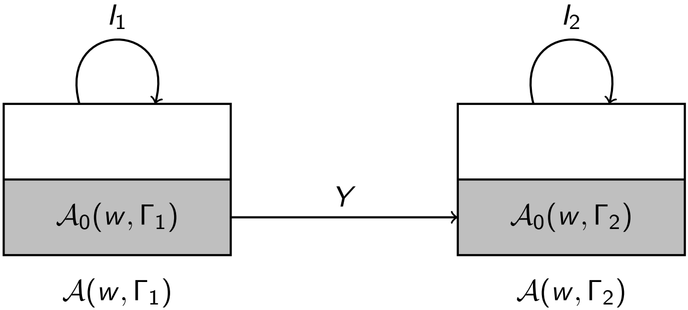
(2) *there exist involutions I_k on $\mathcal{A}(w, \Gamma_k) \setminus \mathcal{A}_0(w, \Gamma_k)$ $(k = 1, 2)$ which reverse sign $(-1)^{n(A)}$ and which preserve* $\mathrm{end}(\cdot)$, $\mathrm{down}(\cdot)$, $\mathrm{wt}(\cdot)$, *and* $\mathrm{height}(\cdot)$.

Generalization of QYB moves (2/2)

Theorem (KLN (2021))

(1) *a bijection* $Y : \mathcal{A}_0(w, \Gamma_1) \to \mathcal{A}_0(w, \Gamma_2)$ *which preserves sign* $(-1)^{n(A)}$ *and which preserves* $\mathrm{end}(\cdot)$, $\mathrm{down}(\cdot)$, $\mathrm{wt}(\cdot)$, *and* $\mathrm{height}(\cdot)$,

(2) *involutions* I_k *on* $\mathcal{A}(w, \Gamma_k) \setminus \mathcal{A}_0(w, \Gamma_k)$ $(k = 1, 2)$ *which reverse sign* $(-1)^{n(A)}$ *and which preserve* $\mathrm{end}(\cdot)$, $\mathrm{down}(\cdot)$, $\mathrm{wt}(\cdot)$, *and* $\mathrm{height}(\cdot)$.



$\to$ We obtain a sijection (I_1, I_2, Y): a generalized QYB move.

Generating functions

- $W_{\mathrm{af}} = W \ltimes Q^\vee = \{wt_\xi \mid w \in W,\ \xi \in Q^\vee\}$: the affine Weyl group
- $x = wt_\xi \in W_{\mathrm{af}}$
- Γ: λ-chain ($\lambda \in P$)

Definition

A generating function $\mathsf{G}_\Gamma(x) \in (\mathbb{Z}[q, q^{-1}][P])[W_{\mathrm{af}}] \Leftrightarrow$

$$\mathsf{G}_\Gamma(x) := \sum_{A \in \mathcal{A}(w,\Gamma)} (-1)^{n(A)} q^{-\,\mathrm{height}(A) - \langle \lambda, \xi \rangle} e^{\mathrm{wt}(A)}\, \mathrm{end}(A) t_{\xi + \mathrm{down}(A)}.$$

Preservation of generating functions

Theorem (KLN (2021))

Let $\lambda \in P$, $x \in W_{\mathrm{af}}$. *Take* λ*-chains* Γ_1, Γ_2 *s.t.*

- $\Gamma_1 \xrightarrow{\text{(YB)}} \Gamma_2$ *or*
- $\Gamma_1 \xrightarrow{\text{(D)}} \Gamma_2$ *in which a segment* $(\beta, -\beta)$ *in* Γ_1, *with* β *not a simple root, is deleted.*

Then $\mathrm{G}_{\Gamma_1}(x) = \mathrm{G}_{\Gamma_2}(x)$.

Conclusion

- We obtain a generalization of QYB move $\mathcal{A}(w, \Gamma) \Rightarrow \mathcal{A}(w, \Gamma')$ as a sijection.

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Conclusion

- We obtain a generalization of QYB move $\mathcal{A}(w, \Gamma) \Rightarrow \mathcal{A}(w, \Gamma')$ as a sijection.
- Generating functions are preserved under deformation procedures (YB) and (D) (deletes $(\beta, -\beta)$ with β not a simple root).
- As an application, we give a combinatorial proof of the Chevalley multiplication formula in the equivariant K-group of semi-infinite flag manifolds, first proved by Lenart-Naito-Sagaki.