Rotor-Routing Induces the Only Consistent Sandpile Torsor Structure on Plane Graphs

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Animations: https://youtu.be/2StIAfnONMs

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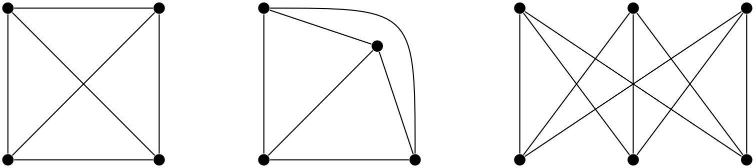
Ribbon Graphs

- A *ribbon graph* G (also called a *combinatorial map*) is a graph along with a choice of cyclic order of edges around each vertex (clockwise for this talk). Ribbon graphs are used to represent graph embeddings.



- A *plane graph* is a ribbon graph with no edge crossings (a planar embedding). Of the ribbon graphs above, only the middle is a plane graph.

Single-Chip Rotor-Routing Algorithm (With Sink)

Input: a ribbon graph G, a spanning tree T, a *sink* vertex s, and a *chip* c on any non-sink vertex.

1. Orient the edges of T toward s. Every vertex $v \in V(G) \setminus s$ has a single outgoing edge called the *rotor at* v.
2. Rotate the rotor at c and then move c along it.
3. Repeat step 2 until c reaches the sink, then remove c.
4. Forget the orientation of the rotors and let T' be their edges.

Output: T'

See Clip 1

Facts about Rotor-Routing

- Rotor-routing was introduced under the name "Eulerian Walkers Model" by Priezzhev, D. Dhar, A. Dhar, and Krishnamurthy in 1996. The following lemmas are implied by their results:

Lemma

The output T' is always a spanning tree.

Lemma

If the single-chip rotor-routing algorithm is performed multiple times, the order of chips does not affect the final tree. See Clip 2

- The 2008 paper "Chip Firing and Rotor-Routing on Directed Graphs" by Holroyd, Levine, Mészáros, Peres, Propp, and Wilson is an excellent survey of rotor-routing and sandpile ideas.

Multiple-Chip Rotor-Routing Algorithm (With Sink)

Input: a ribbon graph G, a spanning tree T, a *sink* vertex s, and a collection $\mathcal{C}$ of *chips* on non-sink vertices.

1. Orient the edges of T toward s. Every vertex $v \in V(G) \setminus s$ has a single outgoing edge called the *rotor at* v.
2. Choose any $c \in \mathcal{C}$. Rotate the rotor at c and then move c along it. If c reaches the sink, remove it from $\mathcal{C}$.
3. Repeat step 2 until $\mathcal{C} = \emptyset$.
4. Forget the orientation of the rotors and let T' be their edges.

Output: T'

The Sandpile Group of a Graph

- Let G be a finite connected graph with vertices $V(G)$.
- A *degree 0 divisor* is an assignment of an integral number of "chips" to each vertex (allowing negative chips) so that there are 0 total chips.
- The degree 0 divisors under pointwise addition form a group called $\text{Div}^0(G)$.
- The *Laplacian matrix* Δ is $D - A$, where D is the *degree matrix* of G and A is the *adjacency matrix* of G.

Definition

The *sandpile group* $\mathcal{S}(G)$ is $\text{Div}^0(G)/\text{im}_{\mathbb{Z}}(\Delta)$. See Clip 3

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Theorem (sandpile matrix-tree theorem for graphs, Biggs 1999)

The size of $\mathcal{S}(G)$ is the number of spanning trees of G.

Sandpile Rotor-Routing Algorithm (With Sink)

Input: a ribbon graph G, a spanning tree T, a *sink* vertex s, and an element of the sandpile group $S \in \mathcal{S}(G)$.

1. Orient the edges of T toward s. Every vertex $v \in V(G) \setminus s$ has a single outgoing edge called the *rotor at* v. Let D be any representative of S such that $D(v) \geq 0$ for $v \neq s$. Let $\mathcal{C}$ be a set of $D(v)$ chips at each $v \neq s$.
2. Choose any $c \in \mathcal{C}$. Rotate the rotor at c and then move c along it. If c reaches the sink, remove it from $\mathcal{C}$.
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The algorithm in the previous slide is well defined. See Clip 4

Rotor-Routing and the Sandpile Group

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- In other words, rotor routing defines a *free transitive action* of $\mathcal{S}(G)$ on the spanning trees of G.

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When is the rotor-routing action preserved after changing the sink vertex?

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Theorem (Chan-Church-Grochow, 2013)

The rotor-routing action is preserved regardless of sink vertex if and only if G is a plane graph. See Clip 5

Sink-Free Rotor-Routing Algorithm

Input: a plane graph G, a spanning tree T, and an element of the sandpile group $S \in \mathcal{S}(G)$.

1. Choose any $s \in V(G)$. Orient the edges of T toward s. Every vertex $v \in V(G) \setminus s$ has a single outgoing edge called the *rotor at* v. Let D be any representative of S such that $D(v) \geq 0$ for $v \neq s$. Let $\mathcal{C}$ be a set of $D(v)$ chips at each $v \neq s$.
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Output: T'

We write that $r_G([D], T) = T'$.

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- We saw that rotor-routing induces a sandpile torsor algorithm, but are there other natural algorithms?

"Other" Sandpile Torsor Algorithms

- in 2012, Baker and Wang used the *Bernardi process* to define another sandpile torsor algorithm.

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Conjecture (Klivans, 2018)

For plane graphs, there is only one sandpile torsor structure.

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- The first challenge to tackling this conjecture is defining *sandpile torsor structure*.

Sandpile Torsor Structure

Proposition (Ganguly-M., 2022+)

Rotor-routing produces 4 closely related sandpile torsor algorithms:

- clockwise rotor-routing,
- counterclockwise rotor-routing,
- inverse clockwise rotor-routing, and
- inverse counterclockwise rotor-routing.

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Two sandpile torsor algorithms have the same *structure* if they differ by inverting the action and/or the ribbon structure.

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- To prevent simple but contrived counterexamples to Klivans' conjecture, we want our algorithm to act *consistently* across different plane graphs.

A Consistency Condition

Theorem (Ganguly-M., 2022+)

*Let G be a plane graph with a spanning tree T, and **incident** vertices c and s. Let $T' = r_G([c-s], T)$.*

1. *For any $e \in E(G)$ (not incident to both c and s), if $e \in T \cap T'$, then*

$$r_G([c-s], T) \setminus e = r_{G/e}([c-s], T \setminus e).$$

See Clip 6

2. *For any $e \in E(G)$, if $e \notin T \cup T'$, then*

$$r_G([c-s], T) = r_{G \setminus e}([c-s], T).$$

See Clip 7

3. *For any $e \in E(G)$, if there is a cut vertex x such that all paths from e to c or s pass through x, then*

$$e \in T \iff e \in T'.$$

See Clip 8

Consistency in General

Definition

A sandpile torsor algorithm is *consistent* if it satisfies the 3 properties on the previous slide.

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Theorem (Ganguly-M.,2022+)

Every consistent sandpile torsor algorithm has the same structure as rotor-routing (i.e. it is unique up to two $\mathbb{Z}_2$ *actions).*

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A sandpile torsor algorithm is *consistent* if it satisfies the 3 properties on the previous slide.

Theorem (Ganguly-M.,2022+)

Every consistent sandpile torsor algorithm has the same structure as rotor-routing (i.e. it is unique up to two $\mathbb{Z}_2$ actions).

- To prove this, we first prove that it suffices to consider a subset of situations where rotor-routing takes just one step.
- We then use induction to reduce to 4 special cases.
- Resolving these cases requires a variety of methods and a great deal of work.

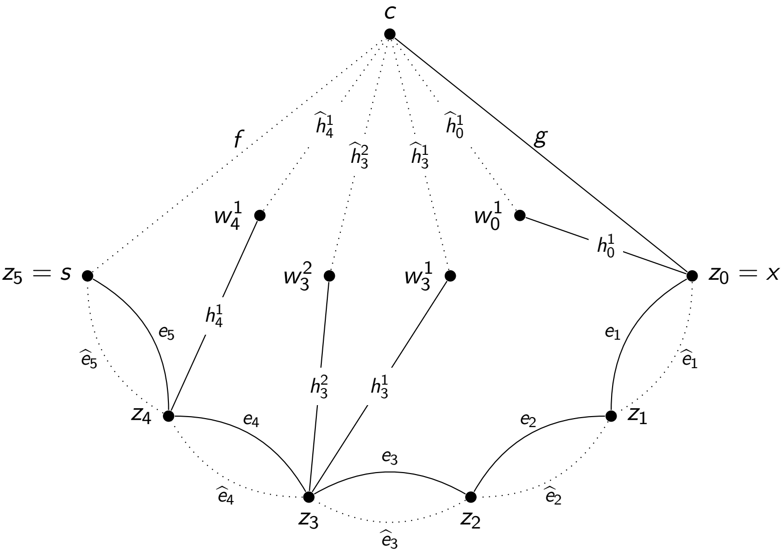
Regular Matroids

- In 2017, Backman, Baker, and Yuen showed how to generalize the Bernardi action to *regular matroids*.
- Instead of a ribbon structure, they require *acyclic circuit and cocircuit signatures*.
- The definitions of consistency and sandpile torsor structure generalize naturally to regular matroids.

Conjecture

- The Backman-Baker-Yuen algorithm is consistent.
- All consistent sandpile torsor algorithms on regular matroids have the same structure.

Thanks for Listening!



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