Rank Polynomials of Fence Posets are Unimodal

joint with Ezgi Kantarci Oğuz

MOHAN RAVICHANDRAN

Boğaziçi University
İstanbul, Turkey

July 19, 2022

IMBM

http://www.imbm.org.tr

For some background on the events at Bogazici, see shorturl.at/bdkrx and shorturl.at/ikW01

For a statement by the AMS, see shorturl.at/kxL58

The Istanbul Center for Mathematical Sciences (ICMS/IMBM) is a research center in Bogazici University that has since 2006 hosted hundreds of research talks as well as summer schools, conferences and workshops.

The center was shut down by the Bogazici University rectorate in May 2022.

The official reason was that the alumni office had run out of office space, but the real reason was to penalize the mathematics and physics departments of Bogazici for speaking up against the erosion of academic freedom and civil rights under the current university administration, appointed in January 2021.

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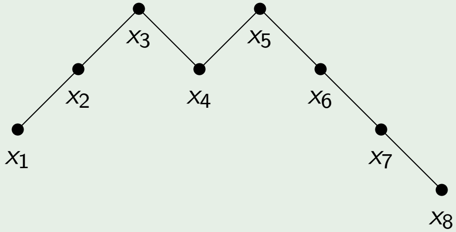
What are fences?

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s)$ be a composition of n. The fence poset of α, denoted $F(\alpha)$ is the poset on $x_1, x_2, \ldots, x_{n+1}$ with the order relations:

$$x_1 \preceq x_2 \preceq \cdots \preceq x_{\alpha_1+1} \succeq x_{\alpha_1+2} \succeq \cdots \succeq x_{\alpha_1+\alpha_2+1} \preceq x_{\alpha_1+\alpha_2+2} \preceq \cdots$$

Example ($\alpha = (2,1,1,3)$)



For a composition of n, we get a poset of $n+1$ nodes.

An ideal of a fence is a down-closed subset: $x \in I$, $y \preceq x \Rightarrow y \in I$.

$$\#I = rank(I)$$

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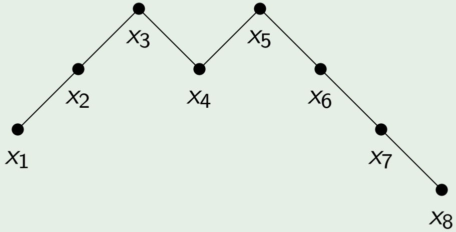

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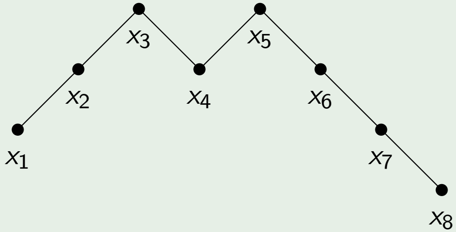

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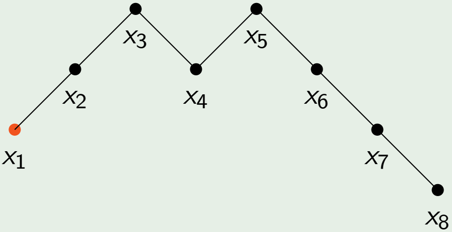

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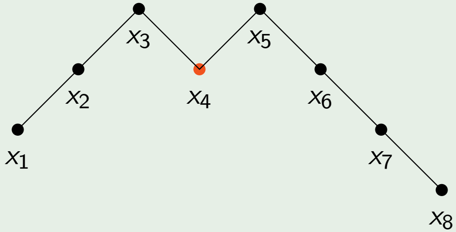

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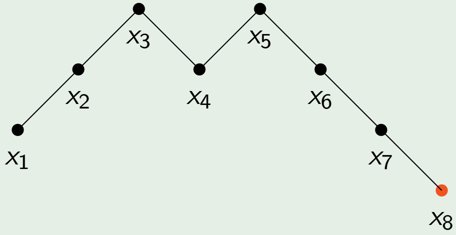

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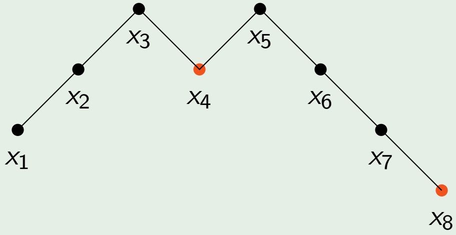

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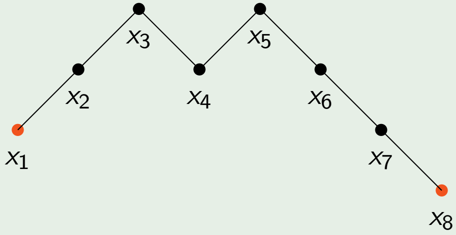

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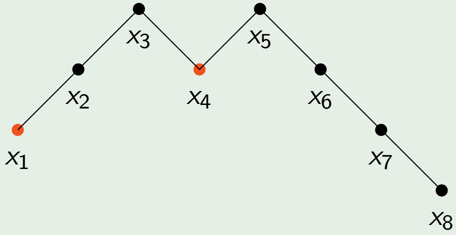

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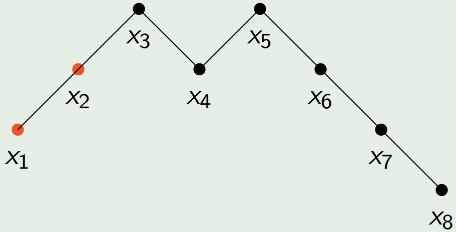

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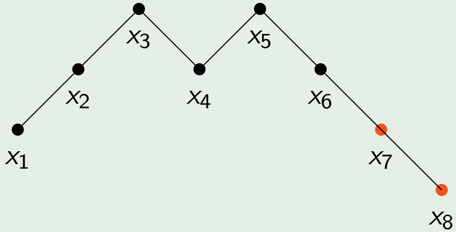

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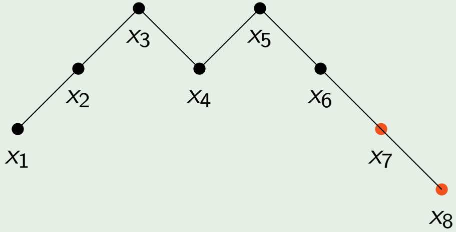

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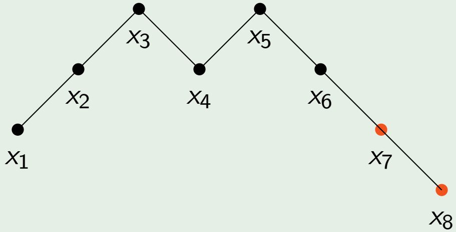

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$1+3q+5q^2+6q^3+6q^4+5q^5+3q^6+2q^7+q^8 \leftarrow$ Rank polynomial.

A q-deformation for rational numbers

Recently, a q-deformation rational numbers was introduced by Morier-Genoud and Ovsienko[1]. Their definition has a *convergence* property, which allows us to extend them to real numbers.

[1] Morier-Genoud and Ovsienko, "q-deformed rationals and q-continued fractions".

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For a given rational number r/s, we first write it as a continued fraction.

$$\frac{r}{s} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_{2m}}}}} = c_1 - \cfrac{1}{c_2 - \cfrac{1}{c_3 - \cfrac{1}{\ddots - \cfrac{1}{c_k}}}}$$

$a_i \in \mathbb{Z},\ a_i \geq 1$ for $i \geq 2$ $\qquad$ $c_i \in \mathbb{Z},\ c_i \geq 2$ for $i \geq 2$

[1] Morier-Genoud and Ovsienko, "*q*-deformed rationals and *q*-continued fractions".

A q-deformation for rational numbers

Then we replace the expansion terms with q-integers (q^{-1}-integers for a_{2k}), and the 1's with powers of q.

$$\left[\frac{r}{s}\right]_q := [a_1]_q + \cfrac{q^{a_1}}{[a_2]_{q^{-1}} + \cfrac{q^{-a_2}}{\ddots + \cfrac{q^{a_{2m-1}}}{[a_{2m}]_{q^{-1}}}}} = [c_1]_q - \cfrac{q^{c_1-1}}{[c_2]_q - \cfrac{q^{c_2-1}}{\ddots - \cfrac{q^{c_{k-1}-1}}{[c_k]_q}}}$$

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Also, when $\frac{r}{s} \geq 0$ the coefficients are non-negative.

Example

$$\frac{32}{9} = 3 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4}}} = 4 - \cfrac{1}{3 - \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{2}}}}$$

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$$\left[\frac{32}{9}\right]_q = \frac{1+3q+5q^2+6q^3+6q^4+5q^5+3q^6+2q^7+q^8}{1+2q+2q^2+2q^3+q^4+q^5}.$$

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In general, if r/s corresponds to $[a_1, a_2, \ldots, a_{2m}]$, we have

$$\left[\frac{r}{s}\right]_q = \frac{\text{Rank polynomial for } (a_1 - 1, a_2, a_3, \ldots, a_{2m} - 1)}{\text{Rank polynomial for } (0, a_2 - 1, a_3, \ldots, a_{2m} - 1)}$$

A closer look at rank sequences for fences

$$\begin{aligned}
(2,1,1,3) &\to (1,3,5,6,6,5,3,2,1)\\
(3,1,1,2) &\to (1,2,3,5,6,6,5,3,1)\\
(1,2,1,3) &\to (1,3,5,6,6,5,4,2,1)\\
(1,1,2,3) &\to (1,3,5,7,7,5,4,2,1)\\
(2,2,3) &\to (1,2,4,5,6,6,4,2,1)\\
(2,3,2) &\to (1,2,4,6,7,6,4,2,1)\\
(2,1,4) &\to (1,2,3,3,4,4,3,2,1)\\
(2,1,2,1,1) &\to (1,3,6,7,8,7,5,3,1)
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(1,2,1,3) &\quad\to\quad (1,3,5,6,6,5,4,2,1)\\
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(2,1,2,1,1) &\quad\to\quad (1,3,6,7,8,7,5,3,1)
\end{aligned}
$$

Conjecture (Morier-Genoud, Ovsienko, 2020)

The rank polynomials of fence posets are unimodal.

What more can we say?

Consider $(2,1,1,3) \to (1,3,5,6,6,5,3,2,1)$.

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Consider $(2, 1, 1, 3) \to (1, 3, 5, 6, 6, 5, 3, 2, 1)$.

We have $1 \leq 1 \leq 2 \leq 3 \leq 3 \leq 5 \leq 5 \leq 6 \leq 6$.

We call such a sequence bottom-interlacing:

$$a_n \leq a_0 \leq a_{n-1} \leq a_1 \leq \ldots \leq a_{\lfloor n/2 \rfloor}. \tag{BI}$$

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We call similarly have top-interlacing sequences:

$$a_0 \leq a_n \leq a_1 \leq a_{n-1} \leq \ldots \leq a_{\lceil n/2 \rceil}. \tag{TI}$$

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For example, the rank sequence $(1, 2, 4, 5, 6, 6, 4, 2, 1)$ of $(2, 2, 3)$ is top interlacing:

$$1 \leq 1 \leq 2 \leq 2 \leq 4 \leq 4 \leq 5 \leq 6 \leq 6.$$

What more can we say?

$$
\begin{aligned}
(2,1,1,3) &\quad\to\quad (1,3,5,6,6,5,3,2,1) \to \text{BI}\\
(3,1,1,2) &\quad\to\quad (1,3,5,6,6,5,3,2,1) \to \text{BI}\\
(1,2,1,3) &\quad\to\quad (1,3,5,6,6,5,4,2,1) \to \text{BI}\\
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(2,3,2) &\quad\to\quad (1,2,4,6,7,6,4,2,1) \to \text{BI}, \text{TI} \ \textit{(symmetric)}\\
(2,1,4) &\quad\to\quad (1,2,3,3,4,4,3,2,1) \to \text{TI}\\
(2,1,2,1,1) &\quad\to\quad (1,3,6,7,8,7,5,3,1) \to \text{BI}
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What more can we say?

$$(2,1,1,3) \rightarrow (1,3,5,6,6,5,3,2,1) \rightarrow \text{BI}$$
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$$(2,3,2) \rightarrow (1,2,4,6,7,6,4,2,1) \rightarrow \text{BI, TI } \textit{(symmetric)}$$
$$(2,1,4) \rightarrow (1,2,3,3,4,4,3,2,1) \rightarrow \text{TI}$$
$$(2,1,2,1,1) \rightarrow (1,3,6,7,8,7,5,3,1) \rightarrow \text{BI}$$

Conjecture (McConville, Sagan, Smyth, 2021[2])

Suppose $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s)$.

(a) If $s = 1$ then $r(\alpha) = (1, 1, \ldots, 1)$ is symmetric.

(b) If s is even, then $r(\alpha)$ is bottom interlacing.

(c) If $s \geq 3$ is odd we have:

  (i) If $\alpha_1 > \alpha_s$ then $r(\alpha)$ is bottom interlacing.

  (ii) If $\alpha_1 < \alpha_s$ then $r(\alpha)$ is top interlacing.

  (iii) If $\alpha_1 = \alpha_s$ then $r(\alpha)$ is symmetric, bottom interlacing, or top interlacing depending on whether $r(\alpha_2, \alpha_3, \ldots, \alpha_{s-1})$ is symmetric, top interlacing, or bottom interlacing, respectively.

[2] McConville, B. E. Sagan, and Smyth, *On a rank-unimodality conjecture of Morier-Genoud and Ovsienko*.

Comments

– While these sequences are indeed unimodal (our main result), they do not satisfy stronger properties like log concavity or real rootedness.

– Indeed, these sequences are often 'barely' unimodal. For instance when $\alpha = (6, 1, 1, 1)$, we have that

$$r[(6, 1, 1, 1)] = (1, 3, 4, 5, 5, 5, 5, 4, 3, 2, 1).$$

– While the peak of the sequence for any composition of n for n odd is always at $\lfloor n/2 \rfloor$ or $\lceil n/2 \rceil$, there seems to be no clear way of deciding between these possibilities.

– There are of course several identities such as

$$R[(\alpha_1, \ldots, \alpha_s) = R[(\alpha_1 - 1, \ldots, \alpha_s) + q^{\alpha_1} R_{\downarrow}[(\alpha_2, \ldots, \alpha_s)],$$

where R is the rank polynomial and $R_{\downarrow}$ is the rank polynomial of fence that starts with a down step. Alas, working with these only leads to frustration and grief.

More comments

- Start with the path graph on n nodes and orient the edges one way or the other, to get a *Type A Dynkin quiver*.

An indecomposable representation of this quiver is essentially a lower ideal of the associated fence poset (with the dimension being the size of the lower ideal). Consequently, we have unimodality for the number of indecomposable reps ranked by dimension.

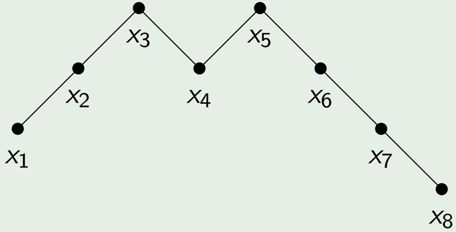
$-$ The poset $F(\alpha)$ has several different descriptions.

$*$ In terms of perfect matchings (alternately in terms of lattice paths) on snake graphs. (Propp).
$*$ In terms of perfect matchings on angles, structures related to cluster algebras (Yurisuka).
$*$ T paths, also structures related to cluster algebras, (Schiffler-Thomas).
$*$ S paths, coming from polygon cluster algebras (Clausen).

Idea

What if we close up the fence?

Example ($\alpha = (2, 1, 1, 3)$)



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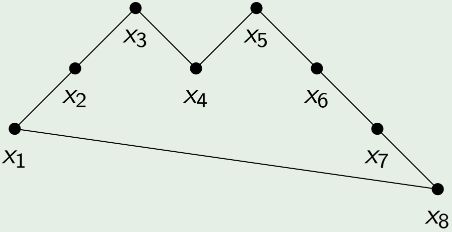

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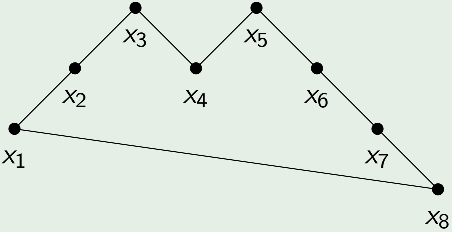

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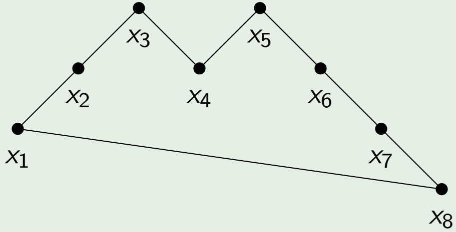

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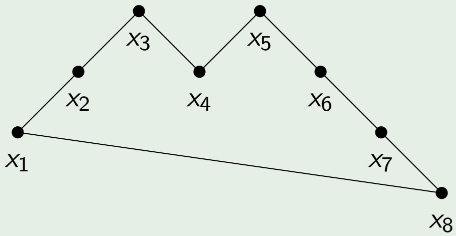

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It is symmetric. Is this always so?

Answer: Yes, but it is not trivial to prove.

Theorem (Kantarcı Oğuz, Ravichandran, 2021[3])

Rank polynomials of circular fence posets are symmetric.

[3] Kantarcı Oğuz and Ravichandran, *Rank Polynomials of Fence Posets are Unimodal*.

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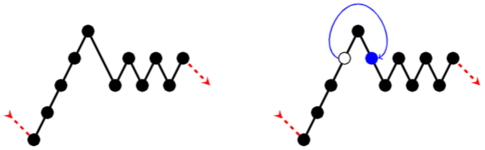
We have one case that is trivially symmetric: $(k, 1, 1, \ldots, 1)$.



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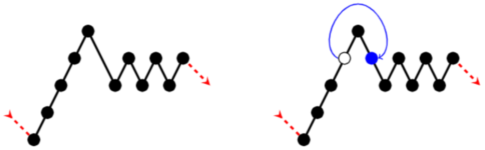

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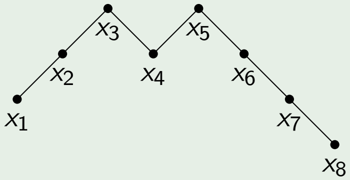

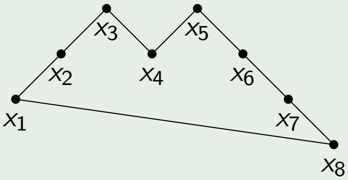

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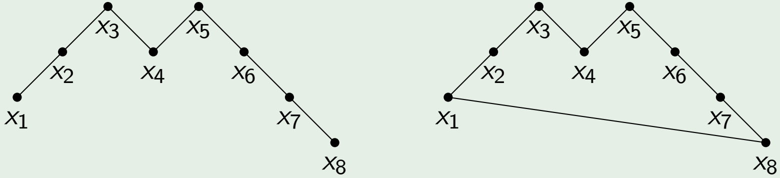

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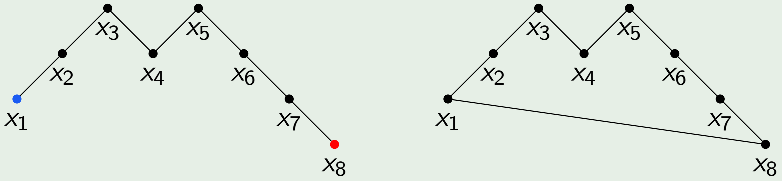

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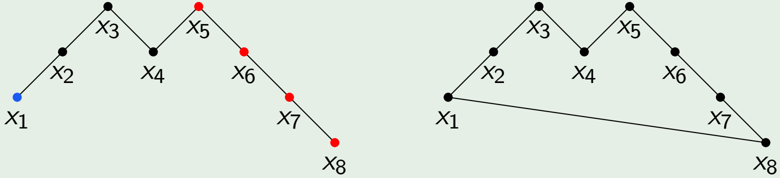

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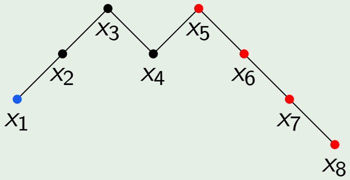

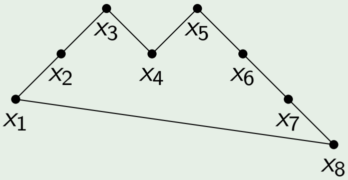

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circular rank polynomial

$q \times$ rank polynomial for $(1, 1)$

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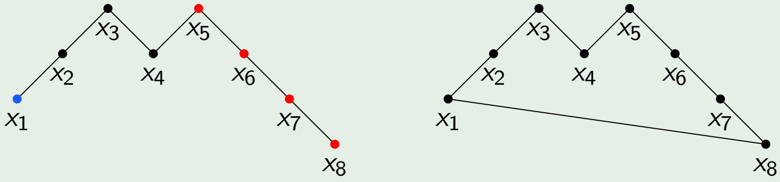

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circular rank polynomial *(symmetric)*

$q \times$ rank polynomial for $(1, 1)$ *(smaller, shifted center)*

What does this tell us about the rank polynomial?

symmetric piece	$(1,2,3,5,5,5,3,2,1)$	$b_0 = b_n,\ b_1 = b_{n-1}, \ldots$
$+$	$+$	
smaller piece, shifted center	$(0,1,2,1,1,0,0,0,0)$	$c_0 \geq c_n,\ c_1 \geq c_{n-1}, \ldots$
$=$	$=$	
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This gives us half of the equations for being bottom interlacing:

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We need a way to shift the pairings to (a_0, a_{n-1}), (a_1, a_{n+1}), $\dots$ to get the rest of the inequalities.

Let us associate another circular fence to our fence.

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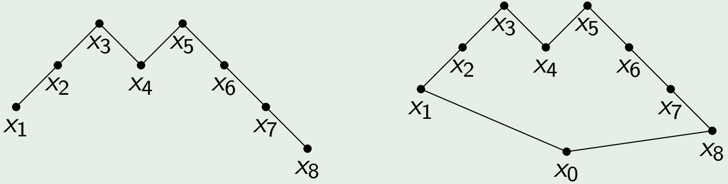
Example (Connecting x_8 and x_1 by a minimal node x_0)



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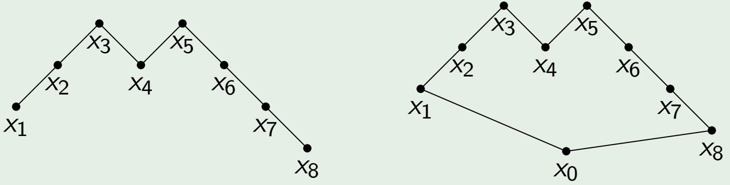

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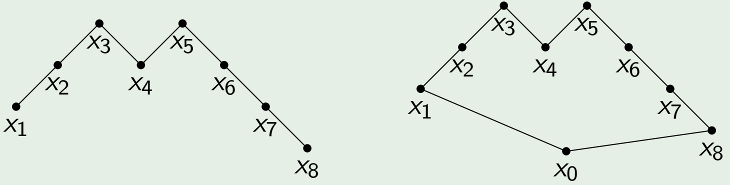

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$q \times$ rank polynomial for $(2, 1, 1, 3)$	circular rank polynomial	rank polynomial for (0)

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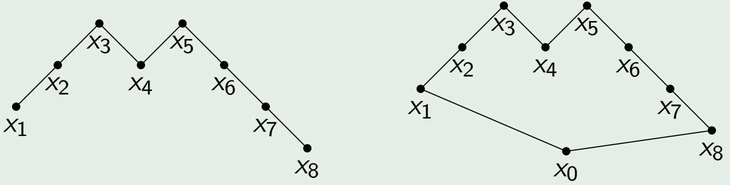

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circular rank polynomial *(symmetric, shifted center)*

rank polynomial for (0) *(smaller, shifted center)*

On the rank polynomial side

symmetric piece larger	$(1,2,3,5,6,6,5,3,2,1)$	$b_0 = b_{n+1},\ b_1 = b_n, \ldots$
$-$	$-$	
smaller piece, shifted center	$(1,1,0,0,0,0,0,0,0)$	$c_0 \geq c_n,\ c_1 \geq c_{n-1}, \ldots$
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This gives us the other half of the bottom-interlacing equations:

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$$+$$

$$a_0 \leq a_{n-1}, \quad a_1 \leq a_{n-2}, \quad a_2 \leq a_{n-3}, \dots$$

$$=$$

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Theorem (Kantarcı Oğuz, Ravichandran, 2021)

Rank polynomials of fence posets are unimodal.

In particular, for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s)$ *we have:*

(a) *If* $s = 1$ *then* $r(\alpha) = (1, 1, \ldots, 1)$ *is symmetric.*

(b) *If* s *is even, then* $r(\alpha)$ *is bottom interlacing.*

(c) *If* $s \geq 3$ *is odd we have:*

- (i) *If* $\alpha_1 > \alpha_s$ *then* $r(\alpha)$ *is bottom interlacing.*
- (ii) *If* $\alpha_1 < \alpha_s$ *then* $r(\alpha)$ *is top interlacing.*
- (iii) *If* $\alpha_1 = \alpha_s$ *then* $r(\alpha)$ *is symmetric, bottom interlacing, or top interlacing depending on whether* $r(\alpha_2, \alpha_3, \ldots, \alpha_{s-1})$ *is symmetric, top interlacing, or bottom interlacing, respectively.*

A remark on our proof

$\boxed{\text{Unimodality for Fences with at most } n \text{ parts.}}$

$$\Downarrow$$

$\boxed{\text{Symmetry for Circular Fences with at most } n+1 \text{ parts.}}$

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Answer: Not always.

For the circular poset $(1, a, 1, a)$ we get a small dip in the middle:

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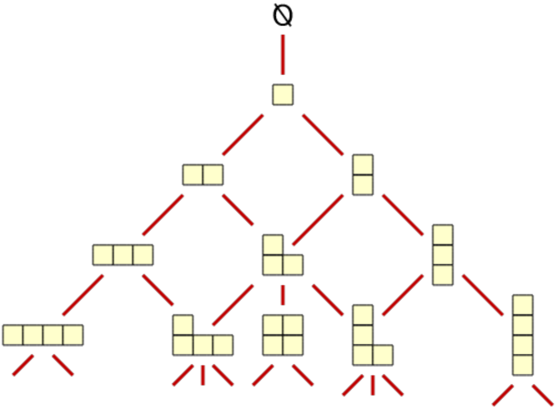
Theorem (Kantarcı Oğuz, Ravichandran, 2022)

For any $\alpha \neq (1, k, 1, k)$ or $(k, 1, k, 1)$ for some k, the rank sequence $\overline{R}(\alpha; q)$ is unimodal.

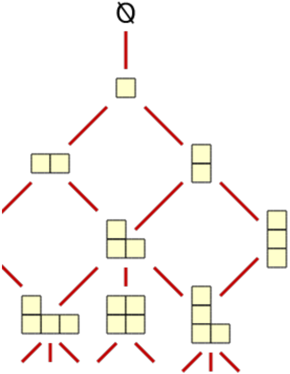
We can also see fences as intervals in the Young's lattice.

Young's Lattice is the lattice of Ferrers diagrams of Partitions ordered by inclusion.



(Image from Wikipedia, created by David Eppstein)

For any partition, we can look at the generating function of the partitions that lay under it.



$$G(\lambda; q) := \sum_{\mu \subset \lambda} q^{|\mu|}$$

$$G\left(\ydiagram{1,2}\,; q\right) = q^3 + 2q^2 + q + 1$$

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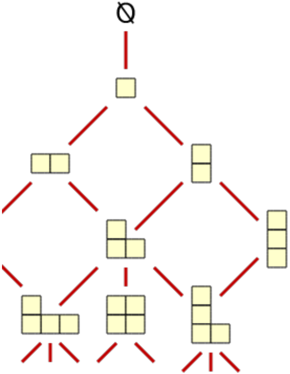

$$G\left(\ydiagram{2,1}; q\right) = q^3 + 2q^2 + q + 1$$

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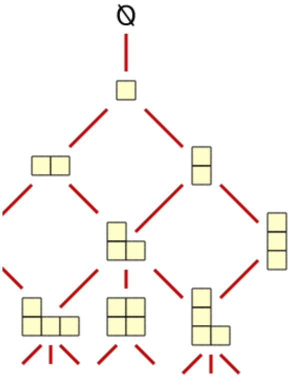

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We can also look at the interval between two partitions.

$$G(\lambda/\nu; q) := \sum_{\nu \subset \mu \subset \lambda} q^{|\mu| - |\nu|}$$

$$G\left(\ydiagram{2,1,1} / \ydiagram{1,1}; q\right) = q^2 + 2q + 1$$

Unimodality of these polynomials were considered by Stanton in 1990[5].

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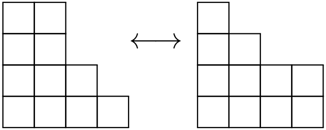

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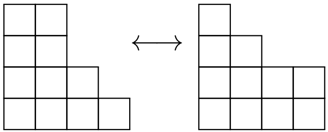

Conjecture (Stanton,1990)

The polynomials corresponding to self-dual partitions are unimodal.

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TABLE I

Partition	i	Values	Partition	i	Values
8 8 4 4	15	31 30 31	11 11 6 6	21	67 66 67
10 9 4 4	17	46 45 46	14 13 4 4	21	76 75 76
10 10 4 4	17	46 45 46	16 12 4 4	23	91 90 91
12 10 4 4	19	61 60 61	14 14 4 4	21	76 75 76
12 11 4 4	19	61 60 61	12 12 8 4	23	81 80 81
12 12 4 4	19	61 60 61	12 10 8 6	23	82 81 82
14 11 4 4	21	76 75 76	8 8 8 6 4 2	23	141 140 141
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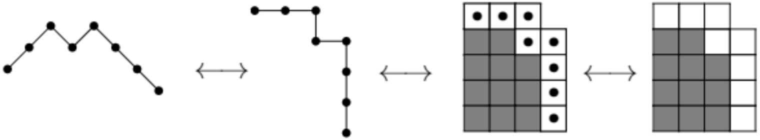
(Table from "Unimodality and Young's Lattice", Stanton)

Given a fence, we can see it as a difference of two partitions α/ν.

Example $((2,1,1,3) \to (4,4,4,4,3)/(3,3,3,2))$



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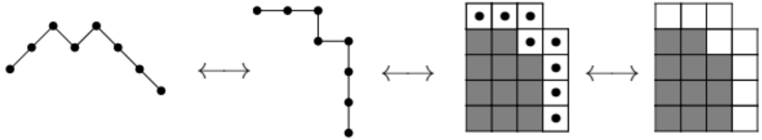

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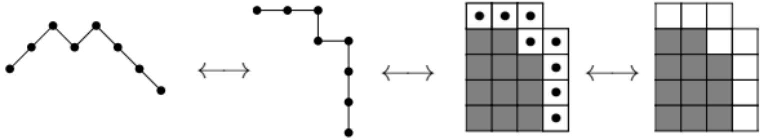

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Rank polynomials actually correspond to a special class of differences called *ribbon diagrams*, where we have no 2×2 box.

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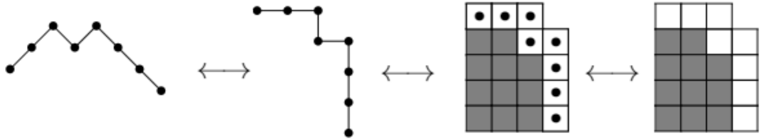
Example $((2,1,1,3) \to (4,4,4,4,3)/(3,3,3,2))$



Note that the ideals of the fence coincide with the partitions that lie between α and ν, so $G(\lambda/\nu)$ agrees with the rank polynomial.

Rank polynomials actually correspond to a special class of differences called *ribbon diagrams*, where we have no 2×2 box.

Polynomials corresponding to ribbon diagrams are unimodal.

Some follow up work

– Let $a = (a_1, \ldots, a_s)$ be a composition of n. The Chainlink polytope[6] is the s dimensional polytope defined as

$$\mathrm{CL}(a) = \{x \in \mathbb{R}^s \mid x_i \in [0, a_i + 1],\ x_i - x_{(i+1)\,\mathrm{mod(s)}} \leq a_i,\ i \in [s]\}.$$

Let $\mathrm{CL}(a)^t$ be the slice of the polytope with respect to the hyperplane $x_1 + \ldots + x_s = t$.

Theorem (Kantarci-Oguz, Ozer, R, 2022)

For any composition a and any real t, we have that

$$|\,\mathrm{CL}(a)^t| = |\,\mathrm{CL}^{m-t}(a)|,$$

where $m = n + s$ as above. Further, when t is an integer, the Ehrhart (Quasi) polynomials are equal as well.

$$\mathrm{Ehr}_{\mathrm{CL}(a)^t} = \mathrm{Ehr}_{\mathrm{CL}^{m-t}(a)}\,.$$

[6] **CL**.

More follow up work

In recent work, Leclerc and Morier-Genoud considered a q deformation of $PSL(2,\mathbb{Z})$. The matrices

$$R_q = \begin{bmatrix} q & 1 \\ 0 & 1 \end{bmatrix}, \quad S_q = \begin{bmatrix} 0 & -q^{-1} \\ 1 & 0 \end{bmatrix},$$

generate a group $PSL_q(2,\mathbb{Z})$ isomorphic to $PSL(2,\mathbb{Z})$.

Theorem (Kantarci-Oguz)

For every word of the form $C = R_q^{c_1} S_q R_q^{c_2} S_q \dots R_q^{c_k} S_q$, where the $c_i > 2$, the polynomials $\mathrm{Tr}(C)$ are unimodal.

The proof uses two ingredients

– Relating these to rank polynomials of circular fence posets.

– A new notion of oriented posets, a class of posets for which the rank polynomials can be computed iteratively using linear algebra.[7].

[7] Kantarcı Oğuz, *Oriented Posets and Rank Polynomials*.

Thank you for listening!

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