Combinatorics of scattering amplitudes

joint work with

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I will discuss

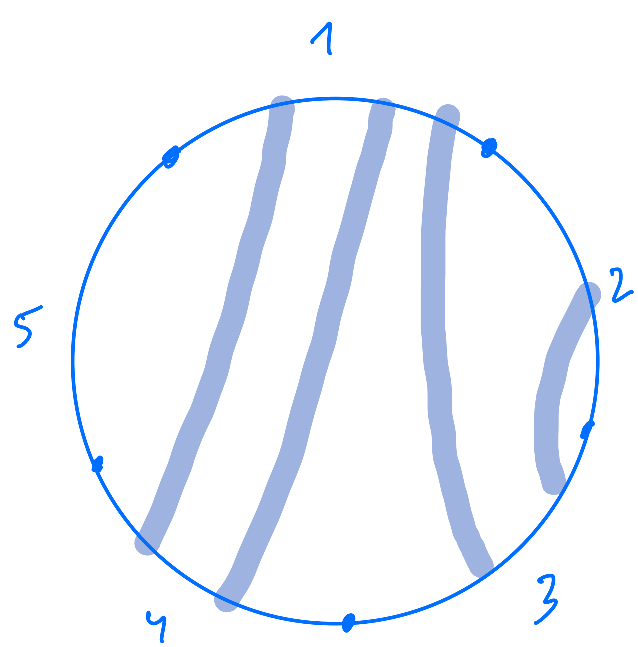
- a problem from quantum field theory
- an approach to solving it using representation theory (with an appearance of something very close to the rank polynomials of fence posets from Mohan Ravichandran's talk)
- generalizations

Let S be a disk, with a finite set of marked points on the boundary.

Let C be the curves running from boundary to boundary, avoiding the marked points, & up to homotopies preserving these properties, & excluding contractible curves.

In more down-to-earth terms, if we number the boundary segments from 1 to n, a choice of γ in C is determined by the two distinct boundary segments of its endpoints.

To illustrate, in the picture below we have two curves representing the class γ_{14}, one representing γ_{13}, and one representing γ_{23}.

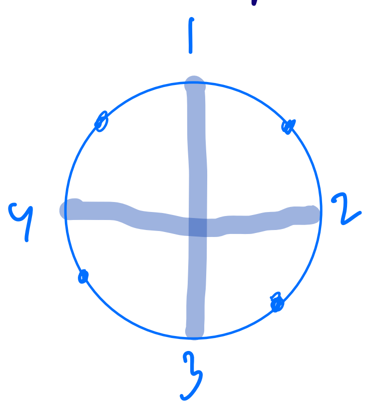


For each $\gamma \in C$, introduce a variable u_γ, & consider the following system of equations:

$$u_\gamma + \prod_{\delta \in C} u_\delta^{c(\gamma,\delta)} = 1$$

where $c(\gamma,\delta)$ is the minimum number of intersection points of γ and δ. Note that $c(\gamma,\gamma)=0$.

Example ($n=4$):



$u_{13} + u_{24} = 1$

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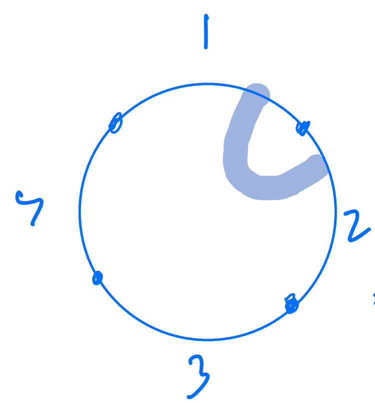


$u_{12} + 1 = 1$

$\Rightarrow u_{12} = u_{23} = u_{34} = u_{41} = 0$

u-equations: $u_\gamma + \prod_{\delta \in C} u_\delta^{c(\gamma,\delta)} = 1$

Write $\mathcal{U}$ for the solutions over $\mathbb{C}$, and $\mathcal{U}_{\geq 0}$ for the solutions in $\mathbb{R}_{\geq 0}$.

What can we say about $\mathcal{U}_{\geq 0}$? In particular, we would like to understand its boundary where some $u_\gamma = 0$.

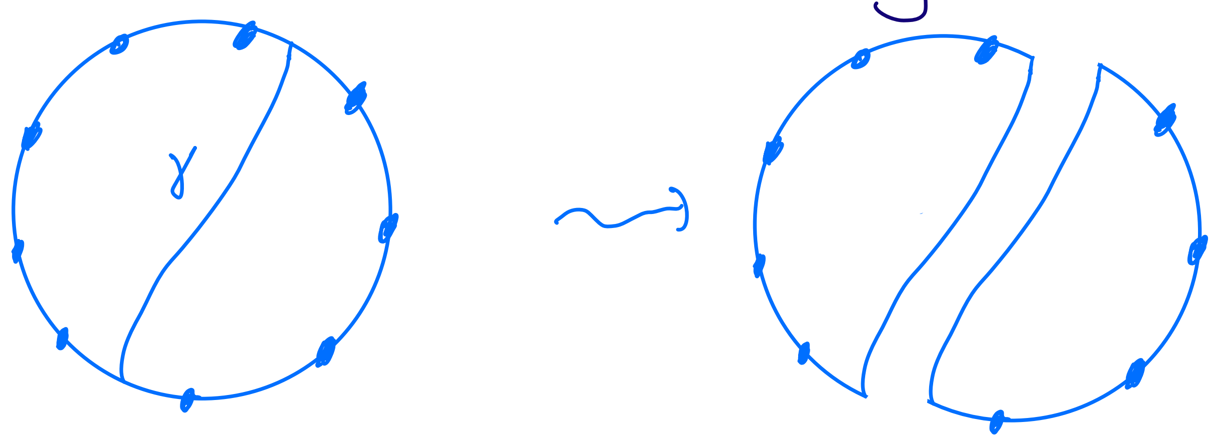
A first observation is that $u_\gamma + \prod_{\delta \in C} u_\delta^{c(\gamma,\delta)} = 1$ implies that for solutions in $\mathcal{U}_{\geq 0}$, we have $0 \leq u_\gamma \leq 1$.

Suppose $u_\gamma = 0$. For any δ with $c(\gamma,\delta) > 0$, we have

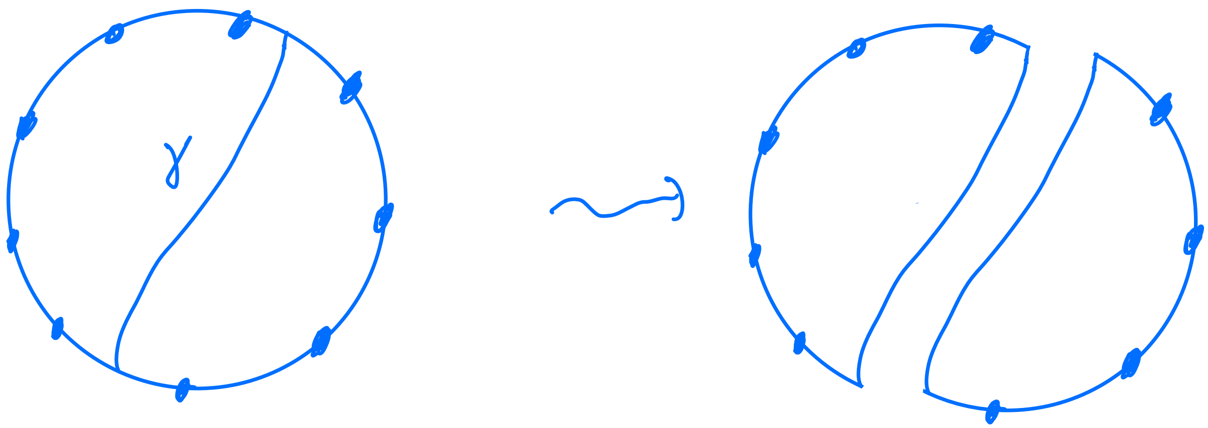
$$u_\delta + \underbrace{\prod_{\alpha \in C} u_\alpha^{c(\delta,\alpha)}}_{0} = 1 \qquad \text{so } u_\delta = 1.$$

That is to say, all the curves crossing γ must be 1.

Cut the disk in two along γ.



$u_\gamma = 0 \Rightarrow u_\delta = 1$ for δ crossing γ.



The u-equations for the remaining variables are simply the u-equations for the two smaller surfaces.

We describe this by saying that the boundaries of $\mathcal{U}_{\geq 0}$ factorize.

In fact, the exact kind of factorization we are seeing is familiar to those who are familiar with the associahedron, or $\overline{\mathcal{M}}_{0,n}$.

Motivation:

In 1969, Koba & Nielsen defined $\mathcal{U}_{7,0}$. Their goal was to define what is now called the tree string amplitude.

Given $p = (p_\gamma)_{\gamma \in C}$ Mandelstam variables

$$A_{tree}(p) = \int_{\mathcal{U}_{7,0}} \prod_{\gamma \in C} u_\gamma^{p_\gamma}$$

$$A_{tree}(p) = \int_{\mathcal{U}_{\geq 0}} \prod_{\gamma \in \mathcal{C}} u_\gamma^{p_\gamma}$$

The point is that if some p_γ goes to zero, the amplitude should factorize as the product of two smaller amplitudes & this is guaranteed by the factorization we saw for the boundary of $\mathcal{U}_{\geq 0}$.

To work with A_{tree}, we need to be able to solve the u-equations explicitly, & I will get to that.

Beforehand, though, I want to say something about where this is leading.

In perturbative quantum field theory, the tree amplitude we have been discussing is the first (or zero-th) term in a series.

The higher order terms should be obtained by applying similar ideas to surfaces of higher genus (rather than the disk). This was grasped already in 1969, but was not pursued successfully at the time.

I will come back at the end to how the techniques I am discussing are relevant to other surfaces.

Quiver representations

A *quiver* is a finite directed graph.

A *representation* of a quiver Q is an assignment of a vector space to each vertex of Q & a linear map between the corresponding vector spaces to each arrow of Q.

Eg.

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\begin{bmatrix}1\\0\end{bmatrix}"] & \mathbb{C}^2 \arrow[r, "\begin{bmatrix}0 & 1\end{bmatrix}"] & \mathbb{C} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{C}^2 \arrow[loop right, "\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}"] \end{tikzcd}$$

Two representations are isomorphic if one can be obtained from the other by a change of basis at the vertices.

Eg. $$\begin{tikzcd} \bullet \arrow[r, "\alpha"] & \bullet \\[-1em] \mathbb{C} & \mathbb{C} \end{tikzcd}$$ are isomorphic for all $\alpha \neq 0$.

More generally, a morphism between two representations M, N of a quiver Q is a collection of linear maps $f_i : M_i \to N_i$ for each vertex i of Q, such that for any arrow $i \xrightarrow{a} j$ of Q, we have that

$$\begin{tikzcd} M_i \arrow[r, "M_a"] \arrow[d, "f_i"'] & M_j \arrow[d, "f_j"] \\ N_i \arrow[r, "N_a"] & N_j \end{tikzcd}$$

commutes.

The direct sum of two representations of Q is formed by taking the direct sums of vector spaces & maps.

A representation is indecomposable if it is not isomorphic to the direct sum of two smaller representations.

For example, the indecomposable representations of $\bullet \to \bullet$ are as follows

$$\mathbb{C} \to 0 \qquad 0 \to \mathbb{C} \qquad \mathbb{C} \xrightarrow{1} \mathbb{C}$$

(1 could be replaced by any non-zero scalar)

Question: what theorem in first-year linear algebra is this?

The fact that the indecomposable representations of $\bullet \longrightarrow \bullet$ are $\underset{\mathbb{C}}{\bullet} \longrightarrow \underset{0}{\bullet}$, $\underset{0}{\bullet} \longrightarrow \underset{\mathbb{C}}{\bullet}$, $\underset{\mathbb{C}}{\bullet} \xrightarrow{1} \underset{\mathbb{C}}{\bullet}$,

amounts to saying that up to multiplication on left & right by invertible matrices, any matrix is equivalent to a partial identity matrix.

$$\underbrace{\begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}}_{n} \Big\} m$$

Let Q be an orientation of a path. Then the indecomposable representations are obtained by taking a connected subset of the vertices of Q, putting one copy of the field at each vertex in the subset, zeros elsewhere, & connecting the copies of the field by isomorphisms.

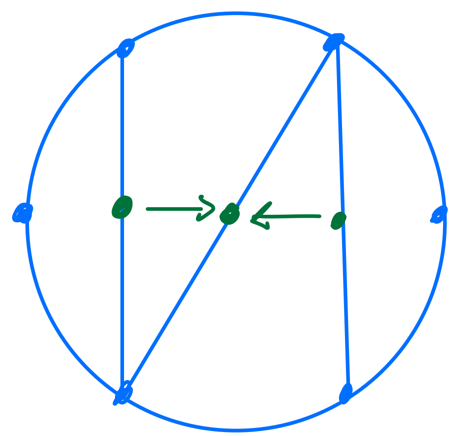
$$
Q: \quad \begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[r] & \bullet & \bullet \arrow[l] \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd}
$$

$$
\begin{tikzcd} 0 & 0 & \mathbb{C} & \mathbb{C} \arrow[l, "1"'] \arrow[r, "1"] & \mathbb{C} & 0 \end{tikzcd}
$$

How is this relevant to the problem we started with?

Fix a triangulation of the disk such that every triangle includes at least one boundary edge.

Now define a quiver by putting a vertex on each diagonal, and joining the vertices corresponding to consecutive diagonals at a marked point by an arrow going clockwise.



The rough idea is that a curve γ in $\mathcal{C}$ corresponds to an indecomposable representation of Q, by using the set of diagonals crossed by γ (which is a consecutive set of vertices of Q) to define a representation.

But $|\mathcal{C}| = \frac{n(n-3)}{2}$, while the number of indecomposable representations of Q is $\frac{(n-2)(n-3)}{2}$. We need $n-3$ "extra" indecomposables.

As in the cluster-algebraic setting it turns out to be natural to consider these objects as shifted projectives $P_i[1]$.

In order to explain how the quiver perspective is helpful, I need to introduce another piece of quiver technology, the Auslander-Reiten quiver.

This is a directed graph with vertex set the indecomposable representations, & arrows given by "irreducible" morphisms which don't factor through any other indecomposable module.

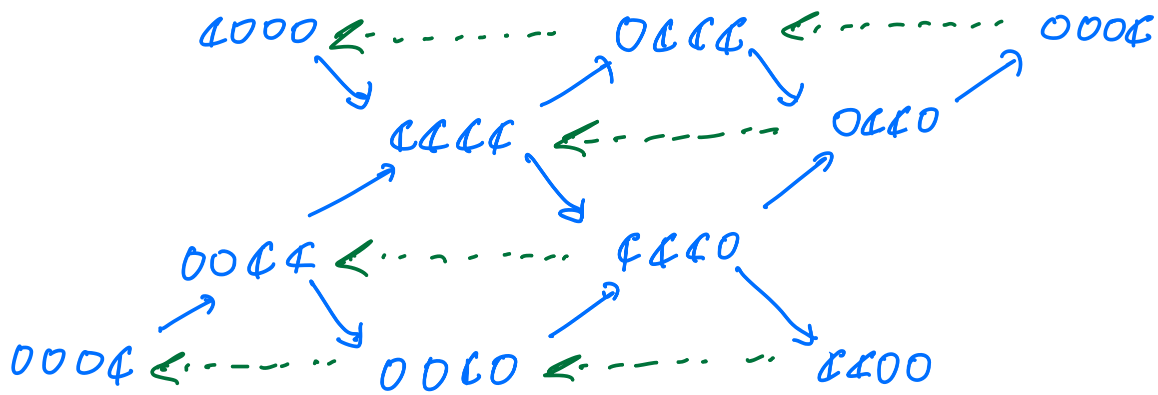
Example $\qquad Q = \bullet \leftarrow \bullet \rightarrow \bullet \rightarrow \bullet$

$$
\begin{tikzcd}[column sep=small, row sep=small]
 & 1000 \arrow[dr] & & 0111 \arrow[dr] & & 0001 \\
 & & 1111 \arrow[ur] \arrow[dr] & & 0110 \arrow[ur] & \\
 & 0011 \arrow[ur] \arrow[dr] & & 1110 \arrow[ur] \arrow[dr] & & \\
0001 \arrow[ur] & & 0010 \arrow[ur] & & 1100 &
\end{tikzcd}
$$

There is a simple combinatorial procedure for constructing the AR quiver. When Q is an oriented path, it always has a similar grid-like structure.

Another important feature of the Auslander-Reiten quiver is the Auslander-Reiten transformation τ which sends each indecomposable to the next representation to the left.

$Q = \bullet \leftarrow \bullet \rightarrow \bullet \rightarrow \bullet$ $\qquad$ $\tau : \dashleftarrow$



The objects on the lefthand edge are the projective representations. Their τ is zero.

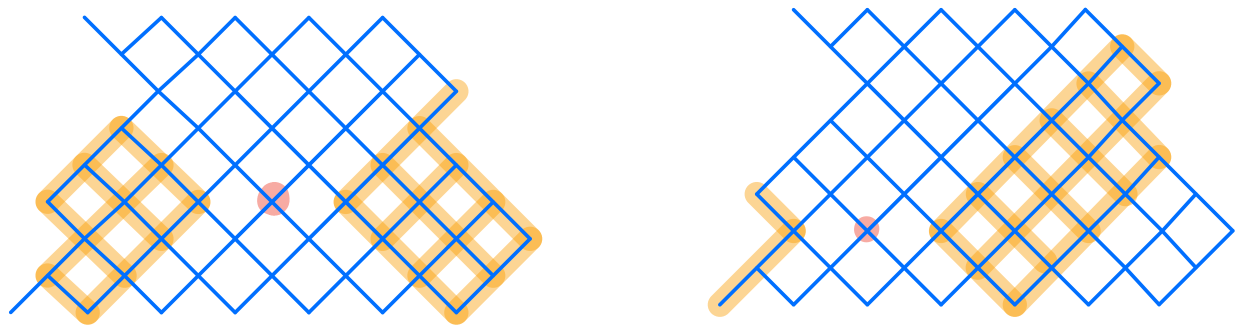
Equipped with an understanding of the Auslander-Reiten translation, we can now describe the intersection number in representation-theoretic terms:

$$c(\gamma, \delta) = \dim \mathrm{Hom}(M_\gamma, \tau M_\delta) + \dim \mathrm{Hom}(M_\delta, \tau M_\gamma)$$

Here M_γ is the representation corresponding to the curve γ (and I am sweeping under the rug the fact that M_γ might not actually be a representation).

This means we can describe the u-equations in purely representation-theoretic terms.

It is also easy to describe this in terms of the AR quiver:



● M_γ ● all M_δ with $c(\gamma,\delta)=1$.

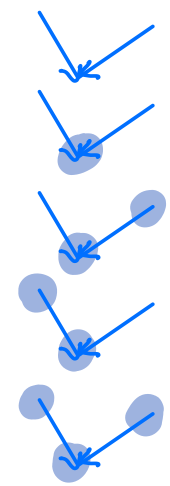
Draw as big a rectangle as fits vertically going left from τM_γ and right from $\tau^{-1} M_\gamma$.

We need one more ingredient for a representation · theoretic expression of a solution to the u equations.

Let M be a representation of Q. A subrepresentation of M is a choice of N_i a subspace of M_i for each vertex i, such that for each $i \xrightarrow{a} j$, we have $M_a(N_i) \subseteq N_j$.

More explicitly, if Q is an orientation of a path, we can view M as defining a fence poset. Then the subrepresentations of M are order ideals in the poset.

E.g. for $M: \mathbb{C} \xrightarrow{1} \mathbb{C} \xleftarrow{1} \mathbb{C}$, the subrepresentations are

0	0	0	
0	$\mathbb{C}$	0	
0	$\mathbb{C}$	$\mathbb{C}$	
$\mathbb{C}$	$\mathbb{C}$	0	
$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$	

We want to keep track of a slightly more refined invariant than the rank polynomial. For N a subrepresentation of M, let

$$y^{\dim N} = \prod_{i \in N} y_i.$$

Define the F-polynomial of M to be $F_M = \sum_{N \subseteq M} y^{\dim N}$.

Now, we can state our solution to the u-equations

$$u_\gamma = 1 - \frac{y^{\dim M_\gamma}}{F_{\tau M_\gamma} F_{M_\gamma}}$$

$$= \frac{F_{E_1} F_{E_2}}{F_{\tau M_\gamma} F_{M_\gamma}}$$

$F\, P_i[1] = 1$

$\dim P_i[1] = 0$

$\tau P_i[1] = I_i$

$$\begin{tikzcd} & E_1 \arrow[dr] & \\ \tau M_\gamma \arrow[ur] \arrow[dr] & & M_\gamma \\ & E_2 \arrow[ur] & \end{tikzcd}$$

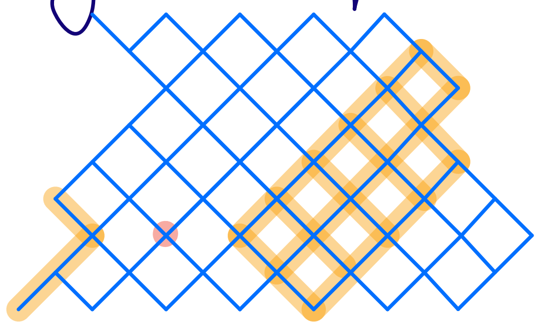
The fact that the two expressions are equal plays a key rôle in the proof. It holds for any finite-dimensional algebra, by a result of Domínguez & Geiss.

By specializing $y_1 \cdots y_n$ to values in $\mathbb{R}_{>0}$, we get a dense subset of all the solutions.

To give a hint as to the proof, I want to recall

$$u_\gamma + \prod_\delta u_\delta^{c(\gamma,\delta)} = 1$$

The δ's appearing in the product are here



and if we substitute

$$u_\delta = \frac{F_{E_1} F_{E_2}}{F_{M_\delta} \, F_{\tau M_\delta}}$$

$$\begin{tikzcd} & E_1 \arrow[dr] & \\ \tau M_\delta \arrow[ur] \arrow[dr] & & M_\delta \\ & E_2 \arrow[ur] & \end{tikzcd}$$

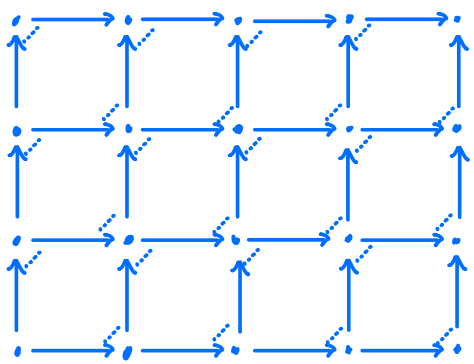
we see that there will be a lot of telescoping. Using the other formula for u_γ completes the proof.

Generalizations

The u-equations can be written (in their representation-theoretic version) for any algebra with finitely many indecomposable modules. A similar (though less explicit) telescoping argument still works.

We can't rule out the possibility that $\mathcal{U}_{\geq 0}$ has other components.

This approach applies, in particular to quivers with relations of the form



Solutions to these u-equations are related to the physics of Cachazo-Early-Guevara-Mizera scattering. I am working out the details with Nick Early.

For other unpunctured surfaces, there is a corresponding gentle algebra, as in works of Palu-Pilaud-Plamondon, among others. These algebras typically admit infinitely many indecomposable modules, so some things have to be modified. (The product in the u-equation would not be well-defined.) Nonetheless, we can again produce solutions for the u-equations in much the same way.

For punctured surfaces, the algebras are only locally gentle. We think we know what to do, but details remain to be worked out. Here, some of our representations are infinite-dimensional, so the F-polynomials become formal power series.

One could hope for u-equations associated to any finite-dimensional algebra. We don't currently know how to tackle them.

Summary: I described a way to reformulate the q-equations in terms of representation theory. While I focussed on the case of a disk, the ideas can be extended to other surfaces, which we expect to have significant consequences for string theory.

Thank you!