Continued fractions, Chen-Stein method and extreme value theory

Parthanil Roy, Indian Statistical Institute
Joint work with Anish Ghosh and Maxim Kirsebom

April 25, 2019

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Note that $7 > 3 > 1$.

Therefore by Euclidean Algorithm, any rational number

$$\omega = p/q \in (0,1)$$

(with $gcd(p,q) = 1$) will have a *terminating (regular) continued fraction expansion*.

Conversely ...

Whenever $A_1, A_2, A_3, A_4 \in \mathbb{N}$,

$$[A_1, A_2, A_3, A_4] := \cfrac{1}{A_1 + \cfrac{1}{A_2 + \cfrac{1}{A_3 + \cfrac{1}{A_4}}}} \in (0, 1)$$

is a rational number.

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More generally, by induction on n,

$$\omega = [A_1, A_2, \ldots A_n]$$

(with $A_1, A_2, \ldots A_n \in \mathbb{N}$) is a rational number in $(0, 1)$.

Non-terminating continued fraction expansion

Theorem

A number $\omega \in (0,1)$ has a unique non-terminating continued fraction expansion

$$\omega = \cfrac{1}{A_1 + \cfrac{1}{A_2 + \cfrac{1}{A_3 + \cdots}}} =: [A_1, A_2, A_3, \ldots]$$

(with each $A_i \in \mathbb{N}$) if and only if $\omega \notin \mathbb{Q}$.

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Examples: $\pi \approx \frac{22}{7}$ and $\pi \approx \frac{355}{113}$.

Why continued fractions?

Continued fractions are important in *algebra, analysis, combinatorics, ergodic theory, geometry, number theory, probability, etc.*.

See, for example, Khintchine (1964).

For an irrational $\omega \in (0, 1)$

$$\omega = \frac{1}{1/\omega} = \frac{1}{[1/\omega] + \{1/\omega\}} =: \frac{1}{A_1(\omega) + T(\omega)}$$

$$= \frac{1}{A_1(\omega) + \frac{1}{A_1(T(\omega)) + T^2(\omega)}}$$

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$$= \cdots$$

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Quick Observation: T, A_1 measurable $\Rightarrow$ each A_n measurable.

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$(\Omega, \mathcal{A}, P, T) =$ the Gauss dynamical system.

A reformulation of Gauss's theorem

Exercise (in *Probability Theory II*): Suppose X is a random variable having probability density function

$$f_X(x) = \frac{1}{(1+x)\log 2}, \quad x \in (0,1).$$

Then show that $\{1/X\} \stackrel{\mathcal{L}}{=} X$.

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Take $\Omega = (0,1)$, $\mathcal{A} = \mathcal{B}_{(0,1)}$, $P(dx) = ((1+x)\log 2)^{-1}\, dx$.

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T preserves $P \Rightarrow \{A_n\}$ is a strictly stationary process. In particular, $A_1, A_2, A_3, \ldots$ are identically distributed.

Two easy observations

- **Direct Computation:** For all $m \in \mathbb{N}$,

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as $n \to \infty$. In particular,

$$nP\left(\frac{A_1 \log 2}{n} > u\right) \to u^{-1}$$

(A_1 is regularly varying with index 1).

If $A_1, A_2, A_3, \ldots$ were independent

then

$$\mathbb{1}_{(A_1 \log 2 > un)},\ \mathbb{1}_{(A_2 \log 2 > un)},\ \mathbb{1}_{(A_3 \log 2 > un)},\ \ldots \overset{iid}{\sim} Ber(p_n),$$

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$$\xrightarrow{\mathcal{L}} \mathcal{E}_\infty^u \sim Poi(u^{-1})$$

as $n \to \infty$.

Doeblin-Iosifescu asymptotics

Theorem (Doeblin (1940), Iosifescu (1977))

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Corollary (Main result of Galambos (1972))

Let $M_n^{(1)} := \max\{A_i \log 2 : 1 \leq 1 \leq n\}$, $n \in \mathbb{N}$. *Then for all* $u > 0$,

$$P\left(\frac{M_n^{(1)}}{n} \leq u\right) \to e^{-u^{-1}}$$

as $n \to \infty$.

The main question

Theorem (Doeblin (1940), Iosifescu (1977))

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Question

What is the rate of convergence in (DI)?

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- Rate of convergence of the scaled maxima for the geodesic flow on the modular surface.

The geodesic flow on the modular surface

The group $SL_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\}$ acts isometrically on $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ by rational transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .z = \frac{az + b}{cz + d}.$$

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Series (1981, 1985): Connected the geodesic flow on $M = \mathbb{H}/SL_2(\mathbb{Z})$ with Gauss dynamical system using a symbolic dynamics.

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$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .z = \frac{az + b}{cz + d}.$$

Series (1981, 1985): Connected the geodesic flow on $M = \mathbb{H}/SL_2(\mathbb{Z})$ with Gauss dynamical system using a symbolic dynamics.

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Our work yields the rate of convergence in Pollicott's result.

The main result

Theorem (Ghosh, Kirsebom, R. (2019))

There exists $\kappa > 0$ *and a sequence* $1 \ll \ell_n \ll n^\epsilon$ *(for all* $\epsilon > 0$*) such that for all* $u > 0$ *and for all* $n \in \mathbb{N}$,

$$d_{TV}(\mathcal{E}_n^u, \mathcal{E}_\infty^u) := \sup_{A \subseteq \mathbb{N} \cup \{0\}} \left| P(\mathcal{E}_n^u \in A) - P(\mathcal{E}_\infty^u \in A) \right| \leq \frac{\kappa}{\min\{u, u^2\}} \frac{\ell_n}{n}.$$

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Corollary

Suppose $M_n^{(k)} := k^{th}$ *maximum of* $\{A_i \log 2 : 1 \leq i \leq n\}$. *For all* $u > 0$ *and for all* $k, n \in \mathbb{N}$*,*

$$\sup_{k \in \mathbb{N}} \left| P\left(\frac{M_n^{(k)}}{n} \leq u\right) - e^{-u^{-1}} \sum_{i=0}^{k-1} \frac{u^{-i}}{i!} \right| \leq \frac{\kappa}{\min\{u, u^2\}} \frac{\ell_n}{n}.$$

Comparison with existing results

- Resnick and de Haan (1989): If $A_1, A_2, \ldots$ were independent, then
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Sketch of proof

Recall $\mathcal{E}_n^u = \sum_{j=1}^n \mathbb{1}_{(A_j \log 2 > un)} \overset{approx}{\sim} Bin\big(n, p_n = P(A_1 \log 2 > un)\big)$.

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- Estimate $d_{TV}(\tilde{\mathcal{E}}_n^u, \mathcal{E}_\infty^u)$ using *second order regular variation*.

How to estimate $d_{TV}(\tilde{\mathcal{E}}_n^u, \mathcal{E}_\infty^u)$?

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Theorem (Philipp (1970))

There exists $C > 0$ and $\theta > 1$ such that for all $m, n \in \mathbb{N}$, for all $F \in \sigma(A_1, A_2, \ldots, A_m)$, and for all $H \in \sigma(A_{m+n}, A_{m+n+1}, \ldots)$,

$$|P(F \cap H) - P(F)P(H)| \leq C\theta^{-n} P(F)P(H).$$

Thank You Very Much