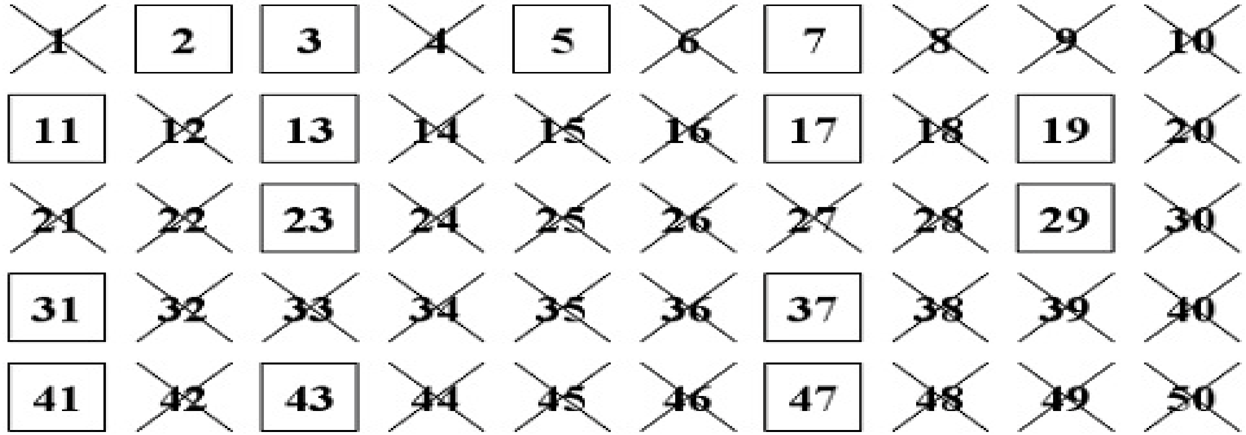

THE RIEMANN HYPOTHESIS

KEN ONO (UNIVERSITY OF VIRGINIA)

SIMONS FOUNDATION

IT IS HARD TO WIN $1 MILLION

IT CAN BE **REALLY HARD TO WIN** $1 MILLION

Millennium Prize problems proposed by the Clay Mathematics Institute.

1. P versus NP
2. The Hodge conjecture
3. The Poincaré conjecture (proved by G.Perelman in 2003)
4. The Riemann hypothesis
5. Yang–Mills existence and mass gap
6. Navier–Stokes existence and smoothness
7. The Birch and Swinnerton-Dyer conjecture

GOD, HARDY, AND THE RIEMANN HYPOTHESIS

On a trip to Denmark, Hardy wrote his friend Harald Bohr:

"Have proof of RH. Postcard too short for proof."

Hardy's Thinking.

God would not let the boat sink on the return and give him the same fame that Fermat had achieved with his "last theorem".

G. H. Hardy (1877-1947)

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HILBERT AND THE RIEMANN HYPOTHESIS

David Hilbert (1862 – 1943)

“If I were to awaken after having slept for a thousand years, my first question would be: Has the Riemann Hypothesis been proven?”

RIEMANN HYPOTHESIS (1859)

Bernhard Riemann (1826-1866)

Conjecture (Riemann)
The nontrivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$.

Question. What does this mean? Why does it matter?

PRIMES

Definition. A **<u>prime</u>** is a natural number > *1* with no positive divisors other than *1* and itself.

Theorem. (Fundamental Theorem of Arithmetic)
Every positive integer >*1* **factors uniquely** (up to reordering) as a product of primes.

2, 3, 5, 7, 11, 13, 17, 19, 23,

29, 31, 37, 41, 43, 47, 53, 59,

61, 67, 71, 73, 79, 83, 89, 97

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PRIMES ARE ORNERY

"Primes grow like weeds... seeming to obey no other law than that of chance... nobody can predict where the next one will sprout...
...Primes are even more astounding, for they exhibit stunning regularity. There are laws governing their behavior, and they obey these laws with almost military precision."

Don Zagier

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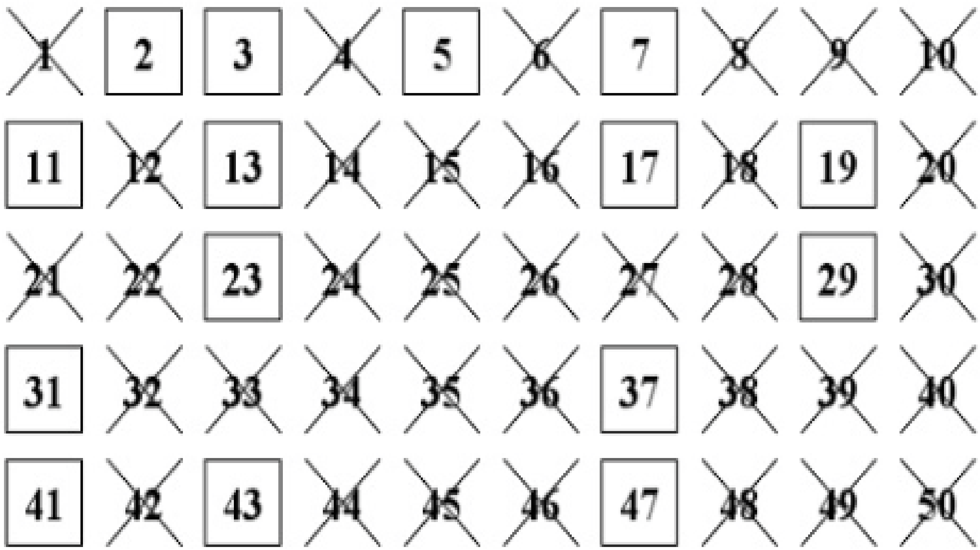
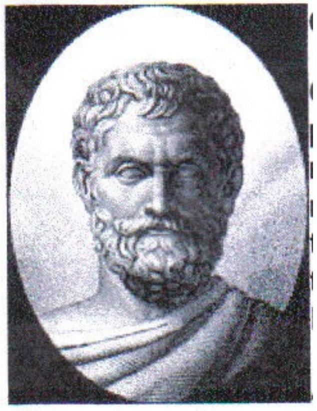
SIEVE OF ERASTOTHENES (~200 BC)





Algorithm for listing the primes up to a given bound.

Problem. This does not reveal much about the primes.

EUCLID (323-283 BC)

Theorem (Euclid)
There are infinitely many primes.

Proof: Suppose that $p_1=2 < p_2 = 3 < \ldots < p_r$ are all of the primes.

Let $P = p_1p_2\ldots p_r+1$ and let p be a prime dividing P.

Then p can not be any of $p_1, p_2, \ldots, p_r$, because otherwise p would divide the difference $P-p_1p_2\ldots p_r=1$, which is impossible.

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EULER (1707-1783)

Geometric Series. If $|r| < 1$, then

$$1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}.$$

Examples. Strange infinite series expressions

$$2 = \frac{1}{1-\frac{1}{2}} = \sum \frac{1}{2^{a_1}}$$

$$3 = \frac{1}{1-\frac{1}{2}} \cdot \frac{1}{1-\frac{1}{3}} = \sum \frac{1}{2^{a_1} 3^{a_2}}$$

$$\frac{15}{4} = \frac{1}{1-\frac{1}{2}} \cdot \frac{1}{1-\frac{1}{3}} \cdot \frac{1}{1-\frac{1}{5}} = \sum \frac{1}{2^{a_1} 3^{a_2} 5^{a_3}}$$

$$\frac{35}{8} = \frac{1}{1-\frac{1}{2}} \cdot \frac{1}{1-\frac{1}{3}} \cdot \frac{1}{1-\frac{1}{5}} \cdot \frac{1}{1-\frac{1}{7}} = \sum \frac{1}{2^{a_1} 3^{a_2} 5^{a_3} 7^{a_4}}$$

EULER (1707-1783)

The Fund. Thm of Arithmetic and geometric series give

$$\sum_n \frac{1}{n^s} = \prod_p \frac{1}{1 - \frac{1}{p^s}}$$

Letting *s=2* (or **any positive even**) Euler obtained formulas such as

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots = \frac{\pi^2}{6}$$

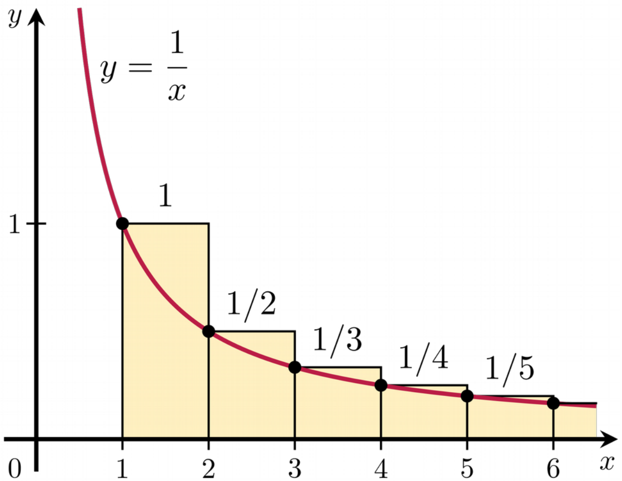
INFINITUDE OF PRIMES APRÉS EULER

Theorem. If π*(n)* is the number of primes < *n*, then

π(*n*) > *-1*+ ln*(n)*.

Proof.

- Let $p_1 = 2, p_2 = 3, p_3 = 5, \ldots$ be the primes, and so $p_j \geq j + 1$.
- Calculus tells us that $\ln(n) = \int_1^n \frac{1}{x}\, dx$.
- $\ln(n) < 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$.
- If $\pi(n) = k$, then Euler's product gives
  $$\ln(n) < \prod_{j=1}^{k} \frac{1}{1 - \frac{1}{p_j}}.$$



INFINITUDE OF PRIMES APRÉS EULER

Proof continued.

- A little algebra and the fact that $p_j \geq j+1$ gives
$$\ln(n) < \prod_{j=1}^{k}\left(1+\frac{1}{j}\right) = \prod_{j=1}^{k}\frac{j+1}{j}.$$
- By telescoping we get
$$\ln(n) < \frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdots\frac{k+1}{k} = k + 1 = \pi(n) + 1.$$
- Therefore, we have $\pi(n) > -1 + \ln(n)$.

GAUSS (1777-1855)

Carl Friedrich Gauss

Conjecture (Gauss).
If we let $\mathrm{Li}(X) := \int_2^X \frac{dt}{\log t}$, *then we have*

$$\pi(X) \sim \mathrm{Li}(X) \sim \frac{X}{\log X}.$$

x	$\pi(x)$	$\frac{x}{ln(x)}$
10^2	25	22
10^3	168	145
10^4	1229	1086
10^5	9592	8686
10^6	78498	72382
10^7	664579	620421
10^8	5761455	5428681
10^9	50847534	48254942
10^{10}	455052511	434294482

ENTER RIEMANN

Bernhard Riemann (1826-1866)

An 8 page paper in 1859

Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.

(Monatsberichte der Berliner Akademie, November 1859.)

$$\Pi \frac{1}{1-\frac{1}{p^s}} = \Sigma \frac{1}{n^s}$$

ENTER RIEMANN

Bernhard Riemann (1826-1866)

An 8 page paper in 1859

- Defined Zeta Function
- Determined many of its properties
- Posed the Riemann Hypothesis
- Strategy to prove Gauss' Conjecture

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RIEMANN'S ZETA FUNCTION

Theorem (Riemann, 1859)

(1) $\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$ *is defined for* $\mathrm{Re}(s) > 1$.

(2)

(3)

(4)

RIEMANN'S ZETA FUNCTION

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(2) Analytic continuation to $\mathbb{C}$ *(simple pole at* $s = 1$*).*

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(3) It satisfies $\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \cdot \zeta(1-s)$.

(4)

RIEMANN'S ZETA FUNCTION

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(3) It satisfies $\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \cdot \zeta(1-s)$.

(4) It has **trivial zeros** *at* $s = -2, -4, -6, -8, \ldots$.

1+2+3+4+5+ . . . = -1/12

Srinivasa Ramanujan (1887-1920)

"Under my theory

1+2+3+4+...= -1/12.

If I tell you this you will at once point out to me the lunatic asylum,,,"

Proof.

(Euler) $\zeta(2) = \frac{\pi^2}{6}$

1+2+3+4+5+ . . . = -1/12

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Proof

(Euler) $\zeta(2) = \frac{\pi^2}{6}$

(Riemann) $\zeta(-1)$ "=" $1 + 2 + 3 + 4 + \ldots$

SIMONS FOUNDATION

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Proof.

(Euler) $\zeta(2) = \frac{\pi^2}{6}$

(Riemann) $\zeta(-1)$ "$=$" $1+2+3+4+\ldots$

(Riemann) $\zeta(-1) = \frac{1}{2} \cdot \frac{1}{\pi^2} \cdot \sin(-\pi/2)\Gamma(2)\zeta(2) = -\frac{1}{12}$. $\square$

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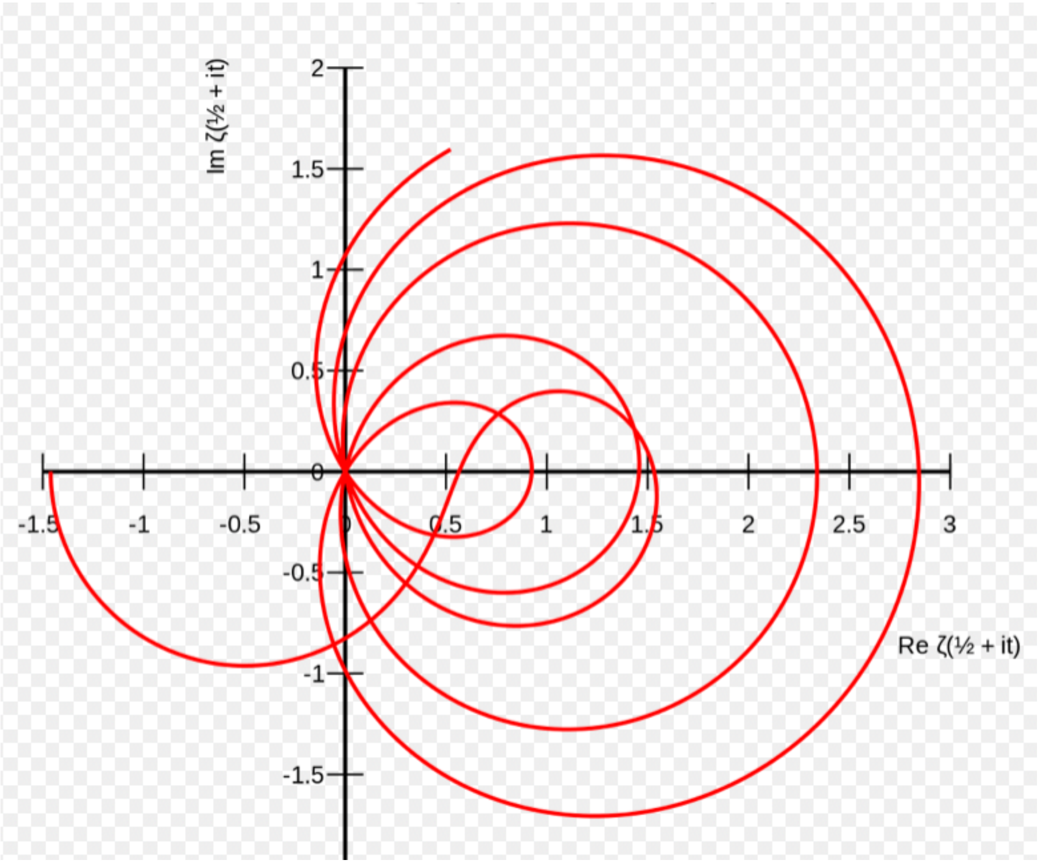
VALUES ON CRITICAL LINE



Spiraling ζ(*½ + it*) for $0 \leq t \leq 50$

Note.

- **ζ**(*½*) = *-1.460354….*
- The first few nontrivial zeros are encountered.

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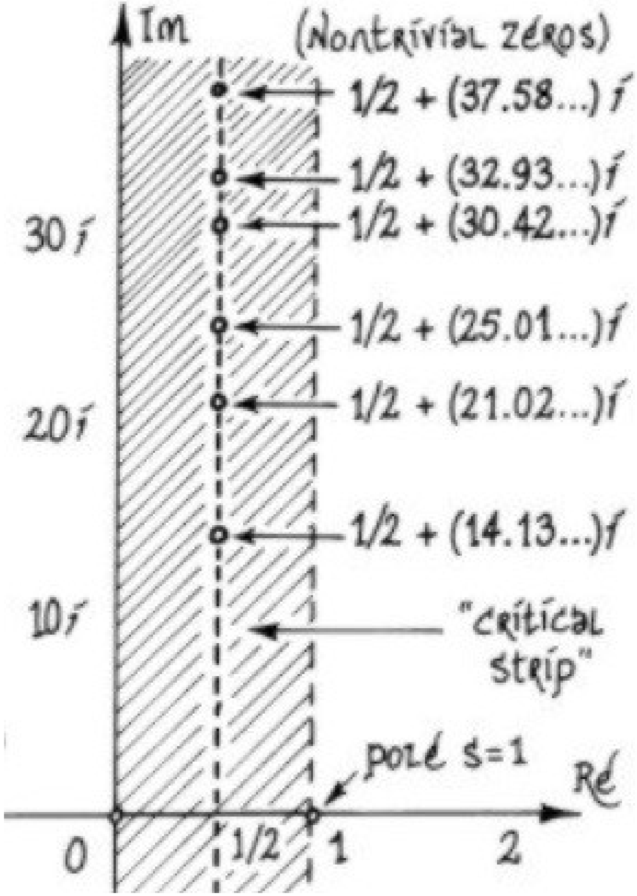
RIEMANN'S HYPOTHESIS

Conjecture (Riemann)
The nontrivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$.

"... it would be desirable to have a rigorous proof of this proposition..."

Bernhard Riemann (1859)



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COUNTING PRIMES

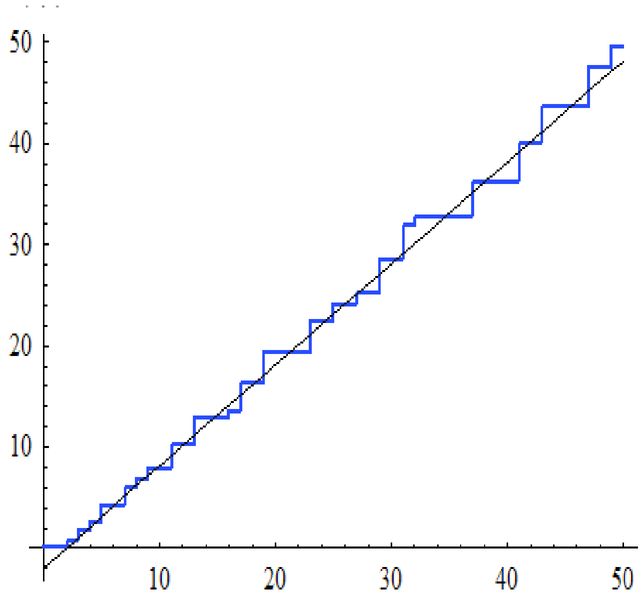
Theorem. (Chebyshev, von Mangoldt)

The Prime Number Theorem is equivalent to

$$\lim_{X \to +\infty} \frac{\Psi(X)}{X} = 1,$$

where we define

$$\Psi(X) := \sum_{p^a \leq X} \log p.$$



Graph of $Y = \Psi(X)$

WHY DO THE NONTRIVIAL ZEROS MATTER?

Theorem. (von Mangoldt)

As a sum over the **nontrivial zeros** *ρ of $\zeta(s)$, we have*

$$\Psi(X) = X - \log(2\pi) - \frac{1}{2}\log(1 - x^{-2}) - \sum_{\rho} \frac{X^{\rho}}{\rho}.$$

Theorem. (Hadamard, de la Vallée-Poussin (1896)

Gauss' Conjecture is true. We have that

$$\pi(X) \sim \mathrm{Li}(X) \sim \frac{X}{\log X}.$$

Proof. We always have Re*(ρ)* < *1*. □

WHY DOES RH MATTER?

Theorem. (von Koch (1901), Schoenfeld (1976))

If RH is true, then for all $X \geq 2657$ *we have*

$$|\pi(X) - \mathrm{Li}(X)| < \frac{\sqrt{X} \cdot \log X}{8\pi}.$$

RH & *Generalized* RH implications include

- Almost every deep question on primes
- Ranks of elliptic curves, Orders of class groups
- Quadratic forms (eg. Bhargava & Conway-Schneeberger style)
- Maximal orders of elements in permutation groups
- Running times for primality tests
- **Thousands of results** proved assuming the truth of RH and GRH…

RAMANUJAN'S TERNARY QUADRATIC FORM

"... *the even numbers which are not of the form* $x^2 + y^2 + 10z^2$ *are the numbers*

$$4^\lambda(16\mu + 6),$$

while the odd numbers that are not of that form, viz.,

$$3, 7, 21, 31, 33, 43, 67, 79, 87, 133, 217, 219, 223, 253, 307, 391 \ldots$$

do not seem to obey any simple law."

Theorem. (O-Soundararajan (1997))
Assuming GRH, the **only positive odds** not of the form $x^2+y^2+10z^2$ are

3, 7, 21, 31, 33, 43, 67, 79, 87, 133, 217, 219, 223, 253, 307, 391, 679, 2719.

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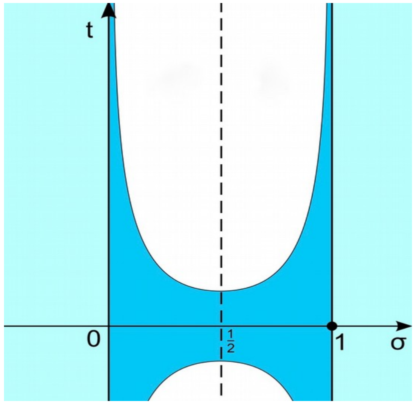
EVIDENCE FOR RH

- The lowest 100 billion nontrivial zeros satisfy RH.

- **Theorem** (Selberg, Levinson, Conrey, Bui, Young,…)
  At least 41% of the infinitely many nontrivial zeros satisfy RH.

- **Theorem (Hadamard, Vallée Poussin, Korobov, Vinogradov)**
  There is a zero-free region for $\zeta(s)$.



PROSPECTS FOR A PROOF

- **(Mertens)** RH is equivalent to the Möbius sum estimate

$$\sum_{n=1}^{X} \mu(n) = O(X^{\frac{1}{2}+\epsilon}).$$

- **Polya's Program**: More on this momentarily.
- **Functional Analysis**: Nyman-Beurling Approach
- **Trace Formulas:** Weil, Selberg, Connes, …
- **Random Matrices:** Dyson, Odlyzko, Montgomery, Keating, Snaith, Katz-Sarnak,…

RANDOM MATRICES

Freeman Dyson

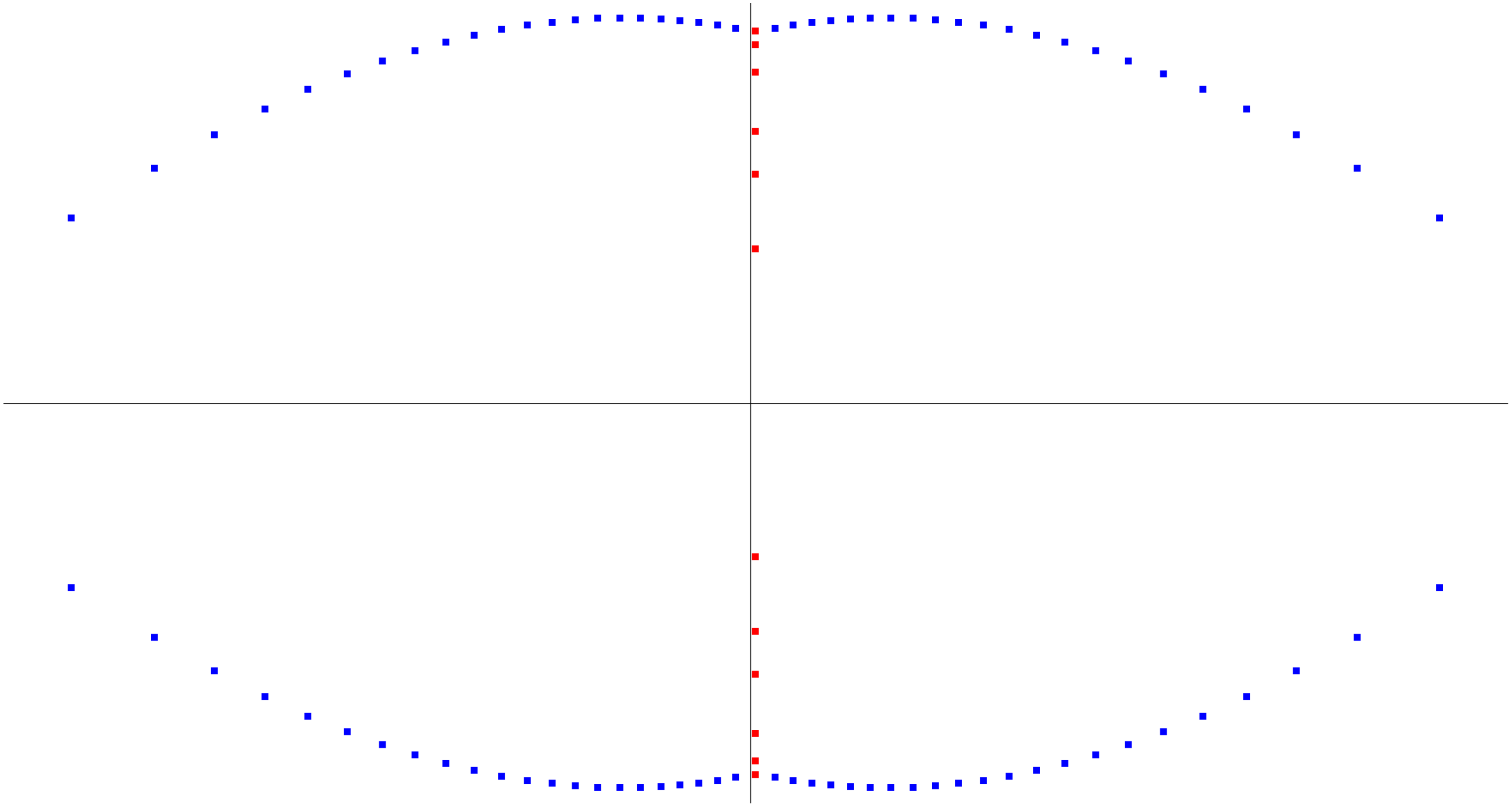
Hugh Montgomery

Andrew Odlyzko

Gaussian Unitary Ensemble (GUE) (Dyson, Montgomery ('70s))

The nontrivial zeros of $\zeta(s)$ appear to be "distributed like" the eigenvalues of random Hermitian matrices.

ROOTS OF THE DEG 100 TAYLOR POLYNOMIAL



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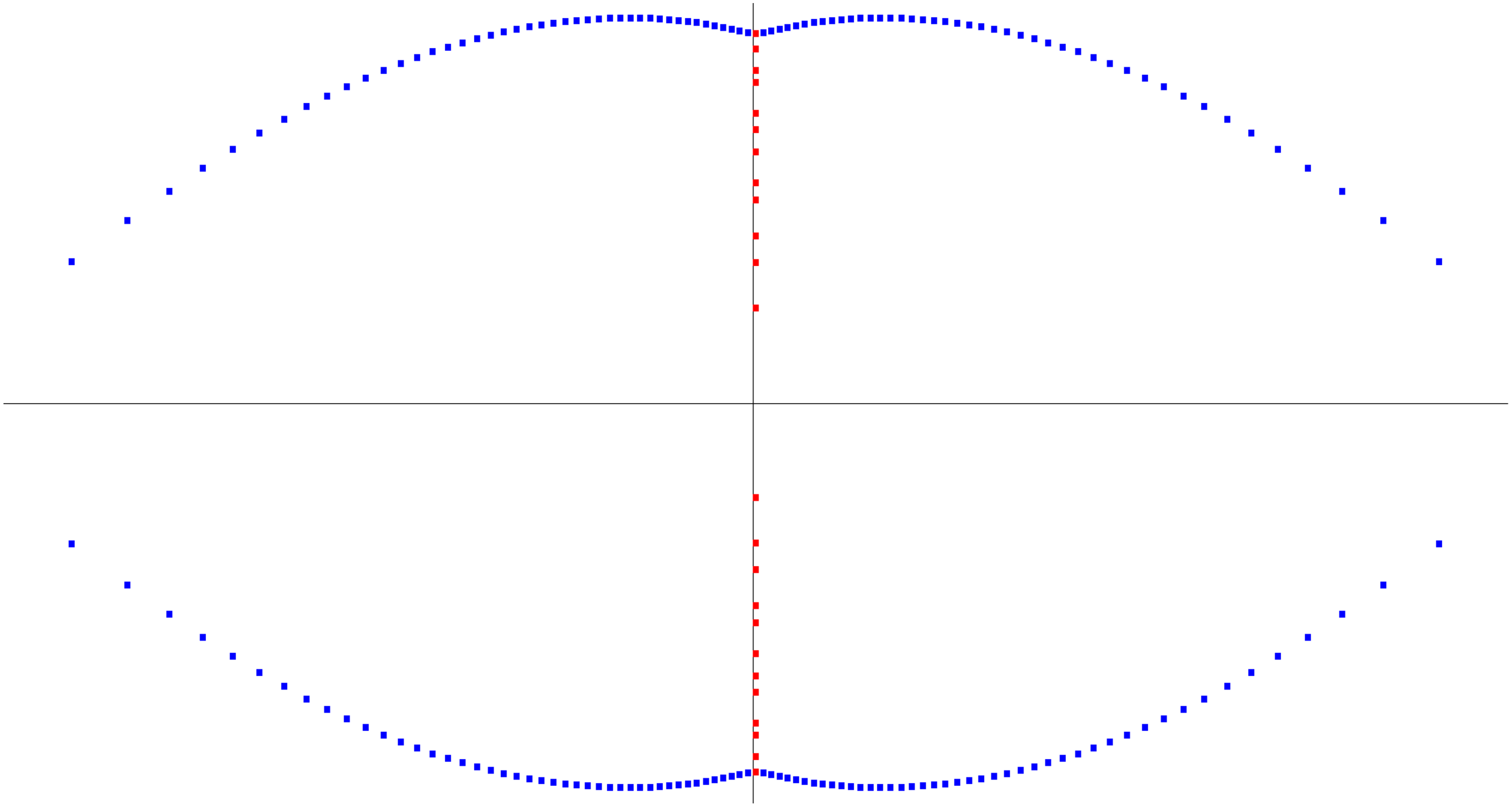
ROOTS OF THE DEG 200 TAYLOR POLYNOMIAL



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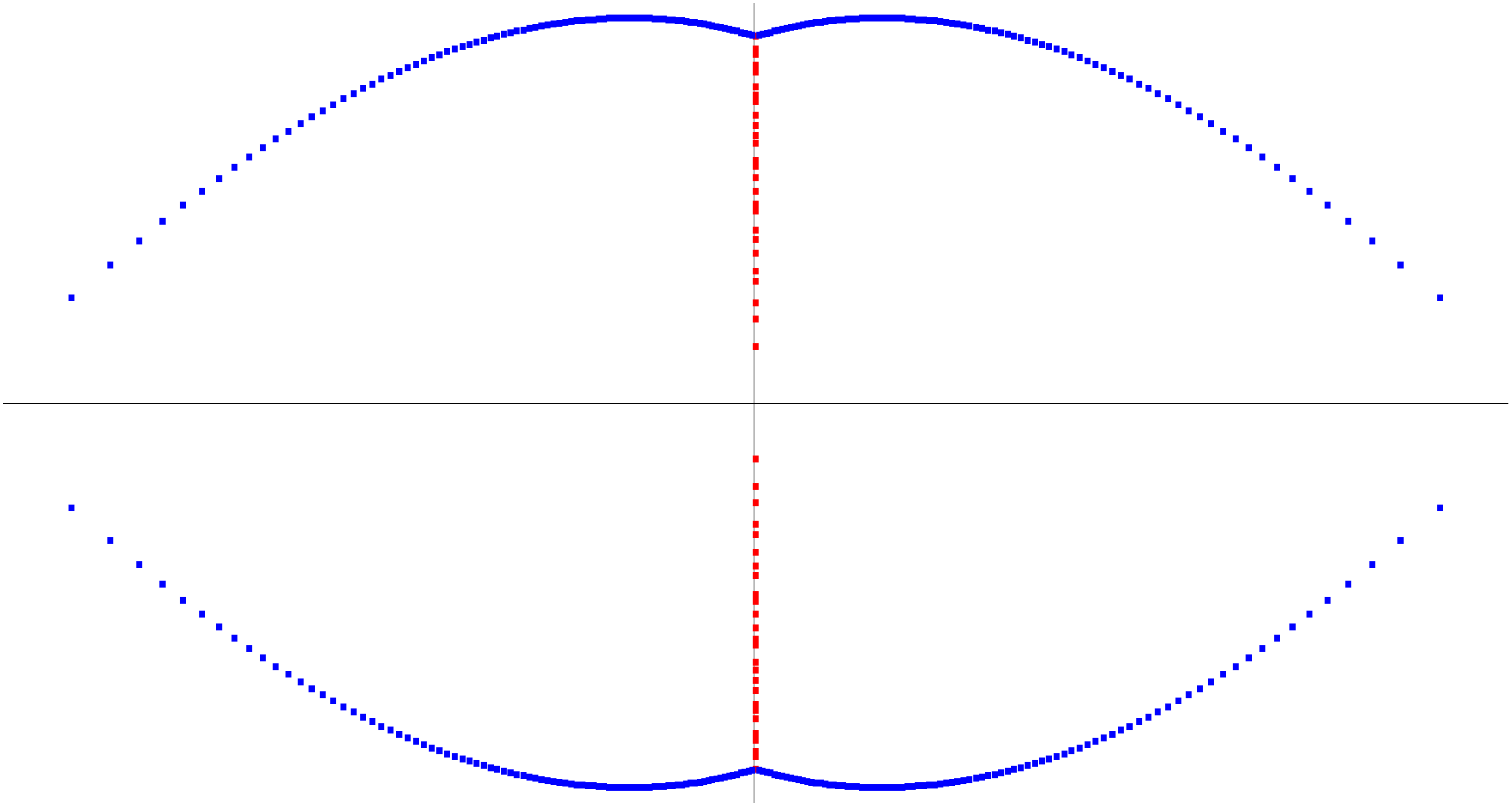
ROOTS OF THE DEGREE 400 TAYLOR POLYNOMIAL



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TAKEAWAY FROM THESE EXAMPLES

- Red roots are good approximations to geniune roots.
- Blue spurious roots are annoying and become more prevalent as the degrees increase.

JENSEN-PÓLYA PROGRAM

J. W. L. Jensen

(1859–1925)

George Pólya

(1887–1985)

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JENSEN-PÓLYA PROGRAM

Definition (Jensen)

If $a : \mathbb{N} \mapsto \mathbb{R}$ is an arithmetic function, then the **Jensen polynomial of degree** d **and shift** n is

$$J_a^{d,n}(X) := \sum_{j=0}^{d} \binom{d}{j} a(n+j) \cdot X^j.$$

Definition

A polynomial $f(X) \in \mathbb{R}[X]$ is **hyperbolic** if all of its roots are real.

JENSEN-PÓLYA PROGRAM

Theorem (Jensen-Pólya (1927))

With $z = -x^2$, define Taylor coefficients $\gamma(n)$

$$\Xi_1(x) = \frac{1}{8} \cdot \Xi\left(\frac{i}{2}\sqrt{x}\right) =: \sum_{n \geq 0} \frac{\gamma(n)}{n!} \cdot x^n.$$

RH **is equivalent to the hyperbolicity** *of all of the* $J_\gamma^{d,n}(X)$.

What was known?

1. *Chasse proved hyperbolicity for $d \leq 2 \cdot 10^{17}$ and $n = 0$.*
2. *The hyperbolicity is known for $d \leq 3$ by work of Csordas, Norfolk and Varga, and Dimitrov and Lucas.*
3. *Nothing for $d \geq 4$.*

OUR WORK ON RH & HERMITE DISTRIBUTIONS

Theorem 1 (Griffin, O, Rolen, Zagier)

For each degree $d \geq 1$ we have that

$$\lim_{n \to +\infty} \widehat{J}_{\gamma}^{d,n}(X) = H_d(X).$$

For each d, all but (possibly) finitely many $J_{\gamma}^{d,n}(X)$ are hyperbolic.

Theorem (Griffin, O, Thorner)

If $1 \leq d \leq 10^{20}$, then $J_{\gamma}^{d,n}(X)$ is hyperbolic **for all** n.

Theorem (Griffin, O, Rolen, Zagier)

GUE is true for the Riemann zeta-function in **derivative aspect**.