# From random permutations and random matrices to random growth: an invitation to the fascinating mathematics of the KPZ universality class

Riddhipratim Basu

International Centre for Theoretical Sciences
Tata Institute of Fundamental Research

Eigenfunctions Seminar, IISc
17 March, 2023

# Ulam and Hammersley

Stanisław Ulam

John Hammersley

# What is universality?

- The large scale behaviour of certain systems are same even though microscopic details differ.

For a sequence of independent and identically distributed random variables $X_1, X_2, \ldots,$ with mean $\mu$ finite variance $\sigma^2$

$$\frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\sigma} \Rightarrow N(0,1).$$

*Central limit theorem*

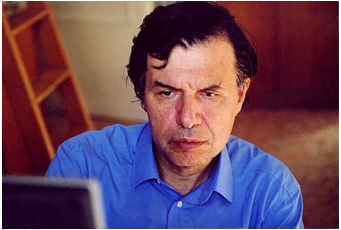
- More sophisticated: Donsker's invariance principle.
- Can be thought of as a one dimensional growth model.
- Many other examples: random matrices etc., not everything is Gaussian.

# A different universal behaviour for planar random growth

- LPP models do not exhibit Gaussian fluctuations.
- Their large scale behaviours are still expected to be universal, but now in a different universality class.

# The Kardar-Parisi-Zhang (KPZ) Universality Class

Mehran Kardar



Georgio Parisi

Yi-Cheng Zhang

# The KPZ equation and the universality class

VOLUME 56, NUMBER 9 PHYSICAL REVIEW LETTERS 3 MARCH 1986

**Dynamic Scaling of Growing Interfaces**

Mehran Kardar
*Physics Department, Harvard University, Cambridge, Massachusetts 02138*

Giorgio Parisi
*Physics Department, University of Rome, I-00173 Rome, Italy*

and

Yi-Cheng Zhang
*Physics Department, Brookhaven National Laboratory, Upton, New York 11973*

(Received 12 November 1985)

A model is proposed for the evolution of the profile of a growing interface. The deterministic growth is solved exactly, and exhibits nontrivial relaxation patterns. The stochastic version is studied by dynamic renormalization-group techniques and by mappings to Burgers's equation and to a random directed-polymer problem. The exact dynamic scaling form obtained for a one-dimensional interface is in excellent agreement with previous numerical simulations. Predictions are made for more dimensions.

PACS numbers: 05.70.Ln, 64.60.Ht, 68.35.Fx, 81.15.Jj

$$\frac{\partial}{\partial t}h(x,t) = \nu\frac{\partial^2}{\partial x^2}h(x,t) + \lambda\left(\frac{\partial}{\partial x}h(x,t)\right)^2 + \xi(x,t).$$

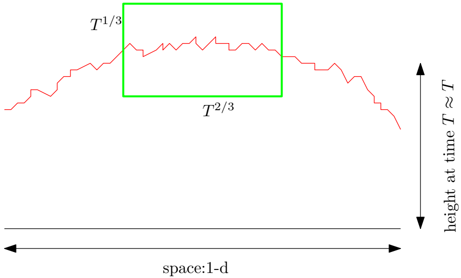
Kardar, Parisi, Zhang (1986)

$\xi :=$ independent space-time white noise.

# KPZ universality: predicted exponents

A non-rigorous renormalization group analysis suggests

- Scaling exponent of $1/3$ for fluctuation.
- Scaling exponent of $2/3$ for correlation length.



# The KPZ equation

$$\frac{\partial}{\partial t}h(x,t) = \frac{\partial^2}{\partial x^2}h(x,t) + (\frac{\partial}{\partial x}h(x,t))^2 + \xi(x,t).$$

- Ill-posed.
- Non-linear term creates the problem.
- Existence, uniqueness, regularity theory developed in Hairer's Fields medal winning works.

Martin Hairer

# Why do physicists care?

- Models are simple to describe and easy to simulate.
- Nonetheless their large scale behaviours empirically match the observed behaviour in many naturally occurring systems of stochastic growth.

# KPZ in real world: examples

1. Mutant bacterial colonies growing in a petri dish.

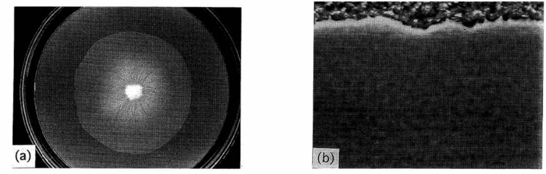



*Image source:* Wakita et. al. , *J. Phys. Soc. Japan*, (1997)

# KPZ in real world: examples

2. Edge of a slowly burning paper.

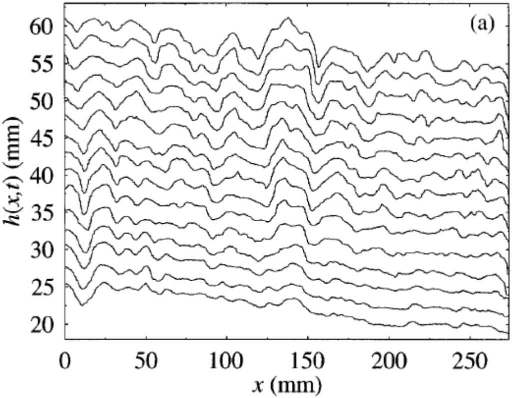



*Image source:* Maunuksela et al., *Phys. Rev. Lett.*, (1997)

# KPZ in real world: examples

3. Interface between dynamic scattering modes.

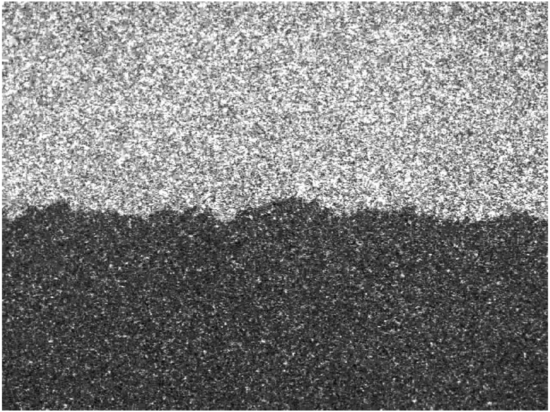


*Image:* Takeuchi et al., *Scientific Reports*, (2011)

# KPZ in real world: examples

4. Coffee ring effect with ellipsoidal particles.

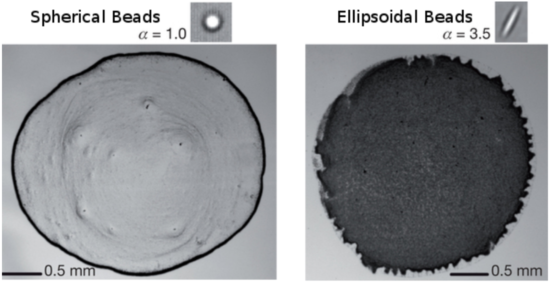



*Image:* Yunker et al., *Nature*, (2011)

# The Game of Tetris

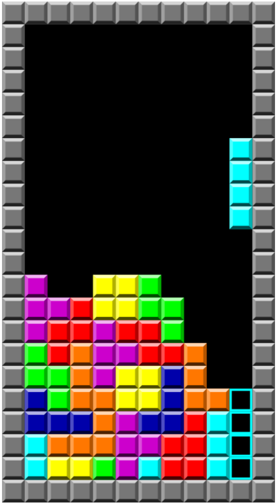


*Image source:* https://en.wikipedia.org/wiki/Tetris

# A mathematical formulation

- At each point of time, a random tetromino is chosen.
- It is given a random orientation.
- The tetromino is then dropped at a randomly chosen location.
- The tetromino sticks to the surface.
- No player intervention.

# How does the top envelope look after a long time?

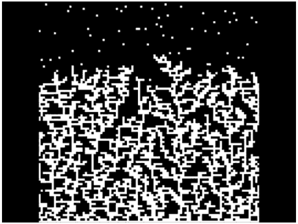


*Image source:*

https://mathsmartinthomas.files.wordpress.com/2017/08/stickytetris.gif

# How does the top envelope look after a long time?

https://www.ams.org/publications/journals/notices/201603/rnoti-p240.pdf

## Questions

- Consider the random interface given by the top envelope at some large time $t$.
- At time $t$, what is the average height of the profile at a given location?
- What is the order of fluctuation around the average?
- What is the correlation length, i.e., how far you need to move away in space so that the heights become independent?

# Why do mathematicians care?

- The problems are *very* hard.
- There are surprising connections to other sub-fields of probability and many different areas of mathematics in general, leading to some very interesting mathematics.
- Random matrix theory, interacting particle systems, partial differential equations, representation theory, algebraic combinatorics,....

# The KPZ revolution (1999– )

- For a handful of models of last passage percolation, there exist surprising bijections that lets one map the problem to a different object.
- Using this one can write down an explicit (but very complicated) formula for the last passage time for these *exactly solvable* models.
- Using such a formula the first rigorous proof of the $n^{1/3}$ fluctuations were given for Possonian last passage percolation by Baik-Deift-Johasson in 1999.
- Many more examples of exactly solvable models have been found since then, and tremendous progress in the understanding of their behaviour.

# The Baik-Deift-Johansson theorem

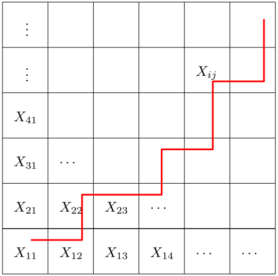
$$\frac{L_n - 2n}{n^{1/3}} \Rightarrow F_2.$$

$F_2$ is the GUE Tracy-Widom distribution from random matrix theory.

# Exponential LPP on $\mathbb{Z}^2$

- Put i.i.d. weights $X_v \sim \text{Exp}(1)$ on each vertex of $\mathbb{Z}^2$.
- Connections to Markovian corner growth, TASEP etc..
- This is an exactly solvable model.
- $\frac{T_{(nx,ny)}}{n} \to (\sqrt{x} + \sqrt{y})^2$.

Rost (1981)




$X_{ij} \sim$ i.i.d. Exponential Variables.

# Exponential LPP on $\mathbb{Z}^2$

- Using a variant of the Robinson- Schensted- Knuth (RSK) correspondence, one can explicitly write down a complicated formula for the distribution of $T_{n,n}$. It turns out that the distribution is the same as the distribution as the largest eigenvalue of a (complex) Gaussian Wishart matrix.; i.e. $X^*X$ where $X$ has i.i.d. complex Gaussian entries.
- Using the formula for the joint distribution of eigenvalues of a Wishart matrix/ a Fredholm determinant formula for the distribution of the largest eigenvalue, one can get the asymptotics of $T_{n,n}$.

# Exponential LPP: exact formula and estimates

- The joint density of eigenvalues is proportional to

$$\prod_{i<j} |\lambda_i - \lambda_j|^2 \prod_i \lambda_i^{(m-n)} e^{-\lambda_i}.$$

- Using this one can show $\frac{T_{\mathbf{0},\mathbf{n}} - 4n}{2^{4/3} n^{1/3}} \Rightarrow F_{\mathrm{GUE}}$. Johansson (1999)
- Moderate deviations are also known.

$$C' e^{-c' x^{3/2}} \leq \mathbb{P}(T_n \geq 4n + x n^{1/3}) \leq C e^{-c x^{3/2}}.$$

$$C' e^{-c' x^{3}} \leq \mathbb{P}(T_n \leq 4n - x n^{1/3}) \leq C e^{-c x^{3}}.$$

Ledoux, Rider (2010)

B., Ganguly, Hegde, Krishnapur (2021)

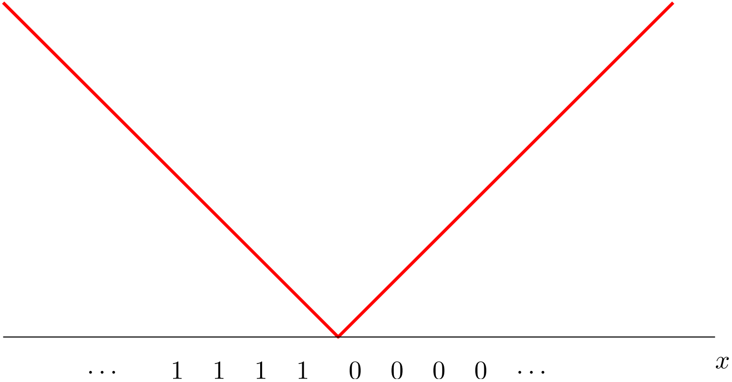
- Much more is known.

# Exactly solvable models

# Mathematical models for the KPZ growth

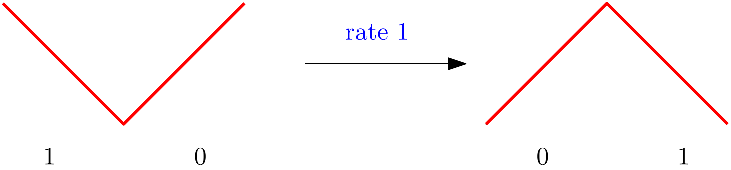
- There are many mathematical models of one dimensional randomly growing interfaces that satisfy the four conditions for the predicted KPZ growth.
- We have not yet managed to rigorously prove the predicted universal behaviour for a large class of such models.
- There are a few exactly solvable models which have remarkable connections to other branches of mathematics for which the predictions have been confirmed.

# Corner Growth Model



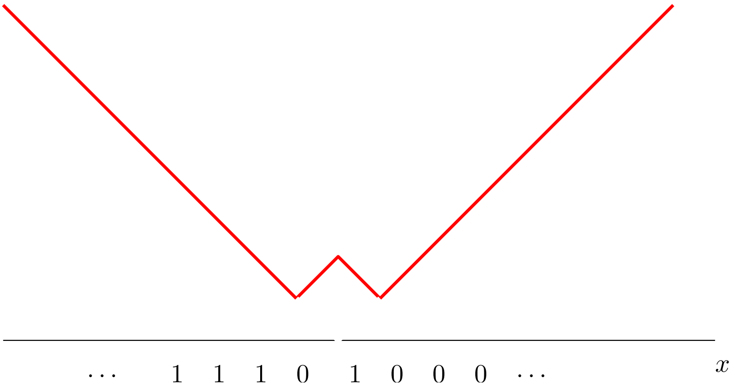

Wedge initial condition

# Corner Growth Model



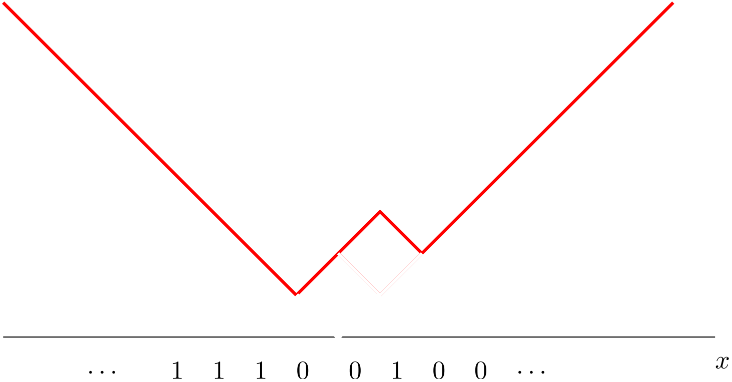

Corners are filled at rate 1

# Evolution in Corner Growth Model



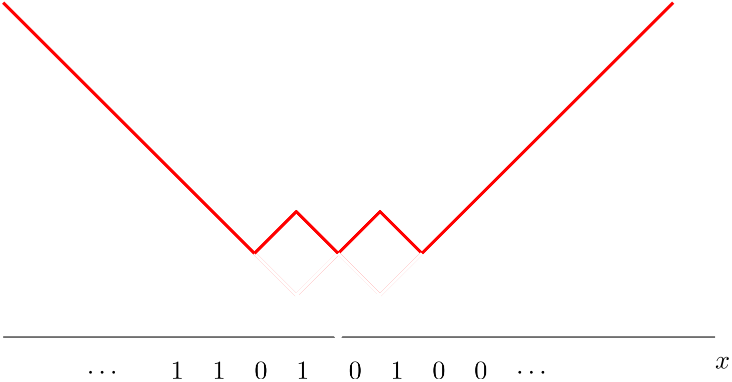

Evolution of height Function

# Evolution in Corner Growth Model



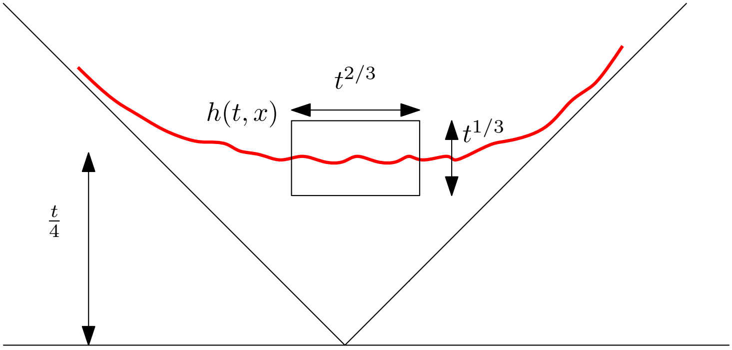

Evolution of height Function

# Evolution in Corner Growth Model




Evolution of height Function

# Interface at a large time



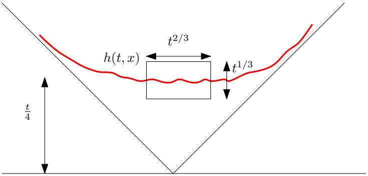

A snapshot at large $t$

# Large $t$ asymptotics

- One point weak convergence:

$$t^{-1/3}\left(h(t,0)-\frac{t}{4}\right)\Rightarrow F$$

as $t\to\infty$ where $F$ is a non-Gaussian universal distribution familiar in random matrix theory.

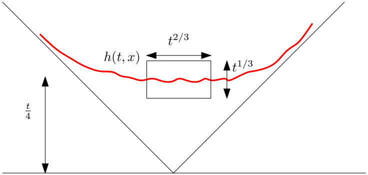


# Large $t$ asymptotics

- Process convergence:

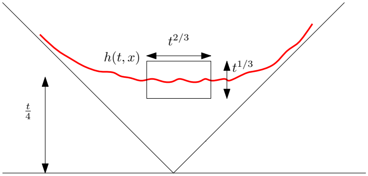
$$t^{-1/3}\left(h(t, xt^{2/3}) - \frac{t}{4}\right) \Rightarrow \mathcal{A}(x)$$

as $t \to \infty$ where $\mathcal{A}(\cdot)$ is a stationary stochastic process on $\mathbb{R}$ shifted by a parabola.



# Much more is known

- Different initial conditions.
- Correlations across time.
- Much more...



# What is so special about this model?

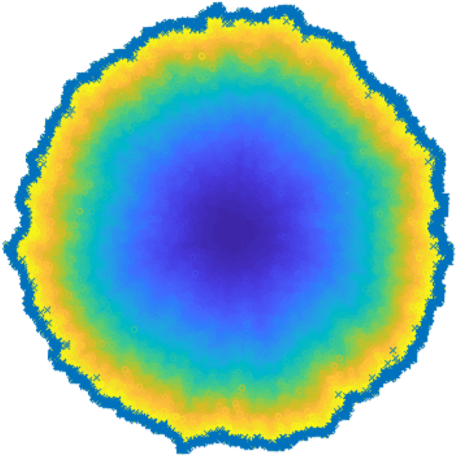
- Using a variant of the Robinson- Schensted- Knuth (RSK) correspondence, one can explicitly write down the density for the time it takes of the height at a given location to reach a given value.
- The formula is complicated, but has a surprising connection to eigenvalues of random matrices.
- Analysis of this (and other similar formulae) gives the one point and the process convergence results.

# Major Challenges: Non-integrable models

# First passage percolation

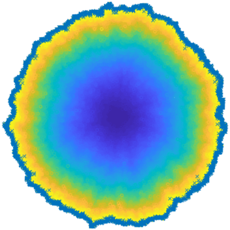
- Consider the following simple model of bacterial growth on $\mathbb{Z}^2$.
- At time 0, the colony consists of only the vertex $(0, 0)$.
- After each unit time, the colony expands by the vertex along a uniformly chosen boundary edge.
- First passage percolation: put i.i.d. weights on edges and consider the weight of the minimum weight path between two vertices.
- It is believed that this model belongs to the KPZ universality class.
- It is however not known to be exactly solvable- no exact formula available.

# First passage percolation



# Much less is known

- Linear growth and law of large numbers is known (shape theorem).
- For fluctuations around the limit shape, one only knows an upper bound of $O(t^{1/2+o(1)})$.



# Summary

- There is non-trivial universal behaviour exhibited by many naturally occurring growing interfaces.
- KPZ universality aims to explain this behaviour.
- The KPZ prediction has been verified for a handful of exactly solvable models based on some remarkable connections.
- Non-integrable models remain a major mathematical challenge.
- An active area of research and lots of interesting mathematics.

# Thank You

Questions?