Non-normal matrices: spectral instability, pseudospectrum, and random perturbation

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Non-normal operators

Normal operator/matrix: $NN^\star = N^\star N$;
Non-normal: $NN^\star \neq N^\star N$.

Examples of non-normal operators/matrices:

- Kramers-Fokker-Planck type operators
- PDE solvability theory
- Damped wave equations
- Open quantum systems
- Scattering theory - long term behavior of a quantum particle
- Linearized operators from models in fluid dynamics
- Evolution driven by non-normal operators

Spectral instability of non-normal operators

For any bounded normal operator N

$$\| (N - z)^{-1} \| = \frac{1}{\text{dist}(z, \text{Spec}(N))}, \qquad z \notin \text{Spec}(N).$$

Spectral instability of non-normal operators

For a non-normal operator N and $z \notin \mathrm{Spec}(N)$ one has
either

$$\|(N-z)^{-1}\| \asymp \frac{1}{\mathrm{dist}(z, \mathrm{Spec}(N))}$$

(zone of spectral stability)

or

$$\|(N-z)^{-1}\| \gg \frac{1}{\mathrm{dist}(z, \mathrm{Spec}(N))}.$$

(zone of spectral instability)

Spectral instability of non-normal operators

Example: Left shift operator on $\mathbb{C}^N$ / Jordan block

$$J_N := \begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & \ddots & \ddots & & \\ & & & 0 & 1 & \\ & & & & 0 & 1 \\ & & & & & 0 \end{bmatrix}, \qquad \mathrm{Spec}(J_N) = \{0\}.$$

Zone of spectral instability: For $z \in D(0,1) := \{w \in \mathbb{C} : |w| < 1\}$

$$\|(J_N - z)v\|_2 = |z|^N \Rightarrow \|(J_N - z)^{-1}\| \geqslant |z|^{-N}$$

$$v := \begin{pmatrix} 1 & z & z^2 & \cdots & z^{N-1} \end{pmatrix}^{\mathsf{T}} \qquad \Rightarrow \qquad \|v\|_2 \asymp 1.$$

Zone of spectral stability: For $z \in \mathbb{C} \setminus \overline{D(0,1)}$.

$$\|(J_N - z)^{-1}\| \asymp 1.$$

Spectral instability of non-normal operators

$$J_N - z := \begin{bmatrix} -z & 1 & & & & \\ & -z & 1 & & & \\ & & \ddots & \ddots & & \\ & & & -z & 1 & \\ & & & & -z & 1 \\ & & & & & -z \end{bmatrix}.$$

Zone of spectral instability: For $z \in D(0,1) := \{w \in \mathbb{C} : |w| < 1\}$

$$\|(J_N - z)v\|_2 = |z|^N \Rightarrow \|(J_N - z)^{-1}\| \geqslant |z|^{-N}$$

$$v := \begin{pmatrix} 1 & z & z^2 & \cdots & z^{N-1} \end{pmatrix}^{\mathsf{T}} \quad \Rightarrow \quad \|v\|_2 \asymp 1.$$

Zone of spectral stability: For $z \in \mathbb{C} \setminus \overline{D(0,1)}$.

$$\|(J_N - z)^{-1}\| \asymp 1.$$

Challenges with non-normal matrices

(i) The eigenvalue analysis in many applications turns out to be misleading.

(ii) The eigenvalues are sensitive to perturbations and thereby often yielding unreliable results.

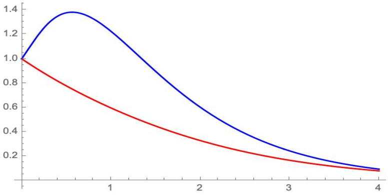
Challenges with non-normal matrices

Example 1. Set $f_A(t) := \|\exp(tA)\|$, $f_B(t) := \|\exp(tB)\|$, $t \geqslant 0$ ($\|\cdot\|$ denotes the operator norm),

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \text{ and } B = \begin{pmatrix} -1 & 5 \\ 0 & -2 \end{pmatrix}.$$

- For large t's the slopes of the curves are determined via an eigenvalue analysis.
- Slopes for $t \asymp 1$?

Challenges with non-normal matrices



$$\color{red}{A} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

$$\color{blue}{B} = \begin{pmatrix} -1 & 5 \\ 0 & -2 \end{pmatrix}$$

• The 'hump'-like structure of the curve $\{f_B(t)\}_{t \geqslant 0}$ cannot be explained solely by the eigenvalues of B.

• Such hump-like structure are ubiquitous in dynamical systems, commonly known as the transient behaviors.

(Example taken from the book by Trefethen and Embree)

Challenges with non-normal matrices

Example 2. Simulate a uniformly random unitary matrix U_N and set $\widehat{J}_N := U_N J_N U_N^*$. $\mathrm{Spec}(J_N) = \mathrm{Spec}(\widehat{J}_N) = \{0\}$.

$$J_N := \begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & \ddots & \ddots & & \\ & & & 0 & 1 & \\ & & & & 0 & 1 \\ & & & & & 0 \end{bmatrix}$$

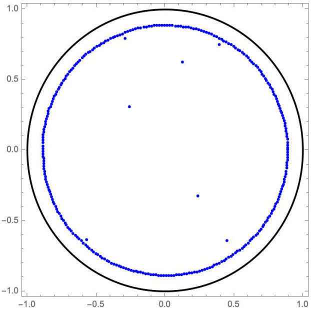
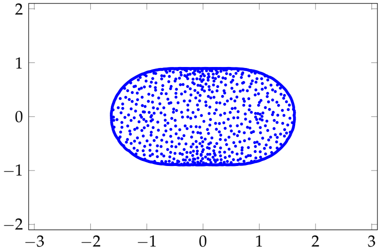


Figure: $N = 1000$. Eigenvalues of $\widehat{J}_N$ computed through Mathematica are plotted in blue and the unit circle $\mathbb{S}^1$ on the complex plane is in black.

Example 3. Simulate a Haar U_N. Compute the eigenvalues of $U_N H_N U_N^*$. $N = 1000$.

$$H_N := J_N + D_N$$
$$D_N = \operatorname{diag}(\{d_i\}_{i=1}^N)$$
$$d_i = -2 + \frac{4i}{N}, i = 1, 2, \ldots, N$$



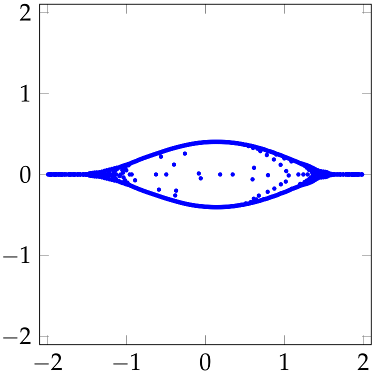
Twisted Toeplitz / Toeplitz with variable coefficients

Example 4. Simulate a Haar U_N. Compute the eigenvalues of $U_N \widetilde{H}_N U_N^*$. $N = 1000$.

$$\widetilde{H}_N := J_N + \widetilde{D}_N$$

$$\widetilde{D}_N = \mathrm{diag}(\{X_i\}_{i=1}^N)$$

$\{X_i\}$ i.i.d. $\mathrm{Unif}[-2, 2]$



Non-periodic one-way model – "limit" of Hatano-Nelson model

(due to Brézin, Feinberg, and Zee)

Eigenvalues move to the 'Hatano-Nelson bubble'

Challenges with non-normal matrices

Remark. Recall $H_N = J_N + D_N$ and $\widetilde{H}_N = J_N + \widetilde{D}_N$, with

$$D_N = \operatorname{diag}(\{d_i\}), \qquad d_i = -2 + \frac{4i}{N}, i = 1, 2, \ldots, N,$$
$$\widetilde{D}_N := \operatorname{diag}(\{X_i\}), \qquad \{X_i\} \text{ i.i.d. } \operatorname{Unif}[-2, 2].$$

Hence

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i(H_N)} \Rightarrow \operatorname{Unif}[-2, 2], \quad \text{and} \quad \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i(\widetilde{H}_N)} \Rightarrow \operatorname{Unif}[-2, 2].$$

However, simulated spectrums of $U_N H_N U_N^*$ and $U_N \widetilde{H}_N U_N^*$ are completely different.

Pseudospectrum

ε-pseudospectrum $(\varepsilon > 0)$

$$(1).\ \mathrm{Spec}_\varepsilon(A) := \mathrm{Spec}(A) \cup \left\{z \in \mathbb{C}\setminus\mathrm{Spec}(A) : \|(A-z)^{-1}\| \geqslant \varepsilon^{-1}\right\}$$

$$(2).\ \mathrm{Spec}_\varepsilon(A) = \bigcup_{\|E\|\leqslant\varepsilon} \mathrm{Spec}(A+E)$$

$(3).$ $z \in \mathrm{Spec}_\varepsilon(A) \Leftrightarrow z \in \mathrm{Spec}(A)$ or $\exists v_z$ s.t. $\|(A-z)v_z\| \leqslant \varepsilon\, \|v_z\|$

$$(1) \Leftrightarrow (2) \Leftrightarrow (3)$$

[Varah '79], [Trefethen, Embree '05]

Pseudospectrum

For any $A \in \mathbb{C}^{N \times N}$ and any $\varepsilon > 0$

$$\text{Spec}_\varepsilon(A) \supset \text{Spec}(A) + D(0, \varepsilon).$$

If $\| \cdot \| = \| \cdot \|_2$ and $A \in \mathbb{C}^{N \times N}$ then

$$A \text{ normal } \Leftrightarrow \text{Spec}_\varepsilon(A) = \text{Spec}(A) + D(0, \varepsilon) \, \forall \varepsilon > 0.$$

More generally, if $A = V \Lambda V^{-1}$ is diagonalizable then

$$\text{Spec}_\varepsilon(A) \subset \text{Spec}(A) + D(0, \varepsilon \kappa(V)), \quad \kappa(V) := \frac{s_{\max}(V)}{s_{\min}(V)}.$$

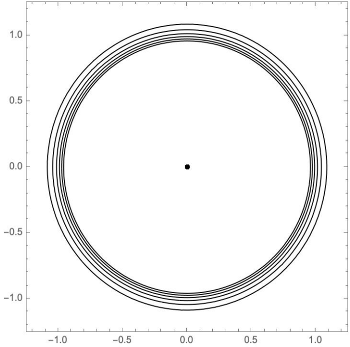
Example 2 (revisited): For any $\delta = \delta_N > 0$ let

$$J_N^{(\delta)} := \begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & \ddots & \ddots & & \\ & & & 0 & 1 & \\ & & & & 0 & 1 \\ \delta & & & & & 0 \end{bmatrix}.$$

Observe: Eigenvalues of $J_N^{(\delta)} = \{\delta^{1/N} e^{2\pi i k/N}, k \in [0, N-1] \cap \mathbb{Z}\}$.
Therefore

- If $\delta = |z|^N$ for some $z \in D(0,1)$ then an exponentially small perturbation of J_N produces eigenvalues that are at a distance $|z|$ from $\mathrm{Spec}(J_N)$. Thus $\mathrm{Spec}_{r^N}(J_N) \supset D(0,r)$ for any $r \in (0,1)$.
- If $\delta \asymp 1$ or if $\delta = O(N^{-\alpha})$ for any $\alpha > 0$ then eigenvalues of $J_N^{(\delta)}$ approaches $\mathbb{S}^1 := \partial D(0,1)$.

Example 2 (continued):



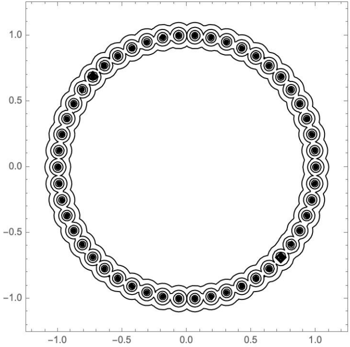


Figure: $N = 50$, $\varepsilon = 10^{-1}, 10^{-1.2}, \ldots, 10^{-2}$. Pseudospectral level lines: J_N on the left panel, $C_N := J_N^{(1)}$ on the right panel.

Examples 3 and 4 (revisited):

$$H_N := \begin{bmatrix} -2+\frac{4}{N} & 1 & & & \\ & -2+\frac{8}{N} & 1 & & \\ & & \ddots & \ddots & \\ & & & 2-\frac{2}{N} & 1 \\ & & & & 2 \end{bmatrix} \widetilde{H}_N := \begin{bmatrix} X_1 & 1 & & & \\ & X_2 & 1 & & \\ & & \ddots & \ddots & \\ & & & X_{N-1} & 1 \\ & & & & X_N \end{bmatrix}$$

$\{X_i\}$ are i.i.d. $\mathrm{Unif}[-2, 2]$.

Examples 3 and 4 (revisited):

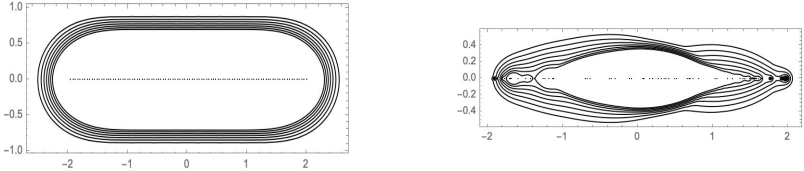
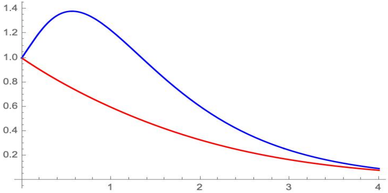


Figure: $N = 100$, $\varepsilon = 10^{-2}, 10^{-2.4}, \ldots, 10^{-4.4}$. Pseudospectral level lines: H_N on the left panel, $\widetilde{H}_N$ on the right panel.

Pseudospectrum

Example 1 (revisited):



$$A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

$$B = \begin{pmatrix} -1 & 5 \\ 0 & -2 \end{pmatrix}$$

Example 1 (revisited):

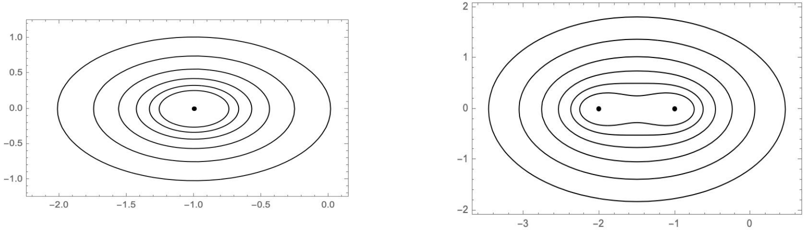


Figure: $\varepsilon = 10^{-0.2}, 10^{-0.4}, \ldots, 10^{-1.2}$. Pseudospectral level lines: A on the left panel, B on the right panel.

Real life implications: Onset of turbulence in the plane Couette flow at a high Reynolds number.

Spectrum of the Navier-Stokes evolution operator linearized about the laminar flow contained in the left half of the plane. For a sufficiently large Reynolds number and a small $\varepsilon > 0$ its ε-pseudospectrum protrudes a distance 'much' greater than ε into the right half plane, and as a result certain perturbations of the plane Couette flow grow transiently at that high Reynolds number eventually decaying due to viscosity.

Move from pseudospectrum to random perturbation

- Pseudospectra are generally harder to characterize and computationally more expensive.
- Random perturbation is an efficient model.
  - The pseudospectrum measures how much one can move the spectrum by a worst-case perturbation.
  - In many physical models the perturbation of an operator is generally induced by sources that are primarily uncontrolled by experimentalists.
  - Natural to study spectral features of disordered perturbations of a non-normal operators/matrices, e.g. open quantum systems.
  - If the simulated $U_N = \mathcal{U}_N + \Delta_N$, where $\mathcal{U}_N$ is a 'true' unitary and Δ_N captures the machine/rounding error then the spectrum of $\widehat{A}_N := U_N A_N U_N^\star$ is same as that of $A_N + \widehat{\Delta}_N$.

Example. For $\boldsymbol{a}(\xi) := \sum_{i=-d_-}^{d_+} a_i \xi^i$, with $\xi \in \mathbb{S}^1$, set

$$T_N(\boldsymbol{a}) := \sum_{i \geqslant 0} a_i J_N^i + \sum_{i<0} a_i (J_N^\star)^i.$$

For $\boldsymbol{a}(\xi) = 2\xi^{-3} - \xi^{-2} + 2\iota\xi^{-1} - 4\xi - 2\iota\xi^2$

$$T_N(\boldsymbol{a}) := \begin{bmatrix} 0 & -4 & -2\iota & & & & & \\ 2\iota & 0 & -4 & -2\iota & & & & \\ -1 & 2\iota & 0 & -4 & -2\iota & & & \\ 2 & -1 & 2\iota & 0 & -4 & -2\iota & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & & & & 0 & -4 & -2\iota \\ & & & 2 & -1 & 2\iota & 0 & -4 \\ & & & & 2 & -1 & 2\iota & 0 \end{bmatrix}.$$

Random perturbations of non-normal matrices

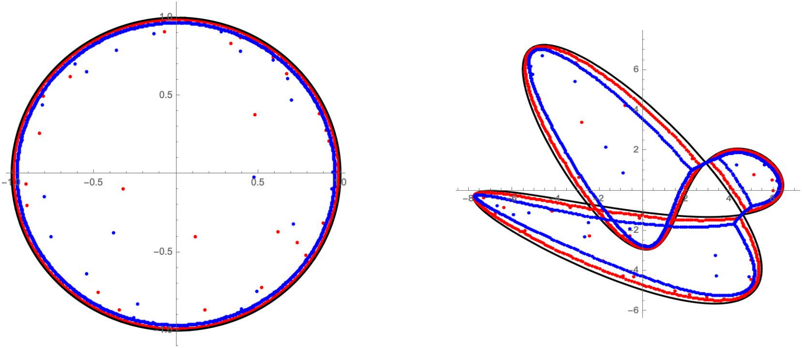


Figure: $N = 1000$. Eigenvalues of $U_N A_N U_N^\star$, U_N a simulated Haar unitary, computed through Mathematica are in blue. Eigenvalues of $A_N + N^{-2} G_N$ are in red, where G_N is the random matrix with i.i.d. standard complex Gaussian entries. Left panel: $A_N = J_N$, and right panel: $A_N = T_N(\boldsymbol{a})$. Symbol curves $\mathbb{S}^1$ (left panel) and $\boldsymbol{a}(\mathbb{S}^1)$ (right panel) in black.

Random perturbations of non-normal matrices

Examples 3 and 4 (revsiting again).

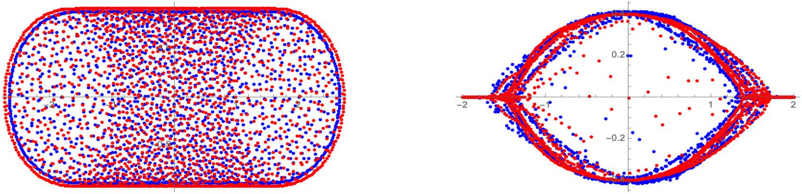


Figure: $N = 2000$. Eigenvalues of $U_N A_N U_N^\star$, U_N a simulated Haar unitary, computed through Mathematica are in blue. Eigenvalues of $A_N + N^{-3} G_N$ are in red, where G_N is the random matrix with i.i.d. standard complex Gaussian entries. Left panel: $A_N = H_N$, and right panel: $A_N = \widetilde{H}_N$. .

Setup:

- A_N an $N \times N$ non-normal matrix.
- E_N is a random matrix with entries that are of $O(1)$.

(e.g. i.i.d. Gaussian entries)

- Consider $A_N + N^{-\gamma} E_N$ for $\gamma > 1/2$.

Observe $\gamma > 1/2$ is necessary. Since $\|E_N\| \asymp N^{1/2}$.

Questions

- Limit of the bulk of the eigenvalues. How does it depend on "$\lim_{N\to\infty} A_N$"? Universal w.r.t. to the distribution of E_N? W.r.t. γ?

$$L_N := \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i}.$$

- Are there outliers?

  stray eigenvalues away from the support of the limiting measure

  If so, what is the limit (of the random point process)? Universal/non-universal?

- How do eigenvectors look like? Localization/delocalization? Quantum unique ergodicity?

Random perturbations of non-self-adjoint operators

- Non-self-adjoint (semiclassical) pseudodifferential operators
  - probabilistic Weyl law

    [Hager '06], [Hager, Sjöstrand '08], [Sjöstrand '08, '09]

    [Bordeaux, Montrieux '08]
  - local eigenvalue statistics

    [Nonenmacher, Vogel '17]
- Twisted Toeplitz matrices/Berezin-Toeplitz quantization of smooth functions on torus

  [Christiansen, Zworski '10], [B., Paquette, Zeitouni '19]

  [Vogel '20]
- Random bi-diagonal matrix/one-way model

  [B., Paquette, Zeitouni '19]

Random perturbations of non-self-adjoint operators

- Non-self-adjoint Toeplitz matrices
  - probabilistic Weyl law/asymptotic eigenvalue density
    [Hager, Davies '09], [Guionnet, Wood, Zeitouni '14]
    [B., Paquette, Zeitouni '19, '20], [Sjöstrand, Vogel '21a, '21b]
    [O'Rourke, Wood '22]
  - rate of convergence, local law
    [O'Rourke, Wood '22]
  - limit of point process induced by outlier eigenvalues
    [Sjöstrand, Vogel '17a, '17b], [B., Zeitouni '20]
  - localization/scarring of eigenvectors
    [B., Vogel, Zeitouni '23]

Spectrum of random perturbation of Toeplitz matrices

$$T_N(\boldsymbol{a}) = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & \cdots & a_{N-1} \\ a_{-1} & a_0 & a_1 & \ddots & & \vdots \\ a_{-2} & a_{-1} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & a_1 & a_2 \\ \vdots & & \ddots & a_{-1} & a_0 & a_1 \\ a_{-(N-1)} & \cdots & \cdots & a_{-2} & a_{-1} & a_0 \end{bmatrix}, \ a_i \in \mathbb{C}.$$

$$T_N(\boldsymbol{a}) = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & \cdots & a_{N-1} \\ a_{-1} & a_0 & a_1 & \ddots & & \vdots \\ a_{-2} & a_{-1} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & a_1 & a_2 \\ \vdots & & \ddots & a_{-1} & a_0 & a_1 \\ a_{-(N-1)} & \cdots & \cdots & a_{-2} & a_{-1} & a_0 \end{bmatrix}, \ a_i \in \mathbb{C}.$$

$T_N(\boldsymbol{a})$ finitely banded if $a_i = 0$ for $i \geqslant d_1 + 1$ and $i \leqslant -(d_2 + 1)$ for some $d_1, d_2 \geqslant 0$.

$$T_N(\boldsymbol{a}) = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & \cdots & a_{N-1} \\ a_{-1} & a_0 & a_1 & \ddots & & \vdots \\ a_{-2} & a_{-1} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & a_1 & a_2 \\ \vdots & & \ddots & a_{-1} & a_0 & a_1 \\ a_{-(N-1)} & \cdots & \cdots & a_{-2} & a_{-1} & a_0 \end{bmatrix}, \; a_i \in \mathbb{C}.$$

- $T_N(\boldsymbol{a})$ can be viewed as a finite dimensional version of an infinite dimensional matrix/operator $T(\boldsymbol{a})$.

$$T_N(\boldsymbol{a}) = \mathbf{1}_{[1,N]\cap\mathbb{N}} T(\boldsymbol{a}) \mathbf{1}_{[1,N]\cap\mathbb{N}}$$

- The symbol of $T(\boldsymbol{a})$/ $T_N(\boldsymbol{a})$ is $\boldsymbol{a}$.

$$\boldsymbol{a}(\xi) := \sum_{k=-\infty}^{\infty} a_k \xi^k, \qquad \xi \in \mathbb{S}^1.$$

▶ If $T(\boldsymbol{a})$ (or equivalently $T_N(\boldsymbol{a})$) if finitely banded then $\boldsymbol{a}$ is a Laurent polynomial.

$$\boldsymbol{a}(\xi) = \sum_{k=-d_2}^{d_1} a_k \xi^k.$$

Examples.

- $T_N(\boldsymbol{a}) = J_N \Leftrightarrow \boldsymbol{a}(\xi) = \xi$.
- $T_N(\boldsymbol{a}) = J_N + J_N^2 \Leftrightarrow \boldsymbol{a}(\xi) = \xi + \xi^2$.
- $T_N(\boldsymbol{a}) = 2(J_N^3)^\star - (J_N^2)^\star + 2\iota J_N^\star - 4J_N - 2\iota J_N^2 \Leftrightarrow \boldsymbol{a}(\xi) = 2\xi^{-3} - \xi^{-2} + 2\iota\xi^{-1} - 4\xi - 2\iota\xi^2$.

Limit of the bulk of the spectrum

Theorem (B., Paquette, Zeitouni '19, '20)

For any $\gamma > \frac{1}{2}$, if E_N satisfies Assumption (A) then the empirical distribution of the eigenvalues of $T_N + N^{-\gamma} E_N$ converges weakly, in probability, to the law of $\boldsymbol{a}(U)$ where $U \sim \mathrm{Unif}(\mathbb{S}^1)$.

(also follows from [O'Rourke, Wood '22])

For any $f \in C_b(\mathbb{C})$

$$\frac{1}{N} \sum_{i=1}^{N} f(\lambda_i) \to \frac{1}{2\pi} \int_0^{2\pi} f(\boldsymbol{a}(e^{\mathrm{i}\theta})) d\theta, \qquad \text{in probability.}$$

Limit of the bulk of the spectrum

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Examples.

$$T_N = J_N,\ \boldsymbol{a}(\xi) = \xi.\ L_N \Rightarrow \text{law of } U, \text{ where } U \sim \mathrm{Unif}(\mathbb{S}^1).$$

$$T_N = J_N + J_N^2,\ \boldsymbol{a}(\xi) = \xi + \xi^2.\ L_N \Rightarrow \text{law of } U + U^2.$$

Limit of the bulk of the spectrum

Theorem (B., Paquette, Zeitouni '19, '20)

For any $\gamma > \frac{1}{2}$, if E_N satisfies Assumption (A) then the empirical distribution of the eigenvalues of $T_N + N^{-\gamma} E_N$ converges weakly, in probability, to the law of $\boldsymbol{a}(U)$ where $U \sim \mathrm{Unif}(\mathbb{S}^1)$.

Assumption (A)

(1)

$$\mathbb{E}\left[\|E_N\|_{\mathrm{HS}}^2\right] = \mathbb{E}\left[\sum_{i,j} |e_{i,j}|^2\right] = O(N^2).$$

(2) (Technical condition) For every $\alpha > 0 \; \exists \, \beta \in (0, \infty)$, such that for any M_N with $\|M_N\| = O(N^\alpha)$,

$$\mathbb{P}\left(s_{\min}(M_N + E_N) \leqslant N^{-\beta}\right) = o(1).$$

Matrices satisfying Assumption (A)

- The entries of E_N are i.i.d. with finite second moment.

  follows from [Tao-Vu '08]

- $E_N = \sqrt{N} U_N$, where U_N is Haar Unitary.

  follows from [Rudelson-Vershynin '14]

- The entries of E_N are independent, satisfy a uniform anti-concentration bound near zero, and have uniform lower bound on the truncated variance.

  [Bordenave-Chafaï '12]

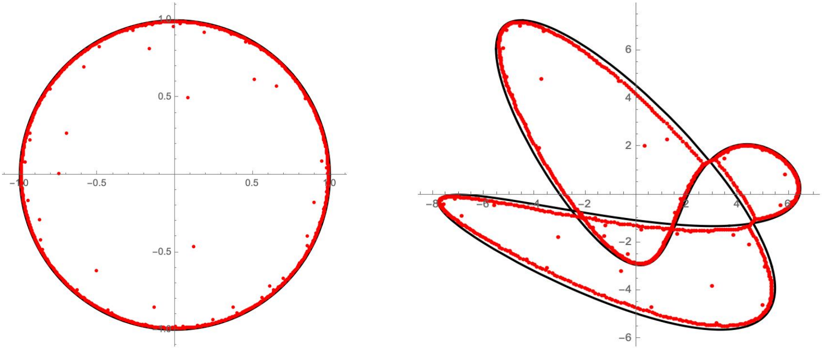
- The entries of E_N have an inhomogeneous variance profile satisfying some appropriate assumptions.

  [Cook '16]

- E_N can also be sparse random matrix.

  [Tao-Vu '08]

Regions of no outliers



Theorem (B., Zeitouni '20)

The entries of E_N are independent entries with zero mean and unit variance. Then for any $\gamma > \frac{1}{2}$, with probability $\to 1$, there are no outliers in any open set

$$U \subsetneq \mathcal{R}_0 := \{z \in \mathbb{C} \backslash \boldsymbol{a}(\mathbb{S}^1) : \mathrm{wind}_{\boldsymbol{a}}(z) = 0\}.$$

Limit of outliers

Theorem (B., Zeitouni '20)

Additionally assume that E_N be a random matrix with i.i.d. entries having zero mean and unit variance and satisfying some anti-concentration bound (e.g. bounded density). Then for any $\gamma > \frac{1}{2}$, the point processes induced by the outlier eigenvalues converge to the zero set of some non-universal (w.r.t. the distribution of the entries of E_N) random analytic function.

Definition of the limiting random analytic function involves skew semistandard Young Tableaux

$T_N = J_N$, entries of E_N are standard complex Gaussian

Limiting random analytic function is a hyperbolic Gaussian analytic function:

$$F(z) = \sum_{\ell=0}^{\infty} g_\ell z^\ell \sqrt{\ell + 1}$$

$\{g_\ell\}$ i.i.d. standard complex Gaussian

Limit of outliers: The Limaçon

$T_N = J_N + J_N^2$, entries of E_N are standard complex Gaussian

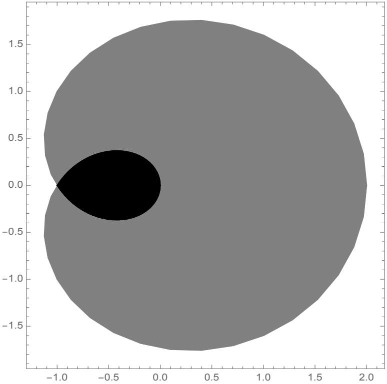


Figure: Three regions: $\mathcal{R}_2$ in black, $\mathcal{R}_1$ in grey, and $\mathcal{R}_0$ in white. For $z \in \mathcal{R}_\ell$ (i) $\mathrm{wind}(z) = \ell$ and (ii) ℓ roots of the equation $\boldsymbol{a}_z(\xi) := \xi + \xi^2 - z = 0$ that are less than one in moduli.

Limit of outliers: The Limaçon

$T_N = J_N + J_N^2$, entries of E_N are standard complex Gaussian

For $z \in \mathcal{R}_1$, the limiting random function is given by

$$F(z) = \sum_{\ell=0}^{\infty} g_\ell \xi_-(z)^\ell \sqrt{\ell + 1}$$

$\{g_\ell\}$ i.i.d. complex standard Gaussian

$\xi_\pm(z)$ are the roots $\boldsymbol{a}_\xi(z) = 0$ with $|\xi_-(z)| < |\xi_+(z)|$

For $z \in \mathcal{R}_2$, the limiting random function is given by

$$F(z) = \sum_{i<j, k<\ell} C_{i,j,k,\ell}(z) \cdot (g_{i,k} g_{j,\ell} - g_{i,\ell} g_{j,k})$$

$\{g_{\ell,\ell'}\}$ i.i.d. complex standard Gaussian

Localization/delocalization of eigenvectors

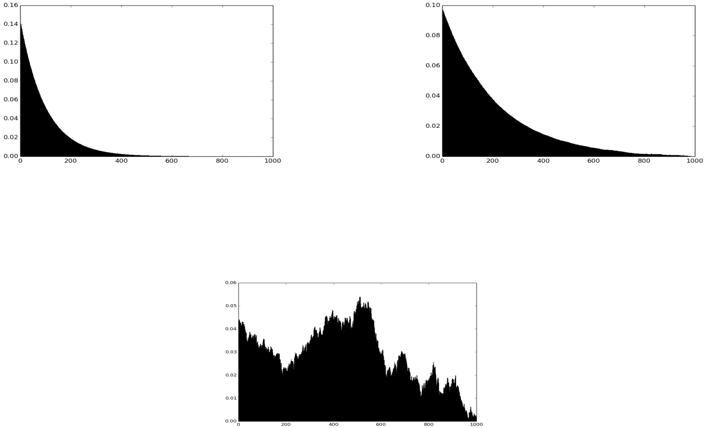


Figure: Moduli of the entries of an eigenvector of $J_N + N^{-\gamma} E_N$: $N = 1000$; top left: $\gamma = 2$, top right: $\gamma = 1.5$, bottom: $\gamma = 1$.

Localization/delocalization of eigenvectors

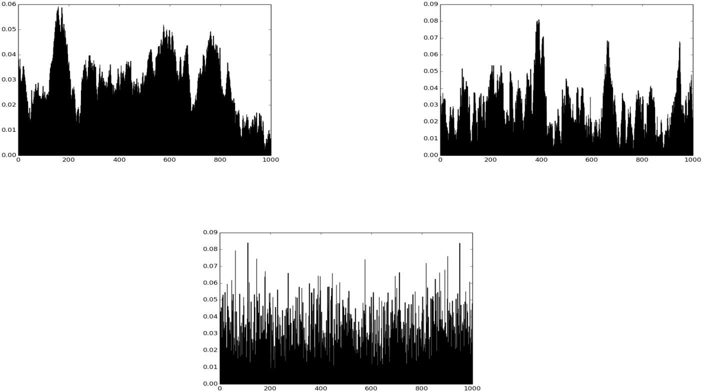


Figure: Moduli of the entries of an eigenvector of $J_N + N^{-\gamma} E_N$: $N = 1000$; top left: $\gamma = 0.9$, top right: $\gamma = 0.75$, bottom: $\gamma = 0.4$.

Localization of eigenvectors for $\gamma > 1$

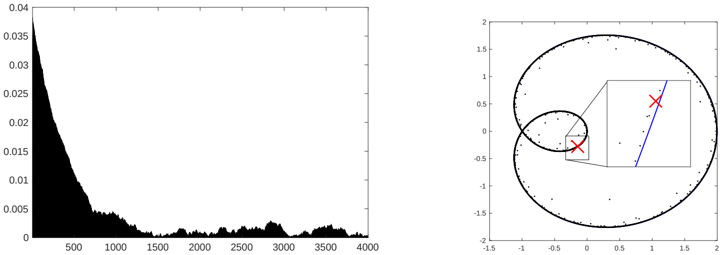
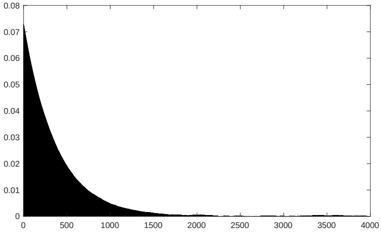
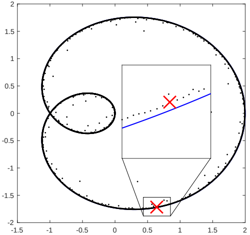


Figure: Eigenvectors (left panel) and eigenvalues (right panel) of $J_N + J_N^2 + N^{-\gamma} E_N$ for $N = 4000$, $\gamma = 1.2$. Plotted are the moduli of the entries of the eigenvector that corresponds to the eigenvalue marked with a red $\times$.

Localization of eigenvectors for $\gamma > 1$





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Localization of eigenvectors for $\gamma > 1$

Theorem (B., Vogel, Zeitouni '23)

For most (right)-eigenvectors v, with probability $\to 1$, as $N \to \infty$, (under some assumptions on E_N) the followings hold:

- *Localization at scale $N/\log N$:* *For any $\ell \in [1, N] \cap \mathbb{Z}$*

$$\|v\|_{\ell^2([1,N-\ell])} \wedge \|v\|_{\ell^2([\ell,N])} \lesssim \exp(-c\ell \log N/N) + N^{-c'}$$

- *Eigenvectors spread out at scale $N/\log N$:*

$$|\mathrm{Supp}(v)| \gtrsim N/\log N$$

$$|\mathrm{Supp}(v)| := \min\{|I| : \|v\|_{\ell^2(I)} \gtrsim 1\}$$

Delocalization of eigenvectors for $\gamma < 1$

We expect a long-range correlation and some form of quantum unique ergodicity.

Work in progress with Vogel and Zeitouni.

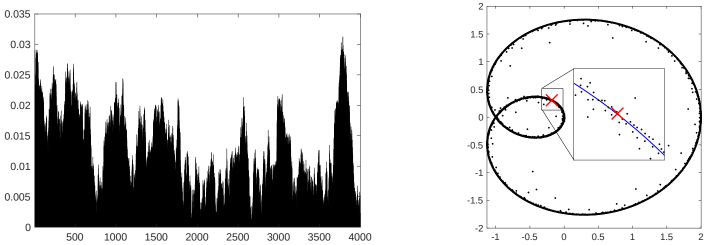

Figure: $T_N = J_N + J_N^2$, $N = 4000$, $\gamma = 0.8$.

Proof ideas for the LSD: Use of $\log$-potential

For a probability measure μ on $\mathbb{C}$, such that $\log(\cdot)$ integrates near infinity, define its $\log$-potential as follows:

$$\mathcal{L}_\mu(z) := \int \log|z - x| d\mu(x), \qquad z \in \mathbb{C}.$$

Facts:

- If $\mathcal{L}_\mu(z) = \mathcal{L}_\nu(z)$ for Lebesgue a.e. $z \in \mathbb{C}$ then $\mu = \nu$.
- If $\{\mu_N\}$ is a tight sequence of (random) probability measures such that $\mathcal{L}_{\mu_N}(z) \to \mathcal{L}_\mu(z)$, for Lebesgue a.e. $z \in \mathbb{C}$, in probability, for some probability measure $\mu \in \mathbb{C}$, then $\mu_N \Rightarrow \mu$, in probability.

$$\int f d\mu_N \to \int f \, d\mu, \text{ as } N \to \infty, \text{ in probability, } f \in C_b(\mathbb{C}).$$

Proof ideas for the LSD: Use of $\log$-potential

Facts:

$$L_N^A := \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i(A_N)}.$$

$$\begin{aligned}\mathcal{L}_{L_N^A}(z) = \frac{1}{N}\sum_{i=1}^N \log|z-\lambda_i(A_N)| &= \frac{1}{N}\sum_{i=1}^N \log|\lambda_i(A_N - z\mathrm{Id}_N)| \\ &= \frac{1}{N}\log\left|\prod_{i=1}^N \lambda_i(A_N - z\mathrm{Id}_N)\right| \\ &= \frac{1}{N}\log|\det(A_N - z\mathrm{Id}_N)|.\end{aligned}$$

Proof ideas for the LSD: Use of $\log$-potential

For a probability measure μ on $\mathbb{C}$, such that $\log(\cdot)$ integrates near infinity, define its $\log$-potential as follows:

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- $$\mathcal{L}_{L_N^A}(z) = \frac{1}{N} \log|\det(A_N - z\mathrm{Id}_N)|.$$

Proof ideas (continued)

Identify the log-potential of the limit: $\mathcal{L}_{\boldsymbol{a}(U)}(z)$

- Recall

$$\boldsymbol{a}(\xi) = \sum_{\ell=-d_2}^{d_1} a_\ell \xi^\ell.$$

- Fix $z \in \mathbb{C}$. Let $\xi_1(z), \ldots, \xi_d(z)$ be the roots of the polynomial $(\boldsymbol{a}(\xi) - z) \cdot \xi^{d_2}$. Here $d := d_1 + d_2$.
- Therefore

$$(\boldsymbol{a}(\xi) - z) \cdot \xi^{d_2} = a_{d_1} \cdot \prod_{\ell=1}^{d} (\xi - \xi_\ell(z)).$$

Proof ideas (continued)

Identify the log-potential of the limit: $\mathcal{L}_{\boldsymbol{a}(U)}(z)$

$$\mathcal{L}_{\boldsymbol{a}(U)}(z) = \int_{\mathbb{S}^1} \log |\boldsymbol{a}(\xi) - z| d\xi = \int_{\mathbb{S}^1} \log |(\boldsymbol{a}(\xi) - z) \cdot \xi^{d_2}| d\xi$$

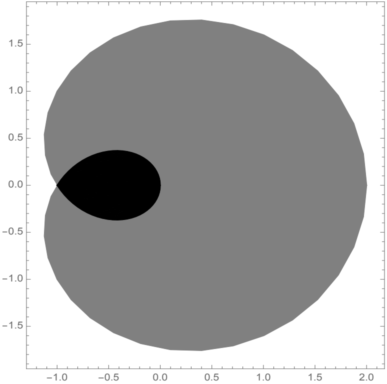
$$= \log |a_{d_1}| + \sum_{\ell=1}^{d} \int_{\mathbb{S}^1} \log |\xi - \xi_\ell(z)| d\xi$$

$$= \log |a_{d_1}| + \sum_{\ell=1}^{d} \log_+ |\xi_\ell(z)|.$$

- The form of the limit depends on the number of the roots that are greater than one in moduli.

Proof ideas for the LSD: The limaçon

Back to the example: $\boldsymbol{a}(\xi) = \xi + \xi^2$.



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Proof ideas (contd.)

Need to show

$$\lim_{N\to\infty} \frac{1}{N} \log|\det(T_N + N^{-\gamma}E_N - z\mathrm{Id}_N)|$$

$$= \begin{cases} 0 & \text{if } z \in \mathcal{R}_2, \\ \log|\xi_1(z)| & \text{if } z \in \mathcal{R}_1, \\ \log|\xi_1(z)| + \log|\xi_2(z)| & \text{if } z \in \mathcal{R}_0. \end{cases}$$

$|\xi_2(z)| \leqslant |\xi_1(z)|$

Proof ideas (contd.)

Need to show

$$\lim_{N\to\infty} \frac{1}{N} \log |\det(T_N + N^{-\gamma} E_N - z\mathrm{Id}_N)| = \begin{cases} 0 & \text{if } z \in \mathcal{R}_2, \\ \log|\xi_1(z)| & \text{if } z \in \mathcal{R}_1, \\ \log|\xi_1(z)| + \log|\xi_2(z)| & \text{if } z \in \mathcal{R}_0. \end{cases}$$

Idea: Expand the determinant

$$\det(T_N + N^{-\gamma} E_N - z\mathrm{Id}_N) = \sum_{\substack{X,Y\subset[N] \\ |X|=|Y|}} (\pm)\cdot \det((T_N - z\mathrm{Id}_N)[X;Y]) \cdot \det(N^{-\gamma} E_N[X^c; Y^c])$$

Proof ideas (contd.)

Need to show

$$\lim_{N\to\infty} \frac{1}{N} \log |\det(T_N + N^{-\gamma} E_N - z\mathrm{Id}_N)| = \begin{cases} 0 & \text{if } z \in \mathcal{R}_2, \\ \log|\xi_1(z)| & \text{if } z \in \mathcal{R}_1, \\ \log|\xi_1(z)| + \log|\xi_2(z)| & \text{if } z \in \mathcal{R}_0. \end{cases}$$

Idea: Expand the determinant and find the dominant term

$$\det(T_N + N^{-\gamma} E_N - z\mathrm{Id}_N) = \sum_{\substack{X,Y\subset[N] \\ |X|=|Y|}} (\pm) \cdot \det((T_N - z\mathrm{Id}_N)[X;Y]) \cdot \det(N^{-\gamma} E_N[X^c;Y^c])$$

Dominant term: test cases

Need to show

$$\frac{1}{N}\log|\det(T_N+N^{-\gamma}E_N-z\mathrm{Id}_N)| \to \log|\xi_1(z)|+\log|\xi_2(z)|, \quad z\in\mathcal{R}_0$$

$$\frac{1}{N}\log\left|\det\left(\begin{bmatrix} -z & 1 & 1 & & & \\ & -z & 1 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -z & 1 & 1 \\ & & & & -z & 1 \\ & & & & & -z \end{bmatrix}\right)\right|$$

$$= \log|z| = \log|\xi_1(z)| + \log|\xi_2(z)|.$$

Dominant term: test cases

Need to show

$$\frac{1}{N}\log|\det(T_N + N^{-\gamma}E_N - z\mathrm{Id}_N)| \to 0, \quad z \in \mathcal{R}_2$$

$$\frac{1}{N}\log\left|\det\left(\begin{bmatrix} -z & 1 & 1 & & & & \\ 0 & -z & 1 & 1 & & & \\ \vdots & 0 & \ddots & \ddots & \ddots & & \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \\ \vdots & \vdots & & & -z & 1 & 1 \\ 0 & 0 & \cdots & \cdots & 0 & -z & 1 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & -z \end{bmatrix}\right)\right| = 0.$$

Dominant term: test cases

Need to show

$$\frac{1}{N}\log|\det(T_N + N^{-\gamma}E_N - z\mathrm{Id}_N)| \to \log|\xi_1(z)|, \quad z \in \mathcal{R}_1$$

$$\frac{1}{N}\log\left|\det\left(\begin{bmatrix} -z & 1 & 1 & & & \\ 0 & -z & 1 & 1 & & \\ \vdots & & \ddots & \ddots & \ddots & \\ \vdots & & & -z & 1 & 1 \\ \vdots & & & & -z & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & -z \end{bmatrix}\right)\right| \asymp \log|\xi_1(z)|.$$

Proof ideas (contd.)

Formally

- $$\begin{aligned} &\det(T_N + N^{-\gamma}E_N - z\mathrm{Id}_N) \\ &= \sum_{\substack{X,Y\subset[N] \\ |X|=|Y|}} (\pm)\cdot\det((T_N - z\mathrm{Id}_N)[X;Y])\cdot\det(N^{-\gamma}E_N[X^c;Y^c]) \\ &= \sum_{k=0}^{N} P_k(z), \end{aligned}$$

  where $P_k(z)$ is the homogeneous polynomial of degree k in the expansion of the determinant in the entries of E_N.

Proof ideas (contd.)

Formally

- For $z \in \mathcal{R}_i$, $i = 0, 1, 2$

$$\sum_{k \neq i} P_k(z) = o(P_i(z)). \qquad \text{(a)}$$

and

$$P_i(z) \asymp \log_+ |\xi_1(z)| + \log_+ |\xi_2(z)|. \qquad \text{(b)}$$

Proof ideas (contd.)

- To prove (a) compute high moments.
- To prove (b) one needs certain anti-concentration bounds.
  - Assume the entries of E_N satisfy required anti-concentration bounds. Prove the convergence of the $\log$-potentials.
  - Show separately that the specific distribution of the entries of E_N do not affect the limiting spectral distribution (replacement principle).

Thank you!