What Fourier and Schur idempotents look like?

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—Joint with Mikael de la Salle and Eduardo Tablate—

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Introduction

Fourier and Schur multipliers

Hilbert transforms

Given $m : \mathrm{G} \to \mathbf{C}$ for some LCA group G, let $T_m f = \big(m\widehat{f}\,\big)^\vee$ be the Fourier multiplier with symbol m. In the classical groups, we find that

$$T_m f(z) = \sum_{k \in \mathbf{Z}^n} m(k)\widehat{f}(k)z^k \quad \text{or} \quad T_m f(x) = \int_{\mathbf{R}^n} m(\xi)\widehat{f}(\xi)e^{2\pi i \langle x, \xi \rangle} d\xi.$$

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Fourier L_p-summability

- **Smooth case**: Which m_j's give $\lim\limits_{j \to \infty} \|f - T_{m_j} f\|_p = 0$?
- **Partial Fourier summation**: What if $m_j = \chi_{\Omega_j}$ for domains Ω_j?
- **Euclidean dilation invariance**: Which Ω make T_{χ_Ω} $L_p(\mathbf{R}^n)$-bded?

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The Hilbert transform in $\mathbf{R}$

$$Hf(x) = \mathrm{p.v.} \int_{\mathbf{R}} \frac{f(y)}{x - y} dy = -i \int_{\mathbf{R}} \mathrm{sgn}(\xi) \widehat{f}(\xi) e^{2\pi i \langle x, \xi \rangle} d\xi$$

Fourier L_p-summability works for dilations of a convex polyhedron Π

What about dilations of the unit ball B?

Original conjecture: $T_{\chi_{\mathrm{B}}}$ is $L_p(\mathbf{R}^n)$-bounded iff $|1/p - 1/2| < 1/2n$.

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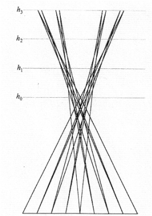
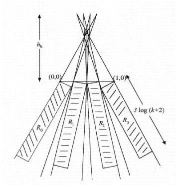


Kakeya construction



Fefferman construction

The same applies to any domain Ω with nonflat smooth boundary $\partial\Omega$

Let (G, μ) be a (unimodular) Lie group with

$$\lambda : \mathrm{G} \to \mathcal{U}(L_2(\mathrm{G}, \mu)) \quad \text{given by} \quad \big[\lambda(g)\varphi\big](h) = \varphi(g^{-1}h).$$

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Define its **group von Neumann algebra** as follows

$$\mathcal{L}\mathrm{G} := \overline{\mathrm{span}\Big\{ f = \int_{\mathrm{G}} \widehat{f}(g)\lambda(g)\, d\mu(g) : \widehat{f} \in \mathcal{C}_{\mathrm{c}}(\mathrm{G}) \Big\}}^{\mathrm{w}} \subset \mathcal{B}(L_2(\mathrm{G},\mu)).$$

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Given $m : \mathrm{G} \to \mathbf{C}$, its **Fourier multiplier** is the map

$$\widehat{T_m f}(g) = \tau(T_m f \lambda(g)^*) = m(g)\tau(f\lambda(g)^*) = m(g)\widehat{f}(g).$$

Fourier multipliers over Lie groups

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Pioneering work: Haagerup '79 + Cowling-Haagerup '85
L_p-**theory**: Strong efforts since 2010 – Junge, Mei, P, Ricard, de la Salle...
Key in geometric group th, functional+harmonic analysis, operator algebras...

What are Schur multipliers?

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$$S_M(A) := \Big(M(j,k)A_{j,k}\Big)_{j,k} \quad \text{for any} \quad A \in M_n.$$

If $M : \mathbf{Z} \times \mathbf{Z} \to \mathbf{C}$, define S_M for infinite matrices $A \in \mathcal{B}(\ell_2(\mathbf{Z}))$.

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If $M : \Omega \times \Omega \to \mathbf{C}$ and $T \in \mathcal{B}(L_2(\Omega, \mu))$ admits a kernel K, define

$$S_M(T)f(x) = \int_\Omega M(x,y)K(x,y)f(y)d\mu(y).$$

No worries: Operator with kernel are dense in the relevant topologies!

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Besides Fourier multipliers:

Which $M : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{C}$ satisfy

$$\big\| S_M(A) \big\|_{S_p(\mathbf{R}^n)} = \mathrm{tr}\Big(\big(S_M(A)^* S_M(A) \big)^{\frac{p}{2}} \Big)^{\frac{1}{p}} \leq C_p \|A\|_{S_p(\mathbf{R}^n)}?$$

Or more generally, same the problem with $M : \mathrm{G} \times \mathrm{G} \to \mathbf{C}$ instead.

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Fourier-Schur transference (2011/2015)

If $1 \leq p \leq \infty$

$$\Big\| S_m : S_p(\mathbf{R}^n) \to S_p(\mathbf{R}^n) \Big\|_{\mathsf{cb}} = \Big\| T_m : L_p(\mathbf{R}^n) \to L_p(\mathbf{R}^n) \Big\|_{\mathsf{cb}}.$$

Moreover, a similar result holds for multipliers in **amenable** groups.

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Moreover, a similar result holds for multipliers in **amenable** groups.

The **logic** of this result is based on...

$$L_\infty(\mathbf{T}) \ni f = \sum_{n \in \mathbf{Z}} \widehat{f}(n) e^{2\pi i n \cdot} \mapsto \Big(\widehat{f}(j-k) \Big)_{j,k} \in \mathcal{B}(\ell_2(\mathbb{Z})).$$

NonToeplitz Schur multipliers

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The Grothendieck-Haagerup characterization

S_M is bounded on $\mathcal{B}(L_2(\mathrm{X}))$ iff S_M is cb-bounded if and only if there exists a Hilbert space $\mathcal{K}$ and uniformly bounded (measurable) families (u_x) and (w_y) in $\mathcal{K}$ satisfying $M(x,y) = \langle u_x, w_y \rangle_{\mathcal{K}}$ for a.e. $x, y \in \mathrm{X}$.

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When $1 < p < \infty$, sufficient regularity conditions are the following...

Hörmander-Mikhlin-Schur multipliers – CGPT 2023

$$\big\| S_M \big\|_{\mathrm{cb}(S_p(\mathbf{R}^n))} \lesssim \sum_{|\gamma| \leq [\frac{n}{2}]+1} \Big\| |x-y|^{|\gamma|} \Big\{ \big| \partial_x^\gamma M(x,y) \big| + \big| \partial_y^\gamma M(x,y) \big| \Big\} \Big\|_\infty$$

Marcinkiewicz thm for Schur multipliers – Yeong-Liu-Mei 2023

$$\big\| S_M \big\|_{\mathrm{cb}(S_p(\mathbf{Z}))} \lesssim \sup_{\substack{j \in \mathbf{Z} \\ \mathcal{J} \in \mathcal{D}(\mathbf{Z})}} \mathbf{Var}_{\mathcal{J}} \big(M(j+\cdot, j) \big) + \mathbf{Var}_{\mathcal{J}} \big(M_(j, j+\cdot) \big)$$

Motivation: Two open problems

Spherical Hilbert transforms

Let $H_{\mathbf{S}}$ be the Schur multiplier with symbol

$$\mathbf{S}^n \times \mathbf{S}^n \ni (x, y) \mapsto \frac{1}{2}\big(1 + \operatorname{sgn}\langle x, y\rangle\big) = \chi_{\langle x,y\rangle > 0}.$$

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Motivation: Are spherical Hilbert transforms S_p-bded for any $p \neq 2$?

Fourier idempotents on Lie groups

Let G be a connected Lie group and $1 < p \neq 2 < \infty$:

- Which smooth domains Ω give $T_{\chi_\Omega} : L_p(\mathcal{L}\mathrm{G}) \to L_p(\mathcal{L}\mathrm{G})$?
- Is there a geometric characterization? A group theoretic one?

This is part of a longstanding search for Fourier L_p-idempotents.

Our main results below give complete answers to the above problems...

Schur idempotents

Local geometry and analytic form

The main result

Let $\Sigma \subset \mathbf{R}^n \times \mathbf{R}^n$ be a $\mathcal{C}^1$-domain.
Given $(x, y) \in \partial\Sigma$, set $\mathbf{n}(x, y) = (\mathbf{n}_1(x, y), \mathbf{n}_2(x, y)) \perp \partial\Sigma$ at (x, y).
A point $(x, y) \in \partial\Sigma$ is called **transverse** when both $\mathbf{n}_1, \mathbf{n}_2$ are nonzero.

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Theorem A (Schur idempotents)

Given $1 < p \neq 2 < \infty$, TFAE for any transverse $(x_0, y_0) \in \partial\Sigma$:

1. S_p**-boundedness.** The Schur idempotent S_Σ whose symbol equals 1 on Σ and 0 elsewhere is bounded on $S_p(L_2(U), L_2(V))$ for some pair of neighbourhoods U, V of x_0, y_0 in $\mathbf{R}^n$.

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2. **Zero-curvature condition.** There are neighbourhoods U, V of the points x_0, y_0 such that the vectors $\mathbf{n}_2(x_1, y)$, $\mathbf{n}_2(x_2, y)$ are parallel for any pair of points $(x_1, y), (x_2, y) \in \partial\Sigma \cap (U \times V)$.

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3. **Triangular truncation representation.** There are neighbourhoods U, V of x_0, y_0 and $\mathcal{C}^1$-functions $f_1 : U \to \mathbf{R}$ and $f_2 : V \to \mathbf{R}$, such that the domain $\Sigma \cap (U \times V) = \big\{(x,y) \in U \times V : f_1(x) > f_2(y)\big\}$.

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$$\Sigma = \Big\{(x, y) : x - y \in \Omega\Big\} \quad \text{for some Euclidean domain } \Omega.$$

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Theorem A holds for differentiable manifolds $M \times N$. This is quite remarkable, since Schur multipliers on general manifolds lack to admit a Fourier transform connection.

Zero-curvature for $\mathcal{C}^2$-domains

Let Σ be a $\mathcal{C}^2$-**domain**:

$$\Sigma \cap (U \times V) = \{(x,y) : F(x,y) > 0\} \text{ for some } F \in \mathcal{C}^2(\mathbf{R}^n \times \mathbf{R}^n).$$

Equivalent curvature condition for $\mathcal{C}^2$-domains

$$\left\langle d_x d_y F(x,y), u \otimes v \right\rangle := u^{\mathrm{t}} \cdot \left(\partial_{x_j} \partial_{y_k} F(x,y)\right)_{j,k} \cdot v = 0$$

for all $(x,y) \in \partial\Sigma \cap (U \times V)$ & $(u,v) \in \ker d_x F(x,y) \times \ker d_y F(x,y)$.

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Vanishing forms of $d_{xx}F$ or $d_{yy}F$: Not valid since

$$\Sigma_{\mathrm{r}} = \big\{(x,y) : x \in \Omega\big\} \quad \text{and} \quad \Sigma_{\mathrm{c}} = \big\{(x,y) : y \in \Omega\big\}$$

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$\mathcal{C}^2$-curvature is (x,y)-symmetric. This holds as well for $\mathcal{C}^1$-domains by Theorem A. Stein's **rotational curvature** $\det[d_x d_y F(x,y)]$ is similar.

Examples and nonexamples I

Thm A holds globally for fully transverse relatively compact domains

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- Fefferman's theorem $\Rightarrow$ No smooth compact Fourier idempotents. However, there are plenty of such (nonToeplitz) Schur idempotents. A funny instance is other form of **ball multiplier**

$$\Sigma_R = \Big\{ (x,y) \in \mathbf{R}^n \times \mathbf{R}^n : |x|^2 + |y|^2 < R^2 \Big\}.$$

These are clearly S_p-**bounded** by condition (3) in Theorem A.
By our lax notion of ∂-flatness: Spheres $\partial\Sigma_R$ have **zero-curvature**!

Examples and nonexamples I

Thm A holds globally for fully transverse relatively compact domains

- Fefferman's theorem $\Rightarrow$ No smooth compact Fourier idempotents. However, there are plenty of such (nonToeplitz) Schur idempotents. A funny instance is other form of **ball multiplier**

$$\Sigma_R = \Big\{ (x,y) \in \mathbf{R}^n \times \mathbf{R}^n : |x|^2 + |y|^2 < R^2 \Big\}.$$

These are clearly S_p-**bounded** by condition (3) in Theorem A. By our lax notion of ∂-flatness: Spheres $\partial\Sigma_R$ have **zero-curvature**!

- **Relative compactness is crucial at this point.** Indeed, any fully transverse $\mathcal{C}^1$-domain of $\mathbf{R} \times \mathbf{R}$ trivially satisfies the zero-curvature condition at every boundary point. But there are Toeplitz examples of such domains arising from Fourier symbols that do not define an S_p-multiplier for any $p \neq 2$: Just pick $\Omega \subset \mathbf{R}$ with T_{χ_Ω} L_p-unbded.

Examples and nonexamples II

Let $-1 < \delta < 1$ and

$$H_{\mathbf{S},\delta}(A) = \left(\chi_{\langle x,y\rangle > \delta} A_{xy}\right)_{x,y \in \mathbf{S}^n}$$

Examples and nonexamples II

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Corollary A (Spherical Hilbert transforms)

If $1 < p \neq 2 < \infty$, the n-dimensional spherical Hilbert transforms $H_{\mathbf{S},\delta}$ are all S_p-bounded for $n = 1$ and S_p-unbounded for $n \geq 2$ and $|\delta| < 1$.

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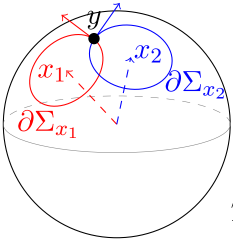

$y \in \partial\Sigma_{x_1} \cap \partial\Sigma_{x_2}$
$T_y\partial\Sigma_{x_1} \neq T_y\partial\Sigma_{x_2}$

Failure of zero-curvature for spherical Hilbert transforms $H_{\mathbf{S},\delta}$
Here $H_{\mathbf{S},\delta} = S_\Sigma$ with $\Sigma = \big\{(x,y) \in \mathbf{S}^n \times \mathbf{S}^n : \langle x,y\rangle > \delta\big\}$ for $n = 2$.

Fourier idempotents

Three Hilbert transforms on Lie groups

Fourier idempotents so far...

Euclidean idempotents

Semispaces and (finite or lacunary) combinations of them.

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How do Fourier L_p-idempotents look for arbitrary Lie groups?

Theorem B1 (Fourier idempotents)

Let G be a connected Lie group.
Let $\Omega \subset \mathrm{G}$ a $\mathcal{C}^1$-domain and $g_0 \in \partial\Omega$.
Then, TFAE for every $1 < p \neq 2 < \infty$:

- χ_Ω defines locally at g_0 a Fourier cb-L_p-multiplier.
- $\partial\Omega = g_0 \exp(\mathfrak{h})$ around g_0 ($\mathfrak{h} =$ codim-1 Lie subalgebra).

Fourier idempotents – The local geometry

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If G is simply connected, this is also equivalent to:

- There is a smooth action $\mathrm{G} \to \mathrm{diff}(\mathbf{R})$ such that

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This unravels what is "Fourier boundary flatness" for Lie groups

Fourier idempotents – The group structure

Consider:

i) The real line $\mathrm{G}_1 = \mathbf{R}$ with $\Omega_1 = (0, \infty)$.

ii) The affine group $\mathrm{G}_2 = \mathrm{Aff}_+(\mathbf{R})$ and $\Omega_2 = \{ax + b : b > 0\}$.

iii) The universal covering $\mathrm{G}_3 = \widetilde{\mathrm{PSL}}_2(\mathbf{R})$ and $\Omega_3 = \{g : \alpha_g(0) > 0\}$.
$\alpha : \widetilde{\mathrm{PSL}}_2(\mathbf{R}) \curvearrowright \mathbf{R}$ by lifting standard action $\mathrm{PSL}_2(\mathbf{R}) \curvearrowright P1(\mathbf{R})$ to universal covers.

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Lie's classification implies the following interesting consequence:

Theorem B2 (Fourier idempotents)

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- χ_Ω defines locally at g_0 a Fourier cb-L_p-multiplier.
- $\Omega = g_0 f^{-1}(\Omega_j)$ near g_0 for a surject hom $f : \mathrm{G} \to \mathrm{G}_j$ & $1 \leq j \leq 3$.

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Just **three Hilbert transforms** on Lie groups: Others = '**Directional**'

Corollary B3 (Nilpotent and Simple Lie groups)

Fourier cb-L_p-idempotents for $1 < p \neq 2 < \infty$:

i) **Simply connected nilpotent Lie groups**
They are locally $= H \circ \varphi$ for $\varphi : \mathrm{G} \to \mathbf{R}$ smooth hom.

ii) **Simple Lie groups not locally isomorphic to** $\mathrm{SL}_2(\mathbf{R})$
These groups do not carry Fourier cb-L_p-idempotents at all.

iii) **Other Lie groups which are locally isomorphic to** $\mathrm{SL}_2(\mathbf{R})$
Unique local Fourier cb-L_p-idempotent up to left/right translation.

Applications for nilpotent and simple Lie groups

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$\mathrm{SL}_2(\mathbf{R})$ **vs** $\mathrm{SL}_2(\mathbf{Z})$**:** $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \mathrm{sgn}(ac + bd)$ is L_p-unbounded!

Thank you!!

Ingredients of the proofs

Lemma A1 (Schur amplification of Meyer's lemma)

Assume $\partial\Sigma$ transverse in $U \times V$ and $S_\Sigma \in \mathcal{B}(S_p(L_2(U), L_2(V)))$.
Given $z_j = (x_j, y) \in \partial\Sigma \cap (U \times V)$, $u_j = \mathbf{n}_2(z_j)$ and $f_j \in L_p(\mathbf{R}^n)$:

$$\Big\| \Big(\sum_j \big| H_{u_j}(f_j) \big|^2 \Big)^{\frac{1}{2}} \Big\|_{L_p(\mathbf{R}^n)} \lesssim \Big\| \Big(\sum_j |f_j|^2 \Big)^{\frac{1}{2}} \Big\|_{L_p(\mathbf{R}^n)}.$$

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Lemma A2 (Local normal form of transverse hypersurfaces)

If $(x_0, y_0) \in \partial\Sigma$ is transverse, there are local diffeomorphisms st

$$\phi(x_0) = \psi(y_0) = 0 \quad \text{and} \quad \phi \times \psi(\Sigma) = \Big\{ \big((s, \tilde{x}), y\big) : s > g(\tilde{x}, y) \Big\}$$

for some $g \in \mathcal{C}^1(\mathbf{R}^{n-1} \times \mathbf{R}^n)$ satisfying $g(0, y) = \langle y, e_1 \rangle$ for every y.

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Lemma A3 (Measurable transformations of Schur S_p-multipliers)

Let $(X, \mu), (X', \mu')$ be atomless σ-finite and $f, g : X \to X'$ be measurable. Then, if $f_*\mu << \mu'$ and $m \in L_\infty(X' \times X')$, we obtain $\|m \circ (f \times g)\|_{MS_p(L_2(X,\mu))} = \|m\|_{MS_p(L_2(X', f_*\mu))} \leq \|m\|_{MS_p(L_2(X',\mu'))}$.

Fourier idempotents

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The result below generalizes P-Ricard-de la Salle's local transference...

Theorem C (Local Fourier-Schur transference)

Let G be a locally compact group and consider a bounded measurable function $m : \mathrm{G} \to \mathbf{C}$. Then, TFAE for $g_0 \in \mathrm{G}$ and every $1 < p < \infty$:

- There is a neighbourhood U of g_0 such that the restriction of T_m to the space of elements of $L_p(\mathcal{L}\mathrm{G})$ Fourier supported by U is cb.
- There are open sets $V, W \subset \mathrm{G}$ with $g_0 \in VW^{-1}$ such that the function $(g,h) \in V \times W \mapsto m(gh^{-1})$ is in $M_{\mathrm{cb}} S_p(L_2(V), L_2(W))$.

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Lie algebra analysis

Zero-curvature $\rightsquigarrow \partial\Omega = g_0 \exp(\mathfrak{h})$.
Lie's classification of codim-1 Lie subalgebras $\rightsquigarrow$ 3 Hilbert transforms

Is transversality essential in Theorem A?

A digression on transversality

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Degenerate case $\mathbf{n}_1 \equiv 0$: $\Sigma = \{(x, y) : y \in \Omega\}$.
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Isolated transverse points: No counterexamples so far.
It is likely that such examples can be found, but not easily...
Difficulty: Probably not for domains with analytic boundary.