Hypoelliptic damped wave equations on graded Lie groups with initial data from negative order Sobolev spaces

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This talk is based on the following joint works with Aparajita Dasgupta (IIT Delhi), Shyam Swarup Mondal (ISI Kolkata), Michael Ruzhansky (UGent, Belgium) and Berikbol Torebek (UGent Belgium):

- A. Dasgupta, V. Kumar, S. S. Mondal and M. Ruzhansky, Semilinear damped wave equations on the Heisenberg group with initial data from Sobolev spaces of negative order, **J. Evol. Equ.** 24(51), (2024). `https://doi.org/10.1007/s00028-024-00976-5` (Open Access).
- A. Dasgupta, V. Kumar, S. S. Mondal and M. Ruzhansky, Higher order hypoelliptic damped wave equations on graded Lie groups with data from negative order Sobolev spaces. `https://doi.org/10.48550/arXiv.2404.08766`
- V. Kumar, S. S. Mondal, M. Ruzhansky and B. T. Torebek, Blow-up result for semilinear damped wave equations with data from negative order Sobolev spaces: the critical case. `https://doi.org/10.48550/arXiv.2408.05598`

Nonlinear damped wave equation on $\mathbb{R}^n$

We consider the following semilinear damped wave equation:

$$\begin{cases} u_{tt} - \Delta u + u_t = |u|^p, & x \in \mathbb{R}^n, \ t > 0, \\ u(0,x) = \varepsilon u_0(x), & x \in \mathbb{R}^n, \\ u_t(0,x) = \varepsilon u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1}$$

where $1 < p < \infty$, Δ is the Laplacian on $\mathbb{R}^n$ and $\varepsilon > 0$ is a size parameter.

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$$T_\varepsilon \begin{cases} = \infty & \text{if } p > p_{\text{crit}}(n), \\ \simeq \exp\left(C\varepsilon^{-(p-1)}\right) & \text{if } p = p_{\text{crit}}(n), \\ \simeq C\varepsilon^{-\frac{2(p-1)}{2-n(p-1)}} & \text{if } p < p_{\text{crit}}(n). \end{cases}$$

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- The exponent $p_{\text{crit}}(n) := 1 + \frac{2}{n}$ plays a similar role as in the semilinear damped wave equation in the study of the following semilinear heat equation:

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- For $N \geq 2$: [Karch 2000] with $p > 1 + \frac{4}{n}$, [Hayashi-Kaikina-Naumkin 2004] with $p > p_{\text{Fuj}}(n)$.
- This is known as the "diffusion phenomenon" of (linear or nonlinear) damped wave equations.

Damped wave equations on $\mathbb{R}^n$: Initial data from negative Sobolev spaces

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  $(u_0, u_1) \in (H^s(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)) \times \big(H^{s-1}(\mathbb{R}^n) \cap L^m(\mathbb{R}^n)\big)$ with $m \in (1, 2]$.
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- Guo and Wang 2012, Tang-Zhang-Zou 2024: Compressible Navier-Stokes equations and the Boltzmann equation with initial data from negative order Sobolev space.

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with initial data additionally belonging to homogeneous Sobolev spaces of negative order $\dot{H}^{-\gamma}(\mathbb{R}^n)$ with $\gamma > 0$.

- They found a new critical exponent

$$p_{\mathrm{crit}}(n,\gamma) := 1 + \frac{4}{n+2\gamma}, \quad \gamma \in \left(0, \frac{n}{2}\right).$$

• For $p > p_{\mathrm{crit}}(n,\gamma)$, the problem (4) admits a global-in-time Sobolev solution for sufficiently small data of lower regularity.

• For $1 < p < p_{\mathrm{crit}}(n,\gamma)$, the solutions to (4) blow-up in a finite time. In other words, there exists $T > 0$ such that the solution to (4) satisfies $\|u(\cdot, t_m)\|_\infty \to \infty$ as $t_m \to T$.

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- The sharp lifespan estimates for weak solutions to (4) is given by

$$T_\varepsilon \begin{cases} = \infty & \text{if } p > p_{\text{crit}}(n,\gamma), \\ \simeq C\varepsilon^{-\frac{2}{2p'-2-\frac{n}{2}-\gamma}} & \text{if } p < p_{\text{crit}}(n,\gamma), \end{cases}$$

where C is a positive constant independent of ε and p'.

For any $T > 0$, a weak solution of the Cauchy problem

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in $[0,T) \times \mathbb{R}^n$ is a function $u \in L^p_{\text{loc}}([0,T) \times \mathbb{R}^n)$ that satisfies the following integral relation:

$$\int_0^T \int_{\mathbb{R}^n} u(t,x) \left(\partial_t^2 \phi(t,x) - \Delta\phi(t,x) - \partial_t \phi(t,x)\right) \, dx\, dt - \int_{\mathbb{R}^n} u_0(x)\phi(0,x)\, dx$$
$$- \int_{\mathbb{R}^n} u_1(x)\phi(0,x)\, dx + \int_{\mathbb{R}^n} u_0(x)\partial_t\phi(0,x)\, dx = \int_0^T \int_{\mathbb{R}^n} |u(t,x)|^p \phi(t,x)\, dx\, dt, \tag{6}$$

for any $\phi \in \mathcal{C}_0^\infty([0,T) \times \mathbb{R}^n)$.
Global solution: If $T = \infty$, we call u to be a global-in-time weak solution to (5),
Local solution: otherwise, u is said to be a local-in-time weak solution to (5).

Theorem 1 (Berikbol, K., Mondal, Ruzhansky, 2024)

Let $\gamma \in \left(0, \frac{n}{2}\right)$ *and let the exponent p satisfy*

$$p = p_{Crit}(n, \gamma) := 1 + \frac{4}{n + 2\gamma}.$$

We assume that the non-negative initial data $(u_0, u_1) \in \dot{H}^{-\gamma}(\mathbb{R}^n) \times \dot{H}^{-\gamma}(\mathbb{R}^n)$ *satisfies*

$$u_0(x) + u_1(x) \geq C_1 \langle x \rangle^{-n\left(\frac{1}{2} + \frac{\gamma}{n}\right)} (\log(e + |x|))^{-1}, \quad x \in \mathbb{R}^n, \tag{7}$$

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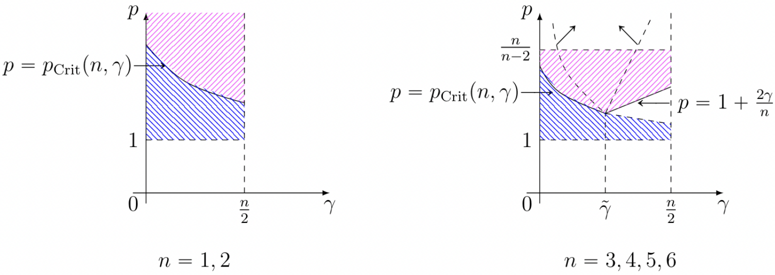
• We prove this theorem using the test function method (Mitidieri and Pohozaev 2001, Zhang 2001).

• We introduce an appropriate test function for the critical case $p = p_{\text{Crit}}$ different from (Chen and Reissig 2023) in the sub-critical case $p < p_{\text{Crit}}(n, \gamma)$.

Figure



Range for global existence of solution for $p > p_{\text{Crit}}(n,\gamma)$

Range for blow-up of weak solution for $p \leqslant p_{\text{Crit}}(n,\gamma)$

Our Goal:

To study the Cauchy problem for the semilinear damped wave equation with power-type nonlinearities:

$$\begin{cases} u_{tt} + \mathcal{R}u + u_t = |u|^p, & x \in \mathbb{G},\ t > 0, \\ u(0,x) = \varepsilon u_0(x), & x \in \mathbb{G}, \\ u_t(0,x) = \varepsilon u_1(x), & x \in \mathbb{G}, \end{cases} \tag{9}$$

where $1 < p < \infty$, $\mathcal{R}$ is a positive Rockland operator of homogeneous degree $\nu \geq 2$ on a graded Lie group $\mathbb{G}$, and the initial data (u_0, u_1) with its size parameter $\varepsilon > 0$ belongs to homogeneous Sobolev spaces of negative order $\dot{H}^{-\gamma}(\mathbb{G}) \times \dot{H}^{-\gamma}(\mathbb{G})$ with $\gamma > 0$.

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- Examples of Rockland operators For $\mathbb{G} = (\mathbb{R}^n, +)$: a Rockland operator $\mathcal{R}$ can be any positive homogeneous elliptic differential operator with constant coefficients, for example, we can consider

$$\mathcal{R} = (-\Delta)^m \text{ or } \mathcal{R} = (-1)^m \sum_{j=1}^{n} a_j \left(\frac{\partial}{\partial x_j}\right)^{2m}, \quad a_j > 0, m \in \mathbb{N},$$

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- Georgiev and Palmieri, 2019: Semilinear damped wave equation on $\mathbb{H}^n$ with power nonlinearity: Critical exponent= Fujita exponent= $1 + \frac{2}{Q}$. Global existence of small data solutions for supercritical case and a blow-up result for (sub)-critical case.
- Palmieri, 2020: The L^2-L^2 decay estimates of for the solution of the homogeneous linear damped wave equation on the Heisenberg group and their derivatives.

Homogeneous Lie groups

A connected and simply connected Lie group $\mathbb{G}$ with $\dim \mathbb{G} = n$ is called a homogeneous Lie group if the Lie algebra $\mathfrak{g} \cong \mathbb{R}^n$ of $\mathbb{G}$ is endowed with a family of dilations $D_r^{\mathfrak{g}}$, $r > 0$, which are vector space automorphisms on $\mathfrak{g}$ satisfying the following two conditions:

- For every $r > 0$, $D_r^{\mathfrak{g}}$ is a map of the form $D_r^{\mathfrak{g}} := \operatorname{Exp}(\ln(r)A)$ for some diagonalisable linear operator $A \equiv \operatorname{diag}[\nu_1, \cdots, \nu_n]$ on $\mathfrak{g}$.
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Dilations and homogeneous Lie groups

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• The eigenvalues $0 < \nu_1 \leq \nu_2 \leq \ldots \leq \nu_n$ of A are called *dilations' weights*. The homogeneous dimension of a homogeneous Lie group $\mathbb{G}$ is given by

$$Q := \mathbf{Tr}(A) = \nu_1 + \cdots + \nu_n.$$

• **Convention**: The weights $0 < \nu_1 \leq \nu_2 \leq \ldots \leq \nu_n$ are jointly rescaled so that the lowest weight $\nu_1 = 1$. This also implies that $Q \geq n$.

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$$\mathfrak{g} = \bigoplus_{i=1}^{s} \mathfrak{g}_i, \quad \text{with} \quad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}. \quad \text{and } \mathfrak{g}_{i+j} = \{0\} \text{ for } i + j > s$$

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> Stratified Lie groups $\subset$ Graded Lie groups $\subset$ Homogeneous groups $\subset$ Nilpotent Lie groups.

- An example of a nine dimensional nilpotent Lie algebra which does not admit any family of dilations.

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Examples of graded Lie groups

• The Abelian group $\mathbb{G} := (\mathbb{R}^n, +)$ with $\mathfrak{g} := \mathbb{R}^n$ with trivial gradation $\mathfrak{g}_1 := \mathbb{R}^n$. The dilation is the canonial dilation

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• The Heisenberg group $\mathbb{H}^n := (\mathbb{R}^{2n+1}, \circ)$ with group operation $\circ$ given by

$$(x, y, t) \circ (x', y', t') = (x + x', y + y', t + t' + \frac{1}{2}(xy' - x'y)),$$

where $(x, y, t), (x', y', t') \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$. The canonical basis for the Lie algebra $\mathfrak{h}_n$ of $\mathbb{H}^n$ is given by the left-invariant vector fields:

$$X_j = \partial_{x_j} - \frac{y_j}{2}\partial_t, \qquad Y_j = \partial_{y_j} + \frac{x_j}{2}\partial_t, \quad j = 1, 2, \ldots n, \text{ and } T = \partial_t, \tag{10}$$

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• The Heisenberg Lie algebra $\mathfrak{h}_n$ admits the decomposition $\mathfrak{h}_n = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where $\mathfrak{g}_1 = \mathbb{R} - \text{span}\{X_j, Y_j\}_{j=1}^n$ and $\mathfrak{g}_2 = \mathbb{R} - \text{span}\{T\}$. This include a family of dilations

$$D_r(x, y, t) = (rx, ry, r^2 t) \qquad r > 0.$$

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$$\mathcal{F}_{\mathbb{G}}(f)(\pi) = \widehat{f}(\pi) := \int_{\mathbb{G}} f(x)\pi(x)^* \, dx = \int_{\mathbb{G}} f(x)\pi(x^{-1}) \, dx. \tag{11}$$

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• The following *Plancherel identity* holds for $f \in \mathcal{S}(\mathbb{G})$

$$\int_{\mathbb{G}} |f(x)|^2\, dx = \int_{\widehat{\mathbb{G}}} \|\widehat{f}(\pi)\|_{\mathrm{HS}(\mathcal{H}_\pi)}\, d\mu(\pi). \tag{12}$$

• The Fourier transform $\mathcal{F}_{\mathbb{G}}$ extends uniquely to a unitary isomorphism from $L^2(\mathbb{G})$ onto the space $L^2(\widehat{\mathbb{G}})$.

• For a left-invariant differential operator T, let us denote by $\pi(T)$, the symbol of T, which is the infinitesimal representation $d\pi(T)$ associated with $\pi \in \widehat{\mathbb{G}}$.

Rockland operator

A positive left-invariant differential operator $\mathcal{R}$ on a homogeneous group $\mathbb{G}$ called the *Rockland operator* if it is homogeneous of positive degree ν, that is,

$$\mathcal{R}(f \circ D_r) = r^{\nu}(\mathcal{R}f) \circ D_r, \quad r > 0, f \in C^{\infty}(\mathbb{G})$$

and the operator $\pi(\mathcal{R})$ is injective on $\mathcal{H}_{\pi}^{\infty}$ for every nontrivial representation $\pi \in \widehat{\mathbb{G}}$, that is,

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Rockland $\implies$ Graded

Existence of a Rockland operator on homogeneous Lie group $\mathbb{G} \implies \mathbb{G}$ is a graded Lie group

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Spectrum of symbol of Rockland operator

- For any $f \in L^2(\mathbb{G})$, we have

$$\mathcal{F}_{\mathbb{G}}(\mathcal{R}f)(\pi) = \pi(\mathcal{R})\widehat{f}(\pi).$$

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- The spectrum of the operator $\pi(\mathcal{R})$ with $\pi \in \widehat{\mathbb{G}}\backslash\{1\}$, is discrete and lies in $(0, \infty)$.
- We can choose an orthonormal basis for $\mathcal{H}_\pi$ such that the infinite matrix associated to the self-adjoint operator $\pi(\mathcal{R})$ has the following representation

$$\pi(\mathcal{R}) = \begin{pmatrix} \pi_1^2 & 0 & \cdots & \cdots \\ 0 & \pi_2^2 & 0 & \cdots \\ \vdots & 0 & \ddots & \\ \vdots & \vdots & & \ddots \end{pmatrix} \qquad (13)$$

where π_i, $i = 1, 2, \ldots,$ are strictly positive real numbers and $\pi \in \widehat{\mathbb{G}}\backslash\{1\}$.

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- The inhomogeneous Sobolev spaces $H^s(\mathbb{G}) := H^s_{\mathcal{R}}(\mathbb{G})$, $s \in \mathbb{R}$, associated to positive Rockland operator $\mathcal{R}$ of homogeneous degree ν, is defined as

$$H^s(\mathbb{G}) := \left\{ f \in \mathcal{D}'(\mathbb{G}) : (I + \mathcal{R})^{s/\nu} f \in L^2(\mathbb{G}) \right\},$$

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- **Hardy-Littlewood-Sobolev inequality**: Let $s > 0$ and $1 < p < q < \infty$ be such that

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Then

$$\|f\|_{L^q(\mathbb{G})} \lesssim \|f\|_{\dot{H}^{p,s}(\mathbb{G})} \simeq \|\mathcal{R}^{\frac{s}{\nu}} f\|_{L^p(\mathbb{G})}. \tag{14}$$

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$$\|u\|_{L^q(\mathbb{G})} \lesssim \|u\|^{\theta}_{\dot{H}^{r,s}(\mathbb{G})} \|u\|^{1-\theta}_{L^2(\mathbb{G})}, \tag{15}$$

for $\theta = \left(\frac{1}{2} - \frac{1}{q}\right) / \left(\frac{s}{Q} + \frac{1}{2} - \frac{1}{r}\right) \in [0, 1]$, provided that $\frac{s}{Q} + \frac{1}{2} \neq \frac{1}{r}$.

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Examples of Rockland operators

- On a graded Lie group $\mathbb{G}$, operators

$$\mathcal{R} := \sum_{j=1}^{n} (-1)^{\frac{\nu_0}{\nu_j}} a_j X_j^{2\frac{\nu_0}{\nu_j}}, \quad \text{with } a_1, a_2, \ldots, a_n > 0$$

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are Rockland operators of degree $\nu = 2\nu_0$.

Theorem 2 (Dasgupta, K., Mondal and Ruzhansky 2024)

Let $\mathbb{G}$ be a graded Lie group of homogeneous dimension Q and let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree ν. Assume that $(u_0, u_1) \in \left(H^s \cap \dot{H}^{-\gamma}\right) \times \left(H^{s-1} \cap \dot{H}^{-\gamma}\right)$ with $s \geq 0$ and $s + \gamma \geq 0$. Then the solution of the linear Cauchy problem

$$\begin{cases} u_{tt} + \mathcal{R}u + u_t = 0, & x \in \mathbb{G},\ t > 0, \\ u(0, x) = u_0(x), & x \in \mathbb{G}, \\ u_t(0, x) = u_1(x), & x \in \mathbb{G}, \end{cases} \tag{16}$$

satisfies the following $\dot{H}^s$-decay estimate

$$\|u(t, \cdot)\|_{\dot{H}^s} \lesssim (1 + t)^{-\frac{s+\gamma}{\nu}} \left(\|u_0\|_{H^s \cap \dot{H}^{-\gamma}} + \|u_1\|_{H^{s-1} \cap \dot{H}^{-\gamma}}\right), \tag{17}$$

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Remark

The additional use of Sobolev spaces of negative order for initial data provides a decay rate $(1+t)^{-\frac{\gamma}{\nu}}$ for any $\gamma > 0$.

Sketch of the proof

Applying the group Fourier transform on $\mathbb{G}$ to the linear system (16) with respect to x, for all $\pi \in \widehat{\mathbb{G}}$, we get

$$\begin{cases} \partial_t^2 \widehat{u}(t,\pi) + \pi(\mathcal{R})\widehat{u}(t,\pi) + \partial_t \widehat{u}(t,\pi) = 0, & \pi \in \widehat{\mathbb{G}},\ t > 0, \\ \widehat{u}(0,\pi) = \widehat{u}_0(\pi), & \pi \in \widehat{\mathbb{G}}, \\ \partial_t \widehat{u}(0,\pi) = \widehat{u}_1(\pi), & \pi \in \widehat{\mathbb{G}}, \end{cases} \tag{18}$$

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For $m, k \in \mathbb{N}$, we introduce the notation

$$\widehat{u}(t,\pi)_{m,k} \doteq (\widehat{u}(t,\pi)e_k, e_m)_{\mathcal{H}_\pi}\ , \tag{19}$$

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where $\{e_m\}_{m\in\mathbb{N}}$ is the same orthonormal basis in the representation space $\mathcal{H}_\pi$ that gives us (13). Then $\widehat{u}(t,\pi)_{m,k}$ solves the following infinite system of ordinary differential equation with respect to t variable

$$\begin{cases} \partial_t^2 \widehat{u}(t,\pi)_{m,k} + \partial_t \widehat{u}(t,\pi)_{m,k} + \beta_{m,\pi}^2 \widehat{u}(t,\pi)_{m,k} = 0, & \pi \in \widehat{\mathbb{G}},\ t > 0, \\ \widehat{u}(0,\pi)_{m,k} = \widehat{u}_0(\pi)_{m,k}, & \pi \in \widehat{\mathbb{G}}, \\ \partial_t \widehat{u}(0,\pi)_{m,k} = \widehat{u}_1(\pi)_{m,k}, & \pi \in \widehat{\mathbb{G}}, \end{cases} \tag{20}$$

where we denote $\beta_{m,\pi}^2 = \pi_m^2$.

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The characteristic equation of the above system is given by

$$\lambda^2 + \lambda + \beta_{m,\pi}^2 = 0.$$

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Using asymptotic expansions of eigenvalues, we consider the following cases:

- When $|\beta_{m,\pi}| < \delta \ll 1$:

$$\begin{aligned} \lambda_1 &= -1 + \mathcal{O}\left(\beta_{m,\pi}^2\right), \\ \lambda_2 &= -\beta_{m,\pi}^2 + \mathcal{O}\left(\beta_{m,\pi}^4\right). \end{aligned} \tag{21}$$

- When $|\beta_{m,\pi}| > N \gg 1$:

$$\begin{aligned} \lambda_1 &= -\frac{1}{2} - i|\beta_{m,\pi}| + \mathcal{O}\left(|\beta_{m,\pi}|^{-1}\right), \\ \lambda_2 &= -\frac{1}{2} + i|\beta_{m,\pi}| + \mathcal{O}\left(|\beta_{m,\pi}|^{-1}\right). \end{aligned} \tag{22}$$

- When $\delta < |\beta_{m,\pi}| < N$:

$$\text{Re}(\lambda_1) < 0 \quad \text{and} \quad \text{Re}(\lambda_2) < 0. \tag{23}$$

Sketch of the proof

Thus, the solution to the homogeneous system (20) is given by

$$\widehat{u}(t,\pi)_{m,k} = K_0(t,\pi)_{m,k}\widehat{u}_0(\pi)_{m,k} + K_1(t,\pi)_{m,k}\widehat{u}_1(\pi)_{m,k}, \tag{24}$$

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where

$$K_0(t,\pi)_{m,k} = \frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2} \tag{25}$$

$$= \begin{cases} \dfrac{\left(-1+\mathcal{O}(\beta^2_{m,\pi})\right)e^{\left(-\beta^2_{m,\pi}+\mathcal{O}(\beta^4_{m,\pi})\right)t} - \left(-\beta^2_{m,\pi}+\mathcal{O}(\beta^4_{m,\pi})\right)e^{\left(-1+\mathcal{O}(\beta^2_{m,\pi})\right)t}}{-1+\mathcal{O}\left(\beta^2_{m,\pi}\right)} & \text{for } |\beta_{m,\pi}| < \delta, \\ \dfrac{\left(i|\beta_{m,\pi}|-\frac{1}{2}+\mathcal{O}\left(|\beta_{m,\pi}|^{-1}\right)\right)e^{\left(-i|\beta_{m,\pi}|-\frac{1}{2}+\mathcal{O}\left(|\beta_{m,\pi}|^{-1}\right)\right)t}}{2i|\beta_{m,\pi}|+\mathcal{O}(1)} & \\ \quad -\dfrac{\left(-i|\beta_{m,\pi}|-\frac{1}{2}+\mathcal{O}\left(|\beta_{m,\pi}|^{-1}\right)\right)e^{\left(i|\beta_{m,\pi}|-\frac{1}{2}+\mathcal{O}\left(|\beta_{m,\pi}|^{-1}\right)\right)t}}{2i|\beta_{m,\pi}|+\mathcal{O}(1)} & \text{for } |\beta_{m,\pi}| > N. \end{cases} \tag{26}$$

$$K_1(t,\pi)_{m,k} = \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \tag{27}$$

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We have the following point-wise estimates for K_0 and K_1:

$$|K_0(t,\pi)_{m,k}| \lesssim \begin{cases} |\beta_{m,\pi}|^2 e^{-ct} + e^{-ct\beta_{m,\pi}^2} & \text{for } |\beta_{m,\pi}| < \delta \ll 1, \\ \mathrm{e}^{-ct} & \text{for } \delta \le |\beta_{m,\pi}| \le N, \\ \mathrm{e}^{-ct} & \text{for } |\beta_{m,\pi}| > N \gg 1, \end{cases} \tag{29}$$

and

Sketch of the proof

$$|K_1(t,\pi)_{m,k}| \lesssim \begin{cases} e^{-ct} + e^{-ct\beta_{m,\pi}^2} & \text{for } |\beta_{m,\pi}| < \delta \ll 1, \\ e^{-ct} & \text{for } \delta \le |\beta_{m,\pi}| \le N, \\ |\beta_{m,\pi}|^{-1} e^{-ct} & \text{for } |\beta_{m,\pi}| > N \gg 1, \end{cases} \tag{30}$$

for some constant $c > 0$.

$$|K_1(t,\pi)_{m,k}| \lesssim \begin{cases} e^{-ct} + e^{-ct\beta_{m,\pi}^2} & \text{for } |\beta_{m,\pi}| < \delta \ll 1, \\ e^{-ct} & \text{for } \delta \leq |\beta_{m,\pi}| \leq N, \\ |\beta_{m,\pi}|^{-1} e^{-ct} & \text{for } |\beta_{m,\pi}| > N \gg 1, \end{cases} \tag{30}$$

for some constant $c > 0$. Now using the Plancherel formula, we obtain

$$\begin{aligned} \|u(t,\cdot)\|_{\dot{H}^s}^2 &= \int_{\widehat{\mathbb{G}}} \|\pi(\mathcal{R})^{\frac{s}{\nu}} \hat{u}(t,\pi)\|_{\mathrm{HS}}^2 \, d\mu(\pi) \\ &\lesssim \sum_{m,k\in\mathbb{N}} \int_{\widehat{\mathbb{G}}} \pi_m^{\frac{4s}{\nu}} \left[|K_0(t,\pi)_{m,k}|^2 |\widehat{u}_0(\pi)_{m,k}|^2 + |K_1(t,\pi)_{m,k}|^2 |\widehat{u}_1(\pi)_{m,k}|^2 \right] d\mu(\pi) \\ &= I_{K_0} + I_{K_1}, \end{aligned} \tag{31}$$

where

$$I_{K_0} := \sum_{m,k\in\mathbb{N}} \int_{\widehat{\mathbb{G}}} \pi_m^{\frac{4s}{\nu}} |K_0(t,\pi)_{m,k}|^2 |\widehat{u}_0(\pi)_{m,k}|^2 d\mu(\pi)$$

and

$$I_{K_1} := \sum_{m,k\in\mathbb{N}} \int_{\widehat{\mathbb{G}}} \pi_m^{\frac{4s}{\nu}} |K_1(t,\pi)_{m,k}|^2 |\widehat{u}_1(\pi)_{m,k}|^2 d\mu(\pi).$$

Sketch of the proof

When $|\beta_{m,\pi}| < \delta \ll 1$, we obtain the following estimates:

$$I_{K_0} \lesssim (1+t)^{-\frac{2(s+\gamma)}{\nu}} \|u_0\|^2_{\dot{H}^{-\gamma}}, \tag{32}$$

and

$$I_{K_1} \lesssim (1+t)^{-\frac{2(s+\gamma)}{\nu}} \|u_1\|^2_{\dot{H}^{-\gamma}}. \tag{33}$$

Now consider the case $|\beta_{m,\pi}| > N \gg 1$.

$$I_{K_0} \lesssim e^{-2ct} \|u_0\|^2_{H^{-s}}, \tag{34}$$

and

$$I_{K_1} \lesssim e^{-2ct} \|u_1\|^2_{H^{s-1}}. \tag{35}$$

Sketch of the proof

The final case when $\delta < |\beta_{m,\pi}| < N$.

$$I_{K_0} \lesssim e^{-2ct} \|u_0\|^2_{H^s}, \tag{36}$$

and

$$I_{K_1} \lesssim e^{-2ct} \|u_1\|^2_{H^{s-1}}. \tag{37}$$

The final case when $\delta < |\beta_{m,\pi}| < N$.

$$I_{K_0} \lesssim e^{-2ct} \|u_0\|^2_{H^s}, \tag{36}$$

and

$$I_{K_1} \lesssim e^{-2ct} \|u_1\|^2_{H^{s-1}}. \tag{37}$$

Combining all the cases for $|\beta_{m,\pi}|$, that is, ((32), (33)), ((34), (35)), and ((36), (37)) along with (31), we obtain $\dot{H}^s$-decay estimate for the solution to linear system (16) as

$$\|u(t,\cdot)\|_{\dot{H}^s} \lesssim (1+t)^{-\frac{s+\gamma}{\nu}} \left(\|u_0\|_{H^s \cap \dot{H}^{-\gamma}} + \|u_1\|_{H^{s-1} \cap \dot{H}^{-\gamma}}\right)$$

for any $t \geq 0$.

Consider the inhomogeneous system

$$\begin{cases} u_{tt} + \mathcal{R}u + u_t = F(t,x), & x \in \mathbb{G},\ t > 0, \\ u(0,x) = u_0(x), & x \in \mathbb{G}, \\ u_t(0,x) = u_1(x), & x \in \mathbb{G}. \end{cases} \tag{38}$$

By applying Duhamel's principle, the solution to the above system can be written as

$$u(t,x) = u_0 * E_0(t,x) + u_1 * E_1(t,x) + \int_0^t F(s,x) * E_1(t-s,x)\, ds,$$

where $*$ denotes the group convolution product on $\mathbb{G}$ with respect to the x variable, and E_0 and E_1 represent the propagators to (38) in the homogeneous case $F = 0$ with initial data $(u_0, u_1) = (\delta_0, 0)$ and $(u_0, u_1) = (0, \delta_0)$, respectively.

A function u is said to be a *mild solution* to (38) on $[0, T]$ if u is a fixed point for the integral operator $N : u \in X_s(T) \mapsto Nu(t,x)$, given by

$$Nu(t,x) := u^{\text{lin}}(t,x) + u^{\text{non}}(t,x), \tag{39}$$

in the energy evolution space $X_s(T) \doteq \mathcal{C}\left([0,T], H^s(\mathbb{G})\right)$, $s \in (0,1]$, equipped with the norm

$$\|u\|_{X_s(T)} := \sup_{t\in[0,T]} \left((1+t)^{\frac{\gamma}{\nu}} \|u(t,\cdot)\|_{L^2} + (1+t)^{\frac{s+\gamma}{\nu}} \|u(t,\cdot)\|_{\dot{H}^s} \right) \tag{40}$$

with $\gamma > 0$, where

$$u^{\text{lin}}(t,x) = u_0 * E_0(t,x) + u_1 * E_1(t,x)$$

is the solution to the corresponding linear Cauchy problem (38), and

$$u^{\text{non}}(t,x) = \int_0^t F(s,x) * E_1(t-s,x)\, ds.$$

Theorem 3 (Dasgupta, K., Mondal and Ruzhansky 2024)

Let $s \in (0,1]$ *and* $\gamma \in \left(0, \frac{Q}{2}\right)$. *Assume that an exponent* p *satisfies*

$$1 < p \leq \frac{Q}{Q-2s} \quad \text{and} \quad p \begin{cases} > p_{Crit}(Q,\gamma,\nu) := 1 + \frac{2\nu}{Q+2\gamma} & \text{if } \gamma \leq \tilde{\gamma}, \\ \geq 1 + \frac{2\gamma}{Q} & \text{if } \gamma > \tilde{\gamma}, \end{cases} \tag{41}$$

where $\tilde{\gamma}$ *denotes the positive root of the quadratic equation* $2\tilde{\gamma}^2 + Q\tilde{\gamma} - \nu Q = 0$, *i.e.,* $\tilde{\gamma} = \frac{-Q+\sqrt{Q^2+8\nu Q}}{4}$. *Then, there exists a small positive constant* ε_0 *such that for any* $(u_0, u_1) \in \mathcal{A}^s := \left(H^s \cap \dot{H}^{-\gamma}\right) \times \left(L^2 \cap \dot{H}^{-\gamma}\right)$ *satisfying* $\|(u_0,u_1)\|_{\mathcal{A}^s} = \varepsilon \in (0, \varepsilon_0]$, *the Cauchy problem for the semilinear damped wave equation*

$$\begin{cases} u_{tt} + \mathcal{R}u + u_t = |u|^p, & x \in \mathbb{G},\ t > 0, \\ u(0,x) = \varepsilon u_0(x), & x \in \mathbb{G}, \\ u_t(0,x) = \varepsilon u_1(x), & x \in \mathbb{G}, \end{cases}$$

has a uniquely determined Sobolev solution $u \in \mathcal{C}\left([0,\infty), H^s\right)$. *Moreover, the solution satisfies the following estimate:*

$$\|u(t,\cdot)\|_{\dot{H}^s_{\mathcal{L}}} \lesssim (1+t)^{-\frac{s+\gamma}{2}} \|(u_0,u_1)\|_{\mathcal{A}^s}.$$

Tools used in the proof:

- We use the group Fourier transform on the graded Lie group $\mathbb{G}$ concerning the spatial variable.
- Sobolev estimates of solutions to the linear Cauchy problem
- Hardy-Littlewood-Sobolev inequality and Gagliardo-Nirenberg inequality
- Banach's fixed point theorem on appropriate space

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- We use the group Fourier transform on the graded Lie group $\mathbb{G}$ concerning the spatial variable.
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- Banach's fixed point theorem on appropriate space

Remarks

- The technical restriction on $1 < p \leq \frac{Q}{Q-2s}$ in the above theorem is due to an application of the Gagliardo-Nirenberg type inequality.

Damped wave equation on graded Lie groups: Global existence

Tools used in the proof:

- We use the group Fourier transform on the graded Lie group $\mathbb{G}$ concerning the spatial variable.
- Sobolev estimates of solutions to the linear Cauchy problem
- Hardy-Littlewood-Sobolev inequality and Gagliardo-Nirenberg inequality
- Banach's fixed point theorem on appropriate space

Remarks

- The technical restriction on $1 < p \leq \frac{Q}{Q-2s}$ in the above theorem is due to an application of the Gagliardo-Nirenberg type inequality.
- Some examples for the admissible range of the exponent p for the global-in-time existence result in certain low homogeneous dimension graded Lie group $\mathbb{G}$ are as follows:
  • When $Q = 1, 2$, we take $s \in (0, 1]$ and $\gamma \in (0, \frac{Q}{2})$ and the exponent satisfies

$$1 + \frac{2\nu}{Q + 2\gamma} < p \begin{cases} < \infty & \text{if } Q \leq 2s, \\ \leq \frac{Q}{Q-2s} & \text{if } Q > 2s. \end{cases}$$

  • When $Q = 3, 4$, we take $s \in (0, 1]$ and $\gamma \in (0, \frac{Q}{2})$ and the exponent satisfies

$$1 + \tfrac{2\nu}{Q+2\gamma} < p \leq \tfrac{Q}{Q-2s} \quad \text{if } 0 < \gamma \leq \tilde{\gamma}$$

$$1 + \tfrac{2\gamma}{Q} \leq p \leq \tfrac{Q}{Q-2s} \quad \text{if } \tilde{\gamma} < \gamma < \tfrac{Q}{2}.$$

Theorem 4 (Dasgupta, K. Mondal and Ruzhansky 2024)

Let $\mathbb{G}$ be a graded Lie group of homogeneous dimension Q and let $\mathcal{R}$ be a positive Rockland operator given by

$$\mathcal{R} := \sum_{j=1}^{n} (-1)^{\frac{\nu_0}{\nu_j}} a_j X_j^{2\frac{\nu_0}{\nu_j}}, \quad \text{with } a_1, a_2, \ldots, a_n > 0, \tag{42}$$

of homogeneous degree $\nu := 2\nu_0$, where ν_0 is any common multiple of dilations weights $\nu_1, \ldots, \nu_n$ on $\mathbb{G}$ and $\{X_1, X_2, \ldots, X_n\}$ is a strong Malcev basis of the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$. Let $\gamma \in \left(0, \frac{Q}{2}\right)$ and the exponent p satisfies $1 < p < p(Q, \gamma, \nu) := 1 + \frac{2\nu}{Q+2\gamma}$. We also assume that the non-negative initial data $(u_0, u_1) \in \dot{H}^{-\gamma} \times \dot{H}^{-\gamma}$ satisfies

$$u_0(x) + u_1(x) \geq C_1 \langle x \rangle^{-Q\left(\frac{1}{2}+\frac{\gamma}{Q}\right)} (\log(e + |x|))^{-1}, \quad x \in \mathbb{G}, \tag{43}$$

where C_1 is a positive constant. Then, there is no global (in-time) weak solution to

$$\begin{cases} u_{tt} + \mathcal{R}u + u_t = |u|^p, & x \in \mathbb{G},\ t > 0, \\ u(0, x) = \varepsilon u_0(x), & x \in \mathbb{G}, \\ u_t(0, x) = \varepsilon u_1(x), & x \in \mathbb{G}. \end{cases}$$

Table

Q	ν	Global Existence	Blow-up
1, 2	≥ 2	$1+\frac{2\nu}{Q+2\gamma} < p \leq \frac{Q}{(Q-2s)_+}$	$1 < p < 1+\frac{2\nu}{Q+2\gamma}$
3	2	$1+\frac{4}{3+2\gamma} < p \leq \frac{Q}{Q-2s}$ if $0<\gamma\leq\tilde{\gamma}$ $1+\frac{2\gamma}{Q} \leq p \leq \frac{Q}{Q-2s}$ if $\tilde{\gamma}<\gamma<\frac{Q}{2}$.	$1 < p < 1+\frac{4}{3+2\gamma}$
3	4	$1+\frac{8}{3+2\gamma} < p \leq \frac{Q}{(Q-2s)}$	$1 < p < 1+\frac{8}{3+2\gamma}$
4, 5, 6	2	$1+\frac{2\nu}{Q+2\gamma} < p \leq \frac{Q}{Q-2s}$ if $0<\gamma\leq\tilde{\gamma}$ $1+\frac{2\gamma}{Q} \leq p \leq \frac{Q}{Q-2s}$ if $\tilde{\gamma}<\gamma<\frac{Q}{2}$.	$1 < p < 1+\frac{2\nu}{Q+2\gamma}$

Table: Ranges of p for global-in-time existence and blow-up of weak solutions for a pair (Q,ν).

Figure

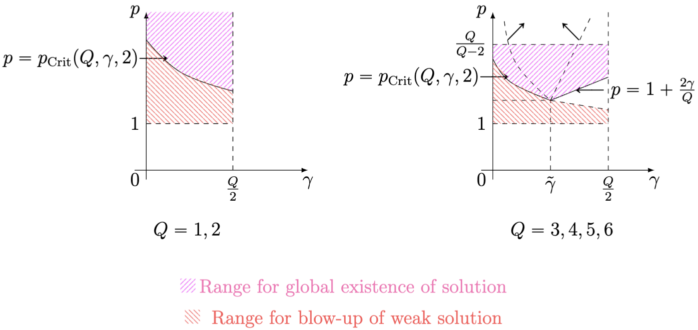

FIGURE 1. Description of the critical exponent in the (γ, p) plane for $\nu = 2$

Sharp lifespan estimates for weak solutions

We define the lifespan T_ε as the maximal existence time for solution of (9), i.e.,

$$T_\varepsilon := \sup \Big\{ T > 0 \ : \ \text{there exists a unique local-in-time solution to the Cauchy} \\ \text{problem (9) on } [0, T) \text{ with a fixed parameter } \varepsilon > 0 \Big\}. \tag{44}$$

We denote $T_{w,\varepsilon}$ and $T_{m,\varepsilon}$ as the lifespan for a weak and mild solution to the Cauchy problem (9), respectively.

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Theorem 5 (Dasgupta, K. Mondal and Ruzhansky 2024)

Let $\mathbb{G}$ be a graded Lie group of homogeneous dimension Q and let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree $\nu \geq 2$. Let $\gamma \in (0, \tilde{\gamma})$ and let the exponent p satisfy $1 < p < p_{Crit}(Q, \gamma, \nu)$ such that

$$1 + \frac{2\gamma}{Q} \leq p \begin{cases} < \infty & \text{if } Q \leq 2, \\ \leq \frac{Q}{Q-2} & \text{if } Q > 2. \end{cases} \tag{45}$$

We also assume that $(u_0, u_1) \in \mathcal{A}^1$ such that $\|(u_0, u_1)\|_{\mathcal{A}^1} < \varepsilon$. Then, there exists a constant ε_0 such that for every $\varepsilon \in (0, \varepsilon_0]$, the lifespan $T_{m,\varepsilon}$ of mild solutions u to the Cauchy problem (9) satisfies the following lower bound condition:

$$T_{m,\varepsilon} \geq C\varepsilon^{-\left(\frac{1}{p-1} - \left(\frac{Q}{2\nu} + \frac{\gamma}{\nu}\right)\right)^{-1}},$$

where the positive constant C is independent of ε, but may depends on p, Q, γ as well as $\|(u_0, u_1)\|_{\mathcal{A}^1}$.

Theorem 6 (Dasgupta, K., Mondal and Ruzhansky 2024)

Let $\mathbb{G}$ be a graded Lie group of homogeneous dimension Q and let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree $\nu \geq 2$. Let $\gamma \in (0, \tilde{\gamma})$ and let the exponent p satisfy $1 < p < p_{Crit}(Q, \gamma, \nu)$. We also assume that $(u_0, u_1) \in \mathcal{A}^1$ such that $\|(u_0, u_1)\|_{\mathcal{A}^1} < \varepsilon$. Then, there exists a constant ε_0 such that for every $\varepsilon \in (0, \varepsilon_0]$, the lifespan $T_{m,\varepsilon}$ of mild solutions u to the Cauchy problem (9) satisfies the following upper bound condition:

$$T_{w,\varepsilon} \leq C\varepsilon^{-\left(\frac{1}{p-1} - \left(\frac{Q}{2\nu} + \frac{\gamma}{\nu}\right)\right)^{-1}}.$$

where the positive constant C is independent of ε, but may depends on p, Q, γ as well as $\|(u_0, u_1)\|_{\mathcal{A}^1}$.

Theorem 6 (Dasgupta, K., Mondal and Ruzhansky 2024)

Let $\mathbb{G}$ be a graded Lie group of homogeneous dimension Q and let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree $\nu \geq 2$. Let $\gamma \in (0, \tilde{\gamma})$ and let the exponent p satisfy $1 < p < p_{Crit}(Q, \gamma, \nu)$. We also assume that $(u_0, u_1) \in \mathcal{A}^1$ such that $\|(u_0, u_1)\|_{\mathcal{A}^1} < \varepsilon$. Then, there exists a constant ε_0 such that for every $\varepsilon \in (0, \varepsilon_0]$, the lifespan $T_{m,\varepsilon}$ of mild solutions u to the Cauchy problem (9) satisfies the following upper bound condition:

$$T_{w,\varepsilon} \leq C\varepsilon^{-\left(\frac{1}{p-1} - \left(\frac{Q}{2\nu} + \frac{\gamma}{\nu}\right)\right)^{-1}}.$$

where the positive constant C is independent of ε, but may depends on p, Q, γ as well as $\|(u_0, u_1)\|_{\mathcal{A}^1}$.

Remark

Therefore, if the exponent p satisfies $1 + \frac{2\gamma}{Q} \leq p \leq \frac{Q}{Q-2}$, then we can claim the sharp estimate for the lifespan T_ε as

$$T_\varepsilon \begin{cases} = \infty & \text{if } p > p_{\text{Crit}}(Q, \gamma, \nu), \\ \simeq C\varepsilon^{-\left(\frac{1}{p-1} - \left(\frac{Q}{2\nu} + \frac{\gamma}{\nu}\right)\right)^{-1}} & \text{if } p < p_{\text{Crit}}(Q, \gamma, \nu), \end{cases}$$

for some $\gamma \in \left(0, \frac{Q}{2}\right)$, where the positive constant C is independent of ε.

Theorem 7 (Berikbol, K., Mondal, Ruzhansky, 2024)

Let $\mathbb{H}^n$ be the Heisenberg group with the homogeneous dimension $Q = 2n + 2$ and let $\Delta_{\mathbb{H}}$ be the sub-Laplacian on $\mathbb{H}^n$. Let $\gamma \in \left(0, \frac{Q}{2}\right)$ and let the exponent p satisfy

$$p = p_{Crit}(Q, \gamma) := 1 + \frac{4}{Q + 2\gamma}.$$

We assume that the non-negative initial data $(u_0, u_1) \in \dot{H}^{-\gamma}_{\Delta_{\mathbb{H}}} \times \dot{H}^{-\gamma}_{\Delta_{\mathbb{H}}}$ satisfies

$$u_0(\eta) + u_1(\eta) \geq C_1 \langle \eta \rangle^{-Q\left(\frac{1}{2} + \frac{\gamma}{Q}\right)} (\log(e + |\eta|))^{-1}, \quad \eta = (x, y, \tau) \in \mathbb{H}^n,$$

where C_1 is a positive constant. Then, there is no global (in-time) weak solution to

$$\begin{cases} u_{tt} - \Delta_{\mathbb{H}} u + u_t = |u|^p, & g \in \mathbb{H}^n, \ t > 0, \\ u(0, g) = u_0(g), & g \in \mathbb{H}^n, \\ u_t(0, g) = u_1(g), & g \in \mathbb{H}^n. \end{cases}$$

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$$u_0(\eta) + u_1(\eta) \geq C_1 \langle \eta \rangle^{-Q\left(\frac{1}{2} + \frac{\gamma}{Q}\right)} (\log(e + |\eta|))^{-1}, \quad \eta = (x, y, \tau) \in \mathbb{H}^n,$$

where C_1 is a positive constant. Then, there is no global (in-time) weak solution to

$$\begin{cases} u_{tt} - \Delta_{\mathbb{H}} u + u_t = |u|^p, & g \in \mathbb{H}^n, \ t > 0, \\ u(0, g) = u_0(g), & g \in \mathbb{H}^n, \\ u_t(0, g) = u_1(g), & g \in \mathbb{H}^n. \end{cases}$$

Conjecture:

The critical case $p := p_{\mathrm{Crit}}(Q, \gamma, \nu)$ belongs to the blow-up range for the damped wave equation on graded Lie group.

Now consider the following Cauchy problem for the heat equation

$$\begin{cases} w_t - \Delta_{\mathbb{H}^n} w = 0, & g \in \mathbb{H}^n, t > 0 \\ w(0,g) = u_0(g) + u_1(g), & g \in \mathbb{H}^n, \end{cases} \tag{46}$$

where $(u_0, u_1) \in \left(H^s_{\Delta_{\mathbb{H}}} \cap \dot{H}^{-\gamma}_{\Delta_{\mathbb{H}}}\right) \times \left(H^s_{\Delta_{\mathbb{H}}} \cap \dot{H}^{-\gamma}_{\Delta_{\mathbb{H}}}\right)$ with $s \geq 0$ and $\gamma \in \mathbb{R}$ such that $s + \gamma \geq 0$.

Now consider the following Cauchy problem for the heat equation

$$\begin{cases} w_t - \Delta_{\mathbb{H}^n} w = 0, & g \in \mathbb{H}^n, t > 0 \\ w(0,g) = u_0(g) + u_1(g), & g \in \mathbb{H}^n, \end{cases} \tag{46}$$

where $(u_0, u_1) \in \left(H^s_{\Delta_{\mathbb{H}}} \cap \dot{H}^{-\gamma}_{\Delta_{\mathbb{H}}}\right) \times \left(H^s_{\Delta_{\mathbb{H}}} \cap \dot{H}^{-\gamma}_{\Delta_{\mathbb{H}}}\right)$ with $s \geq 0$ and $\gamma \in \mathbb{R}$ such that $s + \gamma \geq 0$.
We have the following $\dot{H}^s_{\Delta_{\mathbb{H}}}$-decay estimate for the solution to the Cauchy problem (46) as

$$\|w(t,\cdot)\|_{\dot{H}^s_{\Delta_{\mathbb{H}}}} \lesssim (1+t)^{-\frac{s+\gamma}{2}} \left(\|u_0\|_{H^s_{\Delta_{\mathbb{H}}} \cap \dot{H}^{-\gamma}_{\Delta_{\mathbb{H}}}} + \|u_1\|_{H^s_{\Delta_{\mathbb{H}}} \cap \dot{H}^{-\gamma}_{\Delta_{\mathbb{H}}}}\right) \tag{47}$$

for any $t \geq 0$.

Now consider the following Cauchy problem for the heat equation

$$\begin{cases} w_t - \Delta_{\mathbb{H}^n} w = 0, & g \in \mathbb{H}^n, t > 0 \\ w(0, g) = u_0(g) + u_1(g), & g \in \mathbb{H}^n, \end{cases} \tag{46}$$

where $(u_0, u_1) \in \left(H^s_{\Delta_{\mathbb{H}}} \cap \dot{H}^{-\gamma}_{\Delta_{\mathbb{H}}}\right) \times \left(H^s_{\Delta_{\mathbb{H}}} \cap \dot{H}^{-\gamma}_{\Delta_{\mathbb{H}}}\right)$ with $s \geq 0$ and $\gamma \in \mathbb{R}$ such that $s + \gamma \geq 0$.
We have the following $\dot{H}^s_{\Delta_{\mathbb{H}}}$-decay estimate for the solution to the Cauchy problem (46) as

$$\|w(t, \cdot)\|_{\dot{H}^s_{\Delta_{\mathbb{H}}}} \lesssim (1+t)^{-\frac{s+\gamma}{2}} \left(\|u_0\|_{H^s_{\Delta_{\mathbb{H}}} \cap \dot{H}^{-\gamma}_{\Delta_{\mathbb{H}}}} + \|u_1\|_{H^s_{\Delta_{\mathbb{H}}} \cap \dot{H}^{-\gamma}_{\Delta_{\mathbb{H}}}}\right) \tag{47}$$

for any $t \geq 0$. Recall that, we have the following $\dot{H}^s_{\Delta_{\mathbb{H}}}$-decay estimate for the solution to linear damped wave equation

$$\|u(t, \cdot)\|_{\dot{H}^s_{\Delta_{\mathbb{H}}}} \lesssim (1+t)^{-\frac{s+\gamma}{2}} \left(\|u_0\|_{H^s_{\Delta_{\mathbb{H}}} \cap \dot{H}^{-\gamma}_{\Delta_{\mathbb{H}}}} + \|u_1\|_{H^{s-1}_{\Delta_{\mathbb{H}}} \cap \dot{H}^{-\gamma}_{\Delta_{\mathbb{H}}}}\right), \tag{48}$$

for any $t \geq 0$.

Theorem 8 (Berikbol, K., Mondal, Ruzhansky, 2024)

Let $(u_0, u_1) \in \left(H^s_{\Delta_{\mathbb{H}}} \cap \dot{H}^{-\gamma}_{\Delta_{\mathbb{H}}}\right) \times \left(H^s_{\Delta_{\mathbb{H}}} \cap \dot{H}^{-\gamma}_{\Delta_{\mathbb{H}}}\right)$ *with* $s \geq 0$ *and* $\gamma \in \mathbb{R}$ *such that* $s + \gamma + 2 \geqslant 0$. *Let* u *and* w *be the solutions to the linear Cauchy problems (9) and (46), respectively. Then,* $u - w$ *satisfies*

$$\|u(t,\cdot) - w(t,\cdot)\|_{\dot{H}^s_{\Delta_{\mathbb{H}}}} \lesssim (1+t)^{-\frac{s+\gamma}{2}-1} \left(\|u_0\|_{H^s_{\Delta_{\mathbb{H}}} \cap \dot{H}^{-\gamma}_{\Delta_{\mathbb{H}}}} + \|u_1\|_{H^s_{\Delta_{\mathbb{H}}} \cap \dot{H}^{-\gamma}_{\Delta_{\mathbb{H}}}} \right).$$

Theorem 8 (Berikbol, K., Mondal, Ruzhansky, 2024)

Let $(u_0, u_1) \in \left(H^s_{\Delta_{\mathbb{H}}} \cap \dot{H}^{-\gamma}_{\Delta_{\mathbb{H}}}\right) \times \left(H^s_{\Delta_{\mathbb{H}}} \cap \dot{H}^{-\gamma}_{\Delta_{\mathbb{H}}}\right)$ *with* $s \geq 0$ *and* $\gamma \in \mathbb{R}$ *such that* $s + \gamma + 2 \geqslant 0$. *Let* u *and* w *be the solutions to the linear Cauchy problems (9) and (46), respectively. Then,* $u - w$ *satisfies*

$$\|u(t,\cdot) - w(t,\cdot)\|_{\dot{H}^s_{\Delta_{\mathbb{H}}}} \lesssim (1+t)^{-\frac{s+\gamma}{2}-1} \left(\|u_0\|_{H^s_{\Delta_{\mathbb{H}}} \cap \dot{H}^{-\gamma}_{\Delta_{\mathbb{H}}}} + \|u_1\|_{H^s_{\Delta_{\mathbb{H}}} \cap \dot{H}^{-\gamma}_{\Delta_{\mathbb{H}}}}\right).$$

- We see that the decay is enhanced by a factor of $(1+t)^{-1}$ when we subtract the solution to the damped wave equation by the solution to heat equation (46).
- This concludes that the diffusion phenomenon is also valid in the framework of the negative order Sobolev space $\dot{H}^{-\gamma}_{\Delta_{\mathbb{H}}}$.

THANK YOU!