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Smooth barycentric decompositions of Weyl chambers (a helpful tool to handle inverse spherical Fourier transforms on noncompact symmetric spaces of higher rank)

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Riemannian symmetric spaces of noncompact type (RSSN)

- G semisimple Lie group (noncompact, connected, finite center)
- K maximal compact subgroup of G
- RSSN: $X = G/K$ Cartan-Hadamard manifold
- Cartan decomposition $\sim$ generalized polar decomposition:
$$G = K(\exp \overline{\mathfrak{a}^+})K \ni x = k_1(\exp x^+)k_2$$
- Haar measure: $dx = \text{const.}\, \delta(x^+)\, dk_1\, dx^+ dk_2$
where $\delta(x^+) = \prod_{\alpha\in\Sigma^+} (\sinh\langle\alpha, x^+\rangle)^{m_\alpha} \sim e^{\langle 2\rho, x^+\rangle}$
and $\rho = \sum_{\alpha\in\Sigma^+} \frac{m_\alpha}{2}\alpha \in \mathfrak{a}^+$
- Dimensions:

$\ell = \dim \mathfrak{a}$	rank
$d = \ell + \sum_{\alpha\in\Sigma^+} m_\alpha$	dimension
$D = \ell + 2\lvert\Sigma_r^+\rvert$	pseudo-dimension

Fourier analysis on RSSN [Harish-Chandra, Helgason]

There is a Fourier transform on RSSN $X = G/K$.
For bi–K–invariant functions $f : G \to \mathbb{C}$, it reduces to

Spherical Fourier transform

$$\mathcal{H}f(\lambda) = \int_G dx\, f(x)\, \varphi_{-\lambda}(x) \quad \forall\, \lambda \in \mathfrak{a}$$

Inversion formula

$$f(x) = \text{const.} \int_{\mathfrak{a}} d\lambda\, |\mathbf{c}(\lambda)|^{-2}\, \varphi_\lambda(x)\, \mathcal{H}f(\lambda)$$

- The spherical functions $\varphi_\lambda(x)$ are analogs of Bessel functions for the Euclidean Fourier transform of radial functions
- Behavior of $\varphi_\lambda(x)$ $\begin{cases} \text{a lot of information available} \\ \text{still not fully understood} \end{cases}$
- The Plancherel measure $|\mathbf{c}(\lambda)|^{-2}$ is known explicitly [Gindikin-Karpelevich]

Two main dispersive PDE on RSSN

Schrödinger equation

(S) $$\begin{cases} i\,\partial_t u(t,x) \pm \Delta_x u(t,x) = F(t,x) \\ u(0,x) = f(x) \end{cases}$$

Wave equation

(W) $$\begin{cases} \partial_t^2 u(t,x) - \Delta_x u(t,x) = F(t,x) \\ u(0,x) = f(x),\ \partial_t|_{t=0}\, u(t,x) = g(x) \end{cases}$$

Remarks.

- Similar analysis and properties
- Simpler statements for Schrödinger

Schrödinger equation on RSSN

Homogeneous solution

$$u(t,x) = e^{i\,t\,\Delta} f(x) = (f * s_t)(x)$$

where the Schrödinger kernel

$$s_t(x) = \text{const.} \int_{\mathfrak{a}} d\lambda\, |\mathbf{c}(\lambda)|^{-2}\, \varphi_\lambda(x)\, e^{-i\,(\|\rho\|^2 + \|\lambda\|^2)\,t}$$

is formally the heat kernel with imaginary time

Inhomogeneous solution (Duhamel's formula)

$$u(t,\cdot\,) = e^{i\,t\,\Delta} f - i \int_0^t ds\, e^{i\,(t-s)\Delta} F(s,\cdot\,)$$

Kernel estimates

Rank one [A-Pierfelice 2009]

For $t \in \mathbb{R}^*$ and $r \geq 0$,

$$|s_t(r)| \lesssim e^{-\rho r} \times \begin{cases} (1+r)^{\frac{d-1}{2}} \, |t|^{-\frac{d}{2}} & \text{if } 0 < |t| \leq 1+r \\ (1+r) \, |t|^{-\frac{3}{2}} & \text{if } |t| \geq 1+r \end{cases}$$

Higher rank [A-Meda-Pierfelice-Vallarino-Zhang 2023]

There exists $n > 0$ such that, for $t \in \mathbb{R}^*$ and $x \in G/K$,

$$|s_t(r)| \lesssim e^{-\langle \rho, x^+ \rangle} \, (1+\|x^+\|)^n \times \begin{cases} |t|^{-\frac{d}{2}} & \text{if } 0 < |t| < 1 \\ |t|^{-\frac{D}{2}} & \text{if } |t| \geq 1 \end{cases}$$

Two important inequalities

Dispersive inequality [A-P, A-M-P-V-Z]

Let $2 < q \leq \infty$. Then

$$\|e^{it\Delta}\|_{L^{q'} \to L^q} \lesssim \begin{cases} |t|^{-(\frac{1}{2}-\frac{1}{q})d} & \text{if } 0 < |t| < 1 \\ |t|^{-\frac{D}{2}} & \text{if } |t| \geq 1 \end{cases}$$

Strichartz mixed norm :

$$\|u(t,x)\|_{L_t^p L_x^q} = \Big[\int_{\mathbb{R}} dt \Big(\int_{G/K} dx \, |u(t,x)|^q \Big)^{\frac{p}{q}} \Big]^{\frac{1}{p}}$$

Strichartz inequality [A-P, A-M-P-V-Z]

Solutions to (S) satisfy

$$\|u(t,x)\|_{L_t^{\tilde{p}} L_x^{\tilde{q}}} \lesssim \|f(x)\|_{L_x^2} + \|F(t,x)\|_{L_t^{p'} L_x^{q'}}$$

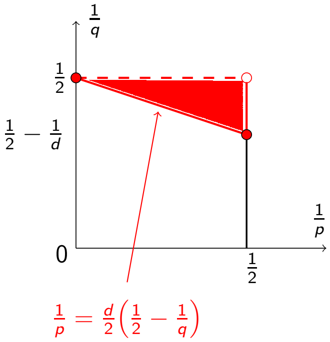
for all admissible couples (p,q), $(\tilde{p},\tilde{q})$

Strichartz inequality (continued)

Definition

A couple (p, q) is admissible if $(\frac{1}{p}, \frac{1}{q})$ belongs to the triangle

$$\left\{ \left(\tfrac{1}{p}, \tfrac{1}{q}\right) \in \left(0, \tfrac{1}{2}\right] \times \left(0, \tfrac{1}{2}\right) \;\middle|\; \tfrac{1}{p} \geq \tfrac{d}{2}\left(\tfrac{1}{2} - \tfrac{1}{q}\right) \right\} \cup \left\{ \left(0, \tfrac{1}{2}\right) \right\}$$



Main tools

- Smooth barycentrix decomposition of Weyl chambers
  $\rightsquigarrow$ kernel estimates
- Improved Hadamard parametrix for the wave operator $\cos t\sqrt{-\Delta}$ on G/K
- Kunze-Stein phenomenon
  $\rightsquigarrow$ dispersive inequality for $|t|$ large
- TT^* argument (Ginibre-Velo, Keel-Tao)
  $\rightsquigarrow$ Strichartz inequality

Suitable version of the Kunze-Stein phenomenon

Let $2 \leq q < \infty$. Then there exists a constant $C > 0$ such that, for every bi-K-invariant (measurable) function κ on G,

$$\| \cdot * \kappa \|_{L^{q'} \to L^q} \leq C \left\{ \int_G dx\, \varphi_0(x)\, |\kappa(x)|^{\frac{q}{2}} \right\}^{\frac{2}{q}}$$

Motivation

The analysis of oscillating integrals such as

$$s_t(x) = \text{const.} \int_{\mathfrak{a}} d\lambda \, |\mathbf{c}(\lambda)|^{-2} \, \varphi_\lambda(x) \, e^{-i\,(\|\rho\|^2 + \|\lambda\|^2)t}$$

requires **integrations by parts**. The Plancherel density

$$|\mathbf{c}(\lambda)|^{-2} = \prod_{\alpha \in \Sigma_{\text{red}}^+} \left| \mathbf{c}_\alpha\!\left(\tfrac{\langle \alpha, \lambda \rangle}{\|\alpha\|^2}\right) \right|^{-2}$$

is a product of one-dimensional differentiable symbols but, in higher rank, it is **not** a differentiable symbol in general. Thus differentiating arbitrarily $|\mathbf{c}(\lambda)|^{-2}$ produces no additional global decay at infinity.

We overcome this problem by splitting up each Weyl chamber $w.\mathfrak{a}^+$ into subsectors $w.S_j$, where we differentiate along a suitable direction $w.\lambda_j$.

Rough barycentric decomposition

$$\sum_{w\in W}\sum_{1\leq j\leq \ell} \mathbb{1}_{w.S_j} = 1 \quad \text{a.e.}$$

- Simple roots : $\alpha_1, \ldots, \alpha_\ell$
- Dual basis of $\mathfrak{a}$: $\{\lambda_1, \ldots, \lambda_\ell\}$

  $\rightsquigarrow$ $\overline{\mathfrak{a}^+} = \mathbb{R}_+\lambda_1 + \ldots + \mathbb{R}_+\lambda_\ell$
- Subsectors of $\overline{\mathfrak{a}^+}$:

  $S_j = \{H \in \overline{\mathfrak{a}^+} \,|\, \langle\alpha_j, H\rangle = \max_{1\leq k\leq\ell}\langle\alpha_k, H\rangle\} \quad \forall\, 1 \leq j \leq \ell$

  $\rightsquigarrow$ $\overline{\mathfrak{a}^+} = \bigcup_{1\leq j\leq\ell} S_j$

  $\rightsquigarrow$ $\mathfrak{a} = \bigcup_{w\in W} w.\overline{\mathfrak{a}^+} = \bigcup_{w\in W}\bigcup_{1\leq j\leq\ell} w.S_j$

Examples of barycentric subdivisions of $\overline{\mathfrak{a}^+}$

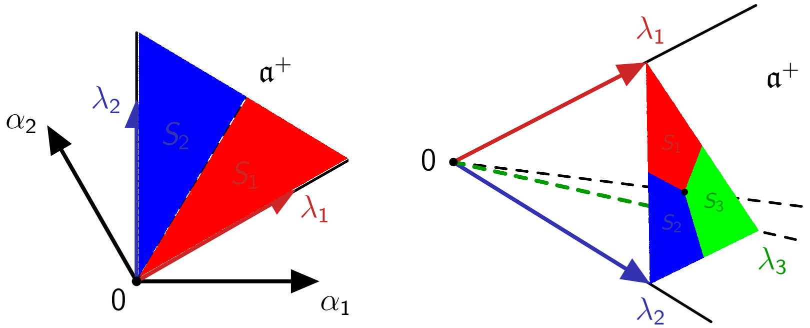


Figure: Root systems A_2 and A_3

Smooth barycentric decomposition

$$\sum_{w\in W}\sum_{1\le j\le \ell}\chi_{w.S_j} = 1 \quad \text{on } \mathfrak{a}\smallsetminus\{0\}$$

- $\chi_{w.S_j}$ is a smooth homogeneous symbol of order 0 on $\mathfrak{a}\smallsetminus\{0\}$
- **Dichotomy :** for every $\alpha\in\Sigma$, $w\in W$ and $1\le j\le \ell$,
  - either $\langle\alpha, w.\lambda_j\rangle = 0$
  - or $\langle\alpha,\lambda\rangle \asymp \|\lambda\| \;\; \forall\, \lambda\in \operatorname{supp}\chi_{w.S_j}$

Application

Away from the origin, each function

$$\chi_{w.S_j}(\lambda)\,|\mathbf{c}(\lambda)|^{-2}$$

behaves as a symbol of order $n-\ell$ under differentiation along $w.\lambda_j$,

i.e., $\quad |\partial^N_{w.\lambda_j}\{\chi_{w.S_j}(\lambda)\,|\mathbf{c}(\lambda)|^{-2}\}| \lesssim |\lambda|^{n-\ell-N} \quad \forall\, |\lambda|\gtrsim 1$

The End

Thank you for your attention