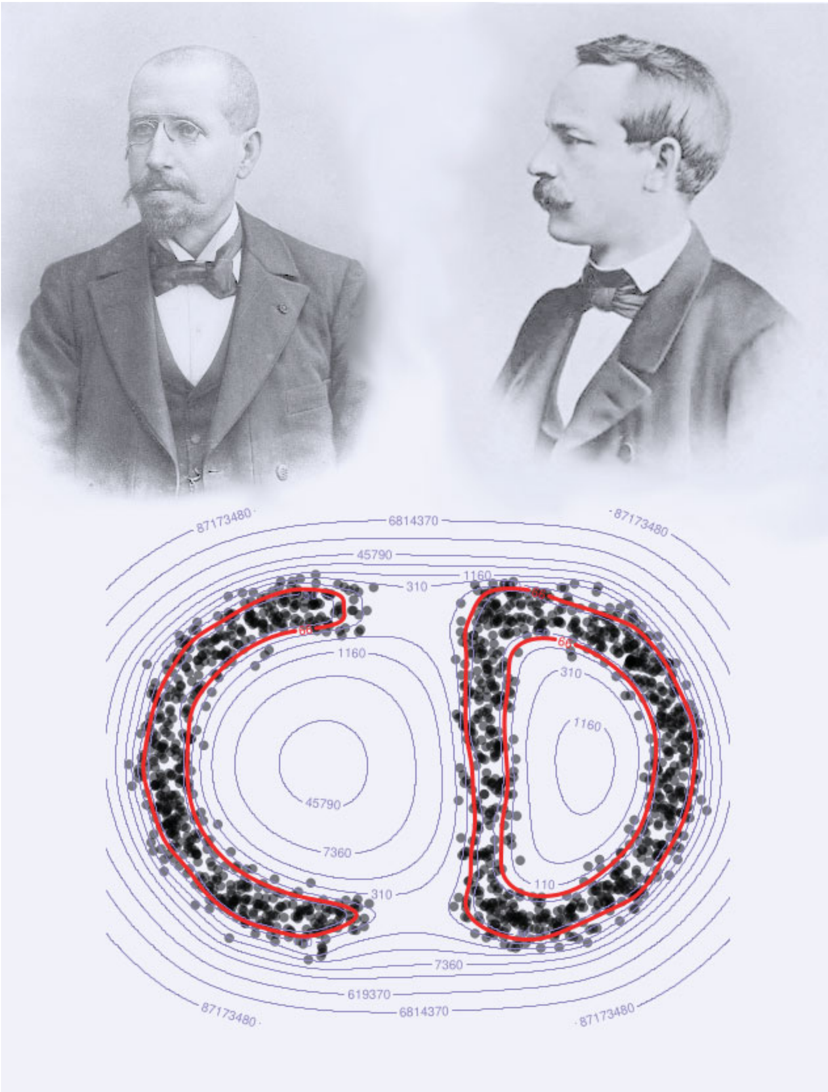
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Christoffel Darboux analysis

Mihai Putinar

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joint work with

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S. Biswas (Calcutta)
B. Gustafsson (Stockholm)
M. Korda (Toulouse)
J.-B. Lasserre (Toulouse)
I. Mezic (UCSB)
E. Pauwels (Toulouse)
E. Saff (Vanderbilt U.)
N. Stylianopoulos (Cyprus)

Complex orthogonal polynomials

$\mu \geq 0$ positive Borel measure, rapidly decreasing on $\mathbb{C}$, of infinite support, so that the *orthogonal polynomials* $P_n(z),\ n \geq 0$, are well defined by

$$P_n(z) = \gamma_n z^n + O(z^{n-1}), \quad \gamma_n > 0,$$

and

$$\langle P_n, P_k \rangle_{2,\mu} = \delta_{nk}.$$

Real moment data (observables) and complex orthogonal polynomials are interchangeable via elementary matrix operations.

Christoffel-Darboux kernel

$$K_N(z,w) = \sum_{k=0}^{N-1} P_k(z)\overline{P_k(w)},$$

is the reproducing kernel in the space $\mathbb{C}_N[z]$ of polynomials of degree less than N:

$$\langle h, K_N(\cdot, w)\rangle_{2,\mu} = h(w), \quad \deg h < N.$$

E. B. Christoffel, *Über die Gaussische Quadratur und eine Verallgemeinerung derselben*, J. Reine Angew. Math. 55 (1858), 61-82.

G. Darboux, *Mémoire sur l'approximation des fonctions de trés-grands nombres, et sur une classe étendue de developpements en série*, Liouville J. (3) 4 (1878), 5-56; 377-416.

Christoffel function

$$|h(w)| \leq \|h\| \|K_N(\cdot, w)\|$$

with optimal solution $K_N(\cdot, w)$:

$$\max \frac{|h(w)|}{\|h\|} = \|K_N(\cdot, w)\|,$$

or equivalently

$$\Lambda_N(\mu, w) := \min \frac{\|h\|_{2,\mu}^2}{|h(w)|^2} = \frac{1}{K_N(w, w)}$$

where $\deg h < N$ and $h(w) \neq 0$.

The asymptotics of Christoffel's function $\Lambda_N(\mu, w)$ were and remain central for many problems of mathematical analysis.

Moment indeterminateness (see Seminar 2 at IISC)

Note that the orthogonal polynomials $P_n(z)$ depend only on the moments of the underlying measure μ:

$$c_{kn} = \int_{\mathbb{C}} z^k \overline{z}^n d\mu(z), \quad k, n \geq 0.$$

Solving the moment problem (i.e. recovery of μ from $(c_{kn})_{k,n=0}^{\infty}$) encounters a natural and difficult obstacle: *is the measure μ unique?*

M. Riesz (1923), R. Nevanlinna (1924): *If $\mathrm{supp}\mu \subset \mathbb{R}$, then the moments determine the measure if and only if*

$$\lim_{N\to\infty} \Lambda_N(\mu, z) = 0,$$

for at least one $z \in \mathbb{C} \setminus \mathbb{R}$ (and hence for all).

Maximal point masses

Note that always $\Lambda_{N+1}(\mu, z) \leq \Lambda_N(\mu, z)$, so

$$\Lambda(\mu, z) = \lim_{N \to \infty} \Lambda_N(\mu, z)$$

exists.

In case $x \in \mathbb{R}$,

$$\mu(\{x\}) \leq \Lambda(\mu, x) = \min \int |h(y)|^2 d\mu(y), \quad h(x) = 1,$$

and the upper bound is attained (by extremal solutions of the moment problem).

The unit circle

$w(t) \geq 0$ integrable weight on $[-\pi, \pi)$. With geometric mean

$$G(w) = \exp(\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln w(t)dt),$$

if w satisfies Szegö's condition

$$\int_{-\pi}^{\pi} \ln w(t)dt > -\infty,$$

and $G(w) = 0$ otherwise.

Szegö (1914) $\Lambda(w(t)dt, 0) = G(w)$.

Formulated equivalently as an extremum problem, with $z = e^{it}$:

$$\lim_{n\to\infty} \min_{A_j} \frac{1}{2\pi}\int_{-\pi}^{\pi} |z^n + A_1 z^{n-1} + \ldots + A_n|^2 w(t)dt = G(w).$$

Density of complex polynomials in Lebesgue space

$\mu \geq 0$ positive Boreal measure on $[-\pi, \pi)$ with Lebesgue decompsoition

$$\mu = w(t)dt + \mu_{sc} + \mu_d.$$

where $w(t)$ has bounded variation.

Kolmogorov (1941), Krein (1945) *The system* $1, z, z^2, \ldots$ *is dense in* $L^p(\mu), p \geq 1$, *if and only if*

$$\int_{-\pi}^{\pi} |\ln w(t)| dt = \infty.$$

Hint: By Szegö's Theorem, if $G(w) > 0$, then e^{-it} cannot be approximated by $1, e^{it}, e^{i2t}, \ldots$.

Accelerated convergence of Fourier series

Let $\mu \geq 0$ be a positive measure supported on a compact set $I \subset \mathbb{R}$. For a continuous function $f \in C(I)$ we set

$$S_N(\alpha, f, x) = \sum_{k=0}^{N-1} \langle f, P_k \rangle P_k(x).$$

Then

$$\sup_{\|f\|_{\infty,I} \leq 1} |S_N(\alpha, f, x)|^2 \leq \|f\|_{2,\alpha}^2 K_N(\alpha; x, x) \leq \alpha(I) K_N(\alpha; x, x).$$

Consequently

$$|f(x) - S_N(\alpha, f, x)| = |f(x) - Q(x) - S_N(\alpha, f - Q, x)| \leq$$

$$\inf_{\deg Q < N} \|f - Q\|_{\infty,K} (1 + \alpha(I) \sqrt{K_N(\alpha : x, x)}).$$

Lebesgue (1905) was the first to use this scheme for studying the convergence of Fourier type developments.

The unit disk

$\mu = dA$ on $\mathbb{D} = \{z \in \mathbb{C} : \ |z| < 1\}$

$$K_N(z,w) \to K(z,w) = \frac{1}{\pi(1 - z\overline{w})^2}, \quad z, w \in \mathbb{D},$$

the Bergman kernel, and

$$K_N(z,z) = \frac{N+1}{\pi} \frac{1}{|z|^2 - 1} |z|^{2N+2}(1 + O(1/N)), \quad |z| > 1.$$

measures how fast a polynomial of degree less than N lifts outside the disk, keeping its square norm average on $\mathbb{D}$ bounded.

Classical asymptotics extended to more general domains by Carleman (1922) and Suetin (1969).

1. Shape reconstruction in 2D

$G = \cup G_j$ an *archipelago*, i.e. a finite union of simply connected domains, with real analytic boundary $\Gamma = \partial G$ and $\mu = \chi_G d\mathrm{Area}$.

"Observables" are finitely many moments

$$a_{mn} = \int z^m \overline{z}^n \; d\mu(z)$$

such as derived from geometric tomography.
They determine the complex orthogonal polynomials $P_n(z), \; 0 \leq n \leq N$, and the CD-kernel

$$K_n(z, w) = \sum_{j=0}^{n-1} P_j(z)\overline{P_j(w)}.$$

Christoffel function as defining function of the archipelago

Normalized Christoffel function

$$\gamma_n(z) = [K_n(z,z)]^{-1/2}$$

satisfies:

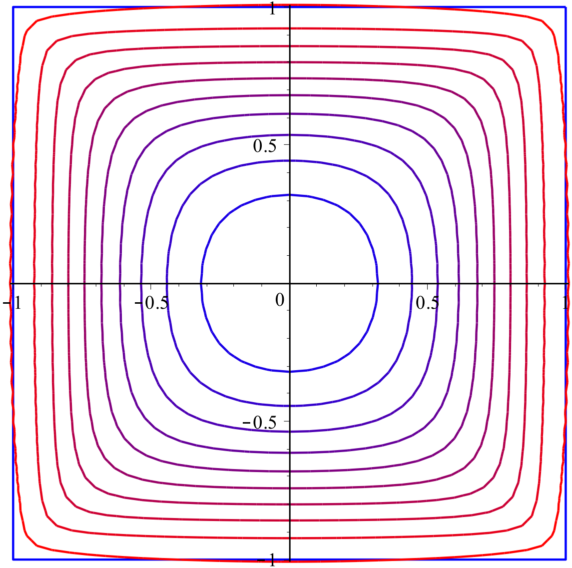
$$\sqrt{\pi}\mathrm{dist}(z,\Gamma) \le \gamma_n(z) \le C\mathrm{dist}(z,\Gamma)$$

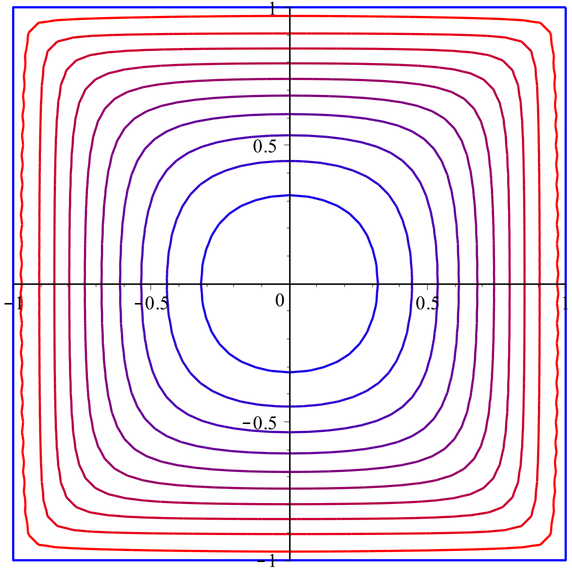
for $z \in G$, close to Γ. Moreover:

$$\gamma_n(z) = O(\frac{1}{n}), \quad z \in \Gamma.$$

Outside $\overline{G}$ this function decreases exponentially to zero. On analytic boundaries one has sharp estimates.

Reconstruction experiments

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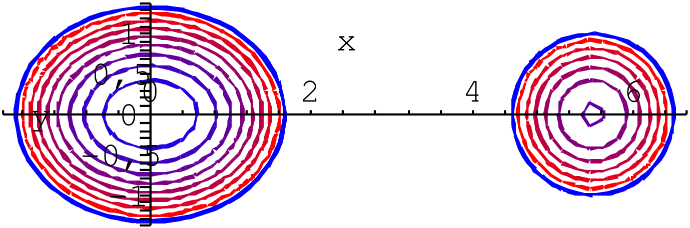
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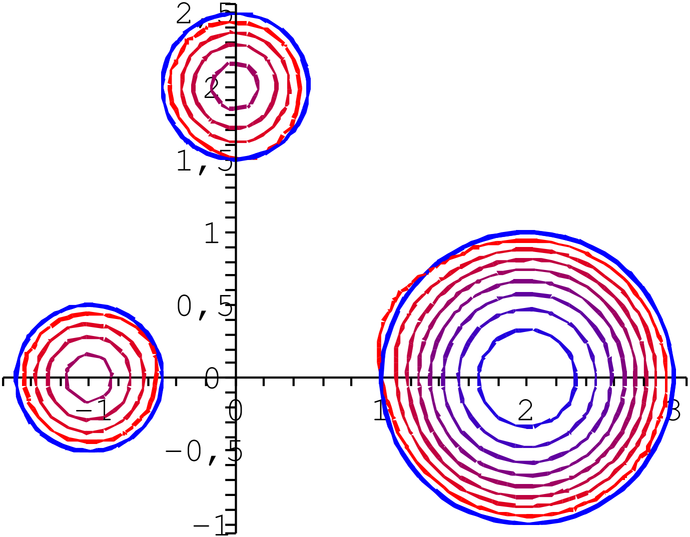
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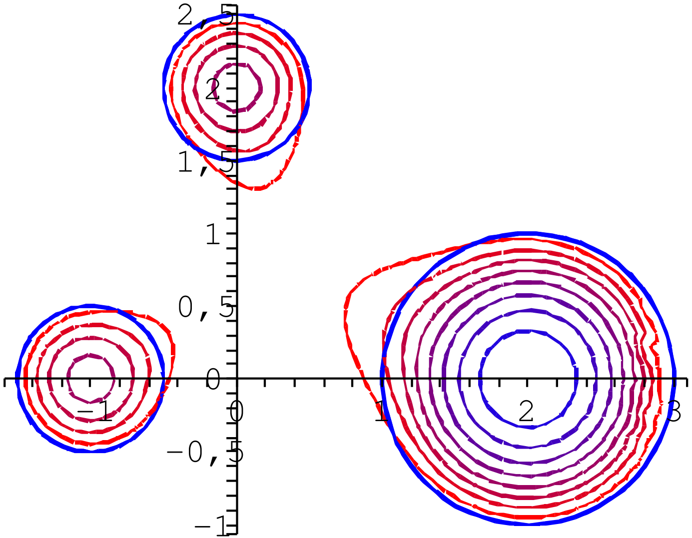
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Nonlinear dynamics

$$x_{k+1} = T(x_k)$$

$$T : X \to X$$

Linear operator

$$Uf = f \circ T$$

$$U : \mathcal{H} \to \mathcal{H}$$

Measure-preserving invertible setting: $\mathcal{H} = L_2(\nu) \Rightarrow U$ unitary $\Rightarrow \sigma(U) \subset \mathbb{T}$

Goal: Understand spectrum of U from data

Spectral theorem

1. $U = \int_{\mathbb{T}} z \, dE(z)$
2. $\mu_f(\cdot) := \langle E(\cdot) f, f \rangle_{L_2(\nu)}$ is a positive measure on $\mathbb{T}$

Fact: If $\overline{\mathrm{span}\{f, Uf, U^{-1}f, U^2 f, U^{-2} f, \ldots\}} = \mathcal{H}$, then μ_f determines U

Reconstructing μ_f from moments

$$\mu = \mu_{\text{at}} + \mu_{\text{ac}} + \mu_{\text{sc}}$$

μ_{at} → Point spectrum (Eigenvalues)

μ_{ac} → AC spectrum $d\mu_{\text{ac}} = \rho(\theta)d\theta$

μ_{sc} → SC spectrum

Point spectrum

$$\lim_{N\to\infty} \frac{1}{K_N(e^{i2\pi\theta}, e^{i2\pi\theta})} = \mu_{\text{at}}(\{e^{i2\pi\theta}\})$$

for all $\theta \in [0, 1]$

AC spectrum [Mate, Nevai, Totik, 91']

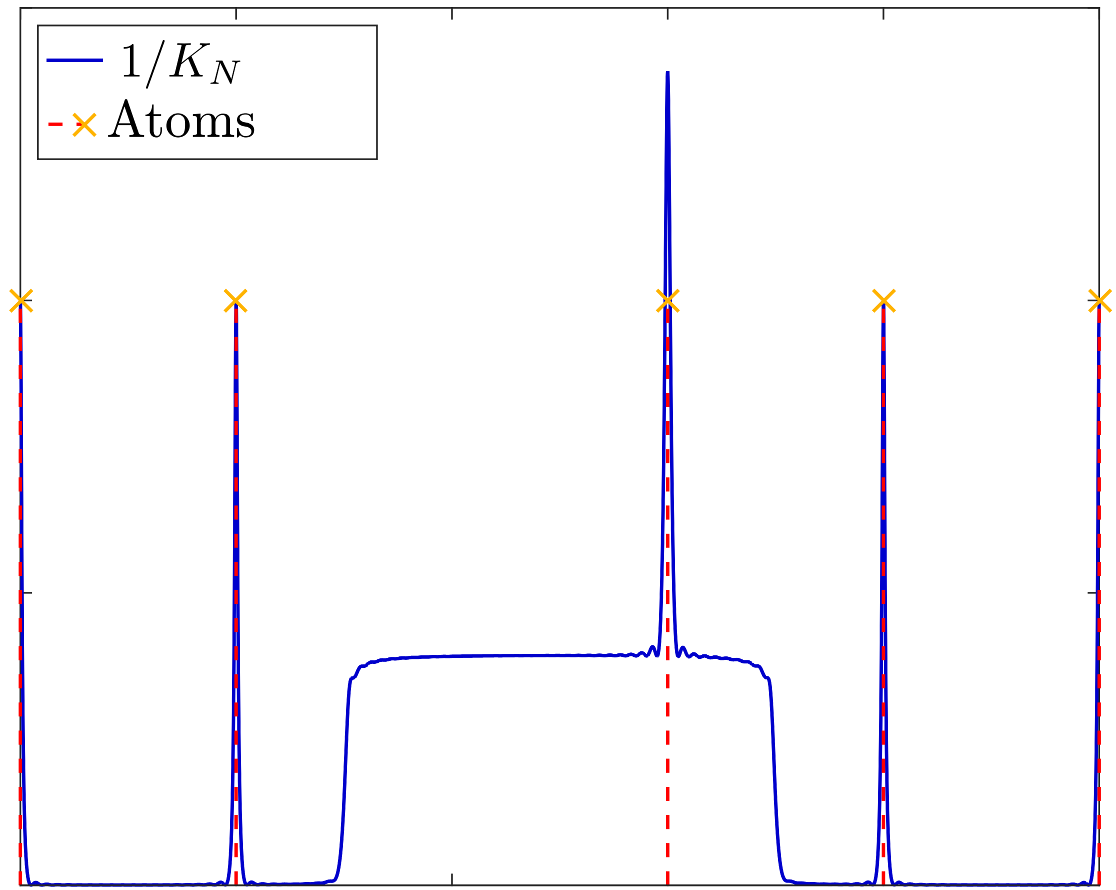
$$\lim_{N\to\infty} \frac{N}{K_N(e^{i2\pi\theta}, e^{i2\pi\theta})} = \rho(\theta)$$

for almost all $\theta \in [0, 1]$

Reconstructing μ_f from moments - Example

$\mu = 4I_{[0.3,0.7]}d\theta + 0.1(\delta_0 + \delta_{0.2} + \delta_{0.6} + \delta_{0.8})$

Point spectrum, N = 100



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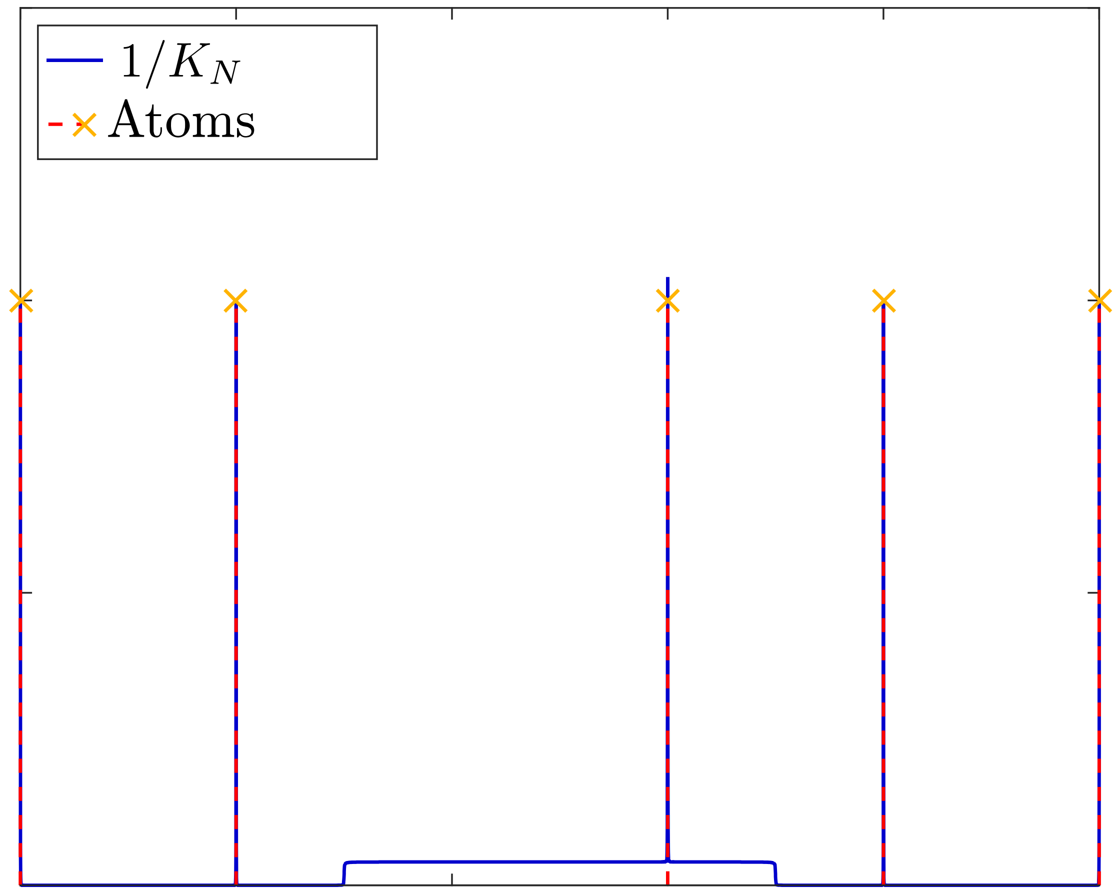

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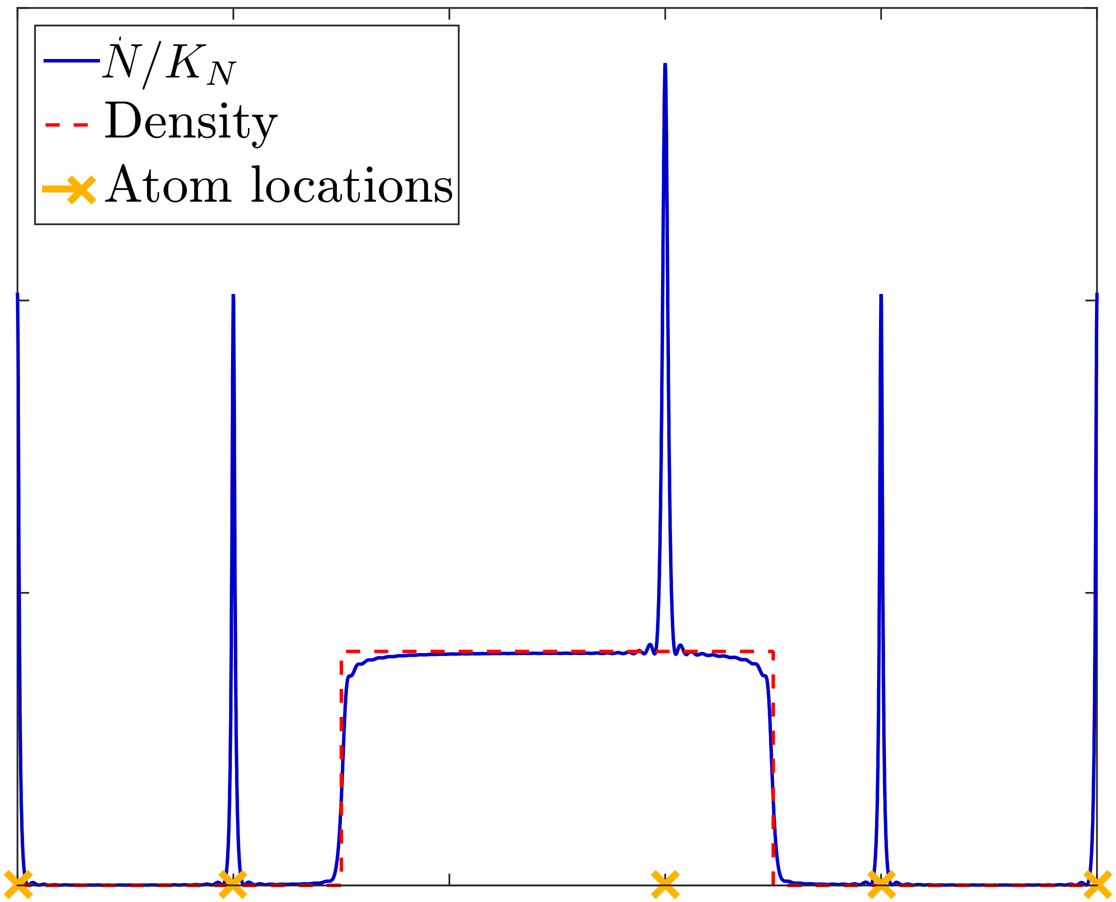
$\mu = 4I_{[0.3,0.7]}d\theta + 0.1(\delta_0 + \delta_{0.2} + \delta_{0.6} + \delta_{0.8})$

AC part, $N = 100$



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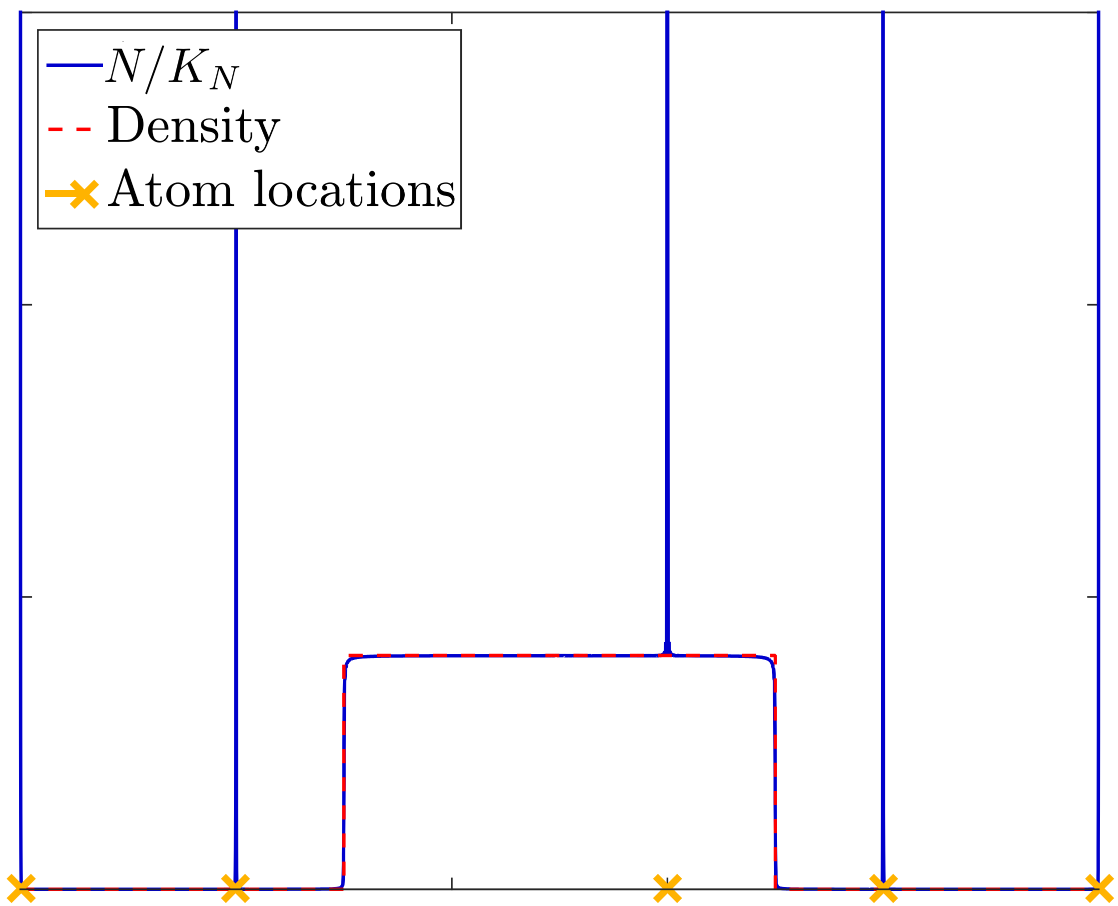
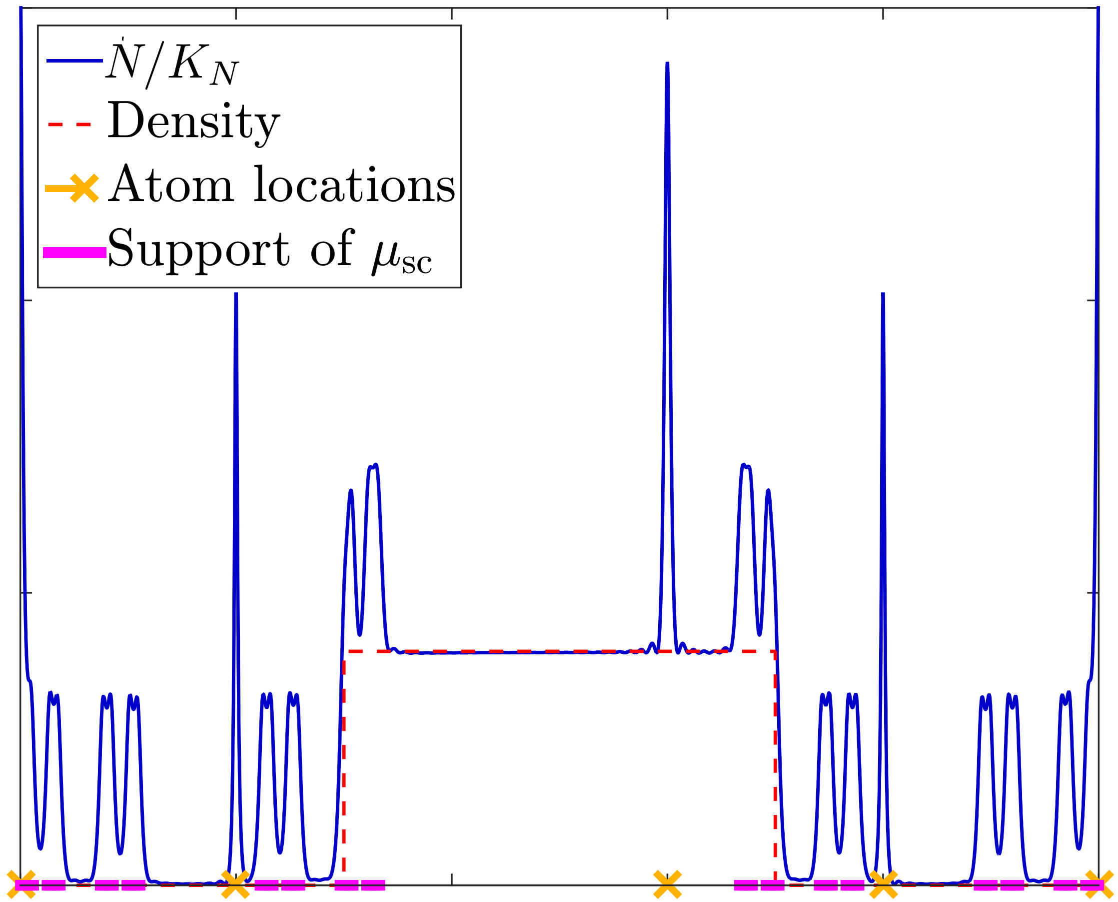


Reconstructing μ_f from moments - Example

$\mu = 4I_{[0.3,0.7]}d\theta + 0.1(\delta_0 + \delta_{0.2} + \delta_{0.6} + \delta_{0.8}) + \text{Cantor}$

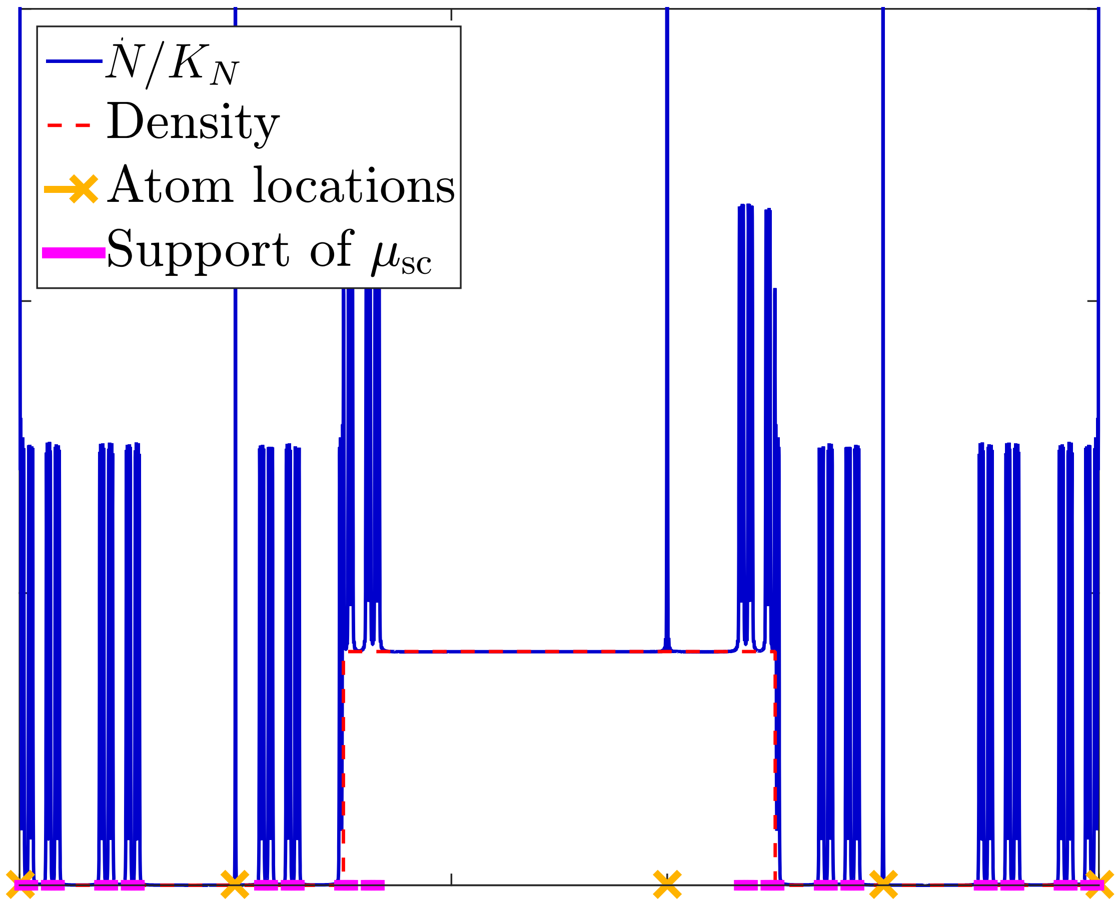
Adding SC spectrum, N = 100



Reconstructing μ_f from moments - Example

$\mu = 4I_{[0.3,0.7]}d\theta + 0.1(\delta_0 + \delta_{0.2} + \delta_{0.6} + \delta_{0.8}) +$ Cantor

Adding SC spectrum, N = 1000



Detecting SC spectrum

$$F_N^{\mathrm{CS}}(t) = \mu_N^{\mathrm{CS}}([0,t])$$

$$F_N^{\mathrm{Q}}(t) = \mu_N^{\mathrm{Q}}([0,t])$$

Converge to $F = \mu([0,t])$

(at points of continuity)

$$F_N^K(t) = \int_0^t \frac{N}{K(e^{i2\pi\theta}, e^{i2\pi\theta})}\, d\theta$$

Converges to $F = \mu([0,t])$ only if μ is AC

Otherwise underestimates F

Detecting SC spectrum

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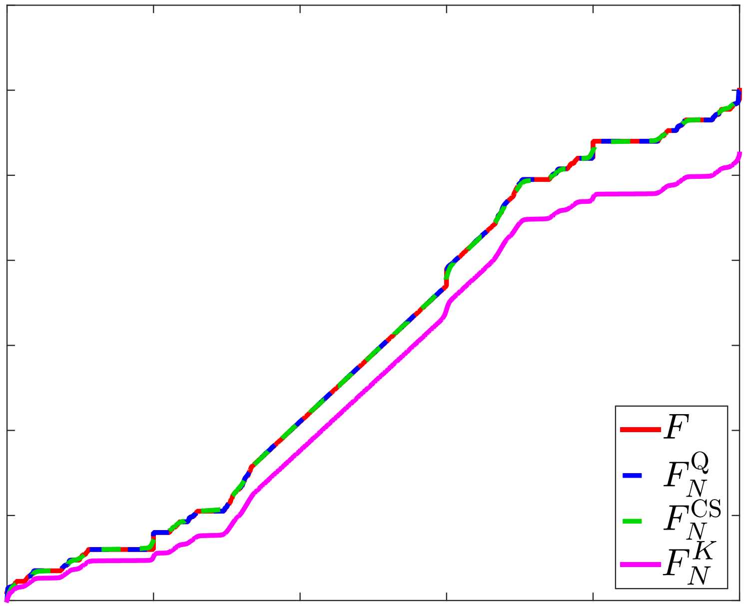
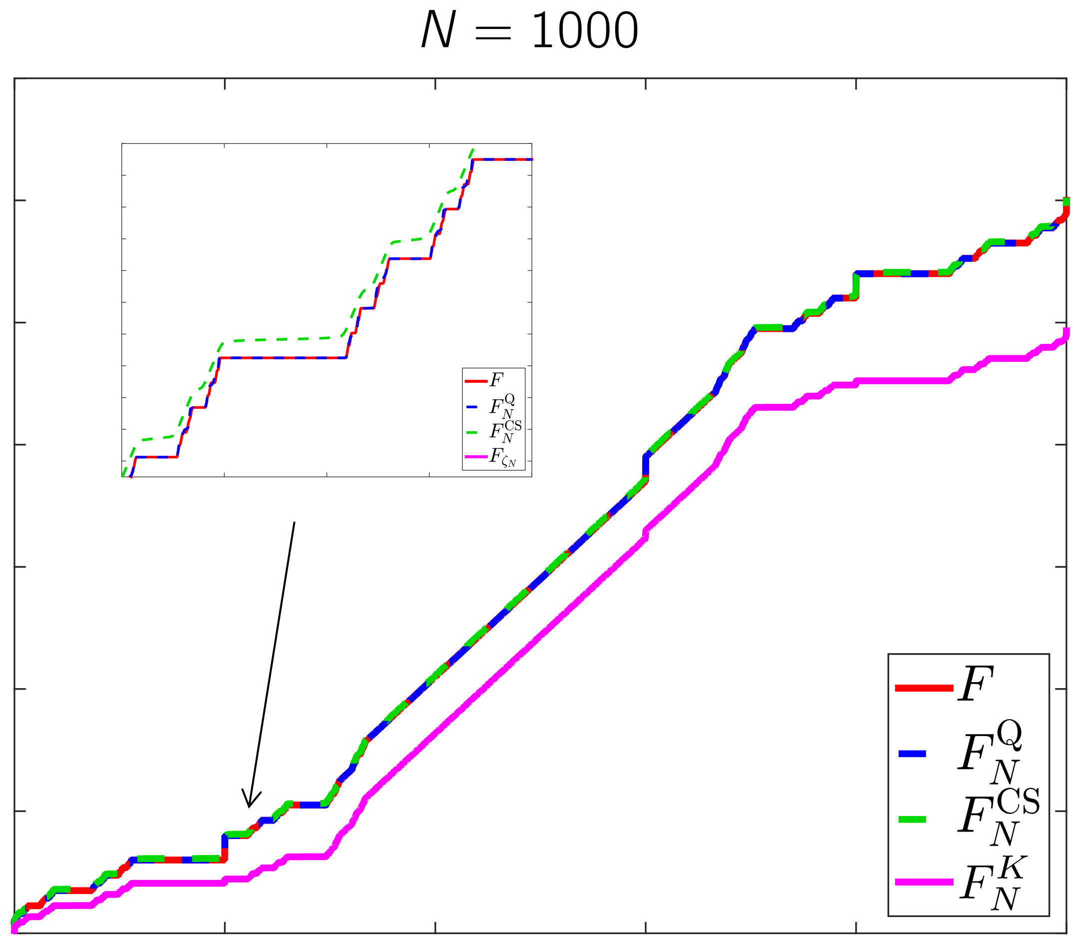


Detecting SC spectrum - CDFs

$\mu = 4I_{[0.3,0.7]}d\theta + 0.1(\delta_0 + \delta_{0.2} + \delta_{0.6} + \delta_{0.8}) +$ Cantor



Detecting SC spectrum

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Converges to $F = \mu([0,t])$ only if μ is AC

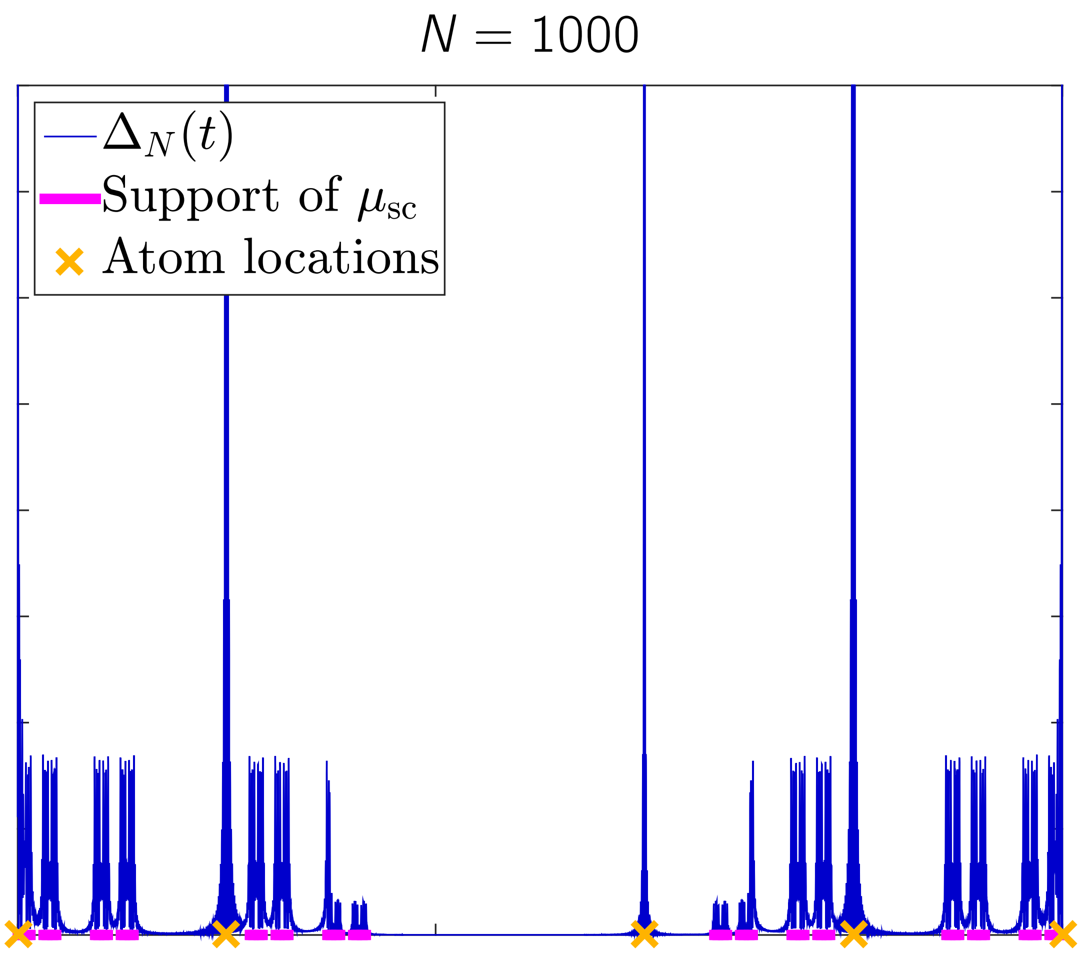
Otherwise underestimates F

Singularity indicator

$$\Delta_N(t) = \frac{F_N^{\mathrm{CS}}(t+\delta) - F_N^{\mathrm{CS}}(t-\delta)}{F_N^{K}(t+\delta) - F_N^{K}(t-\delta)} - 1$$

Detecting SC spectrum – Singularity indicator

$\mu = 4I_{[0.3,0.7]}d\theta + 0.1(\delta_0 + \delta_{0.2} + \delta_{0.6} + \delta_{0.8}) +$ Cantor



Cat map

$$\begin{aligned} x_1^+ &= 2x_1 + x_2 \mod 1 \\ x_2^+ &= x_1 + x_2 \mod 1 \end{aligned}$$

$$N = 100,\ M = 10^5$$

Spectrum known analytically *[Govindarajan et al., 2017]*

$$f_1 = e^{i2\pi(2x_1+x_2)} + \frac{1}{2}e^{i2\pi(5x_1+3x_2)}$$

$$\rho_{f_1} = \frac{5}{4} + \cos(2\pi\theta)$$

$$f_2 = e^{i2\pi(2x_1+x_2)} + \frac{1}{2}e^{i2\pi(5x_1+3x_2)} + \frac{1}{4}e^{i2\pi(13x_1+8x_2)}$$

$$\rho_{f_2} = \frac{21}{16} + (5/4)\cos(2\pi\theta) + \frac{1}{2}\cos(4\pi\theta)$$

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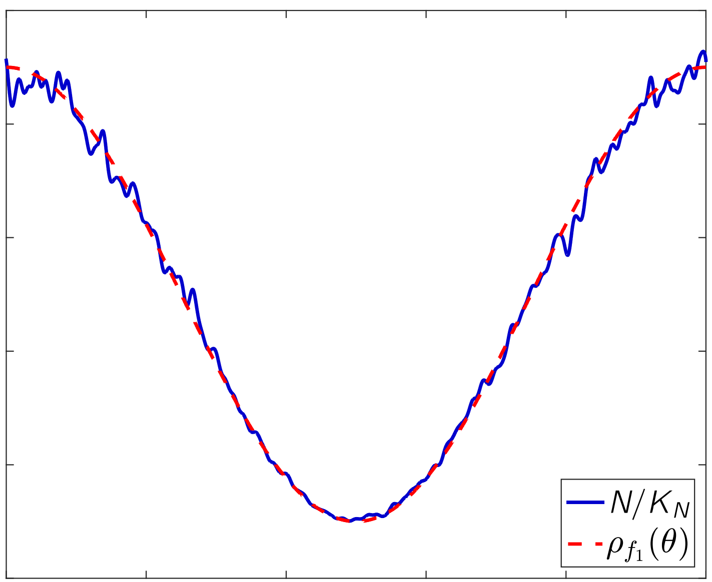
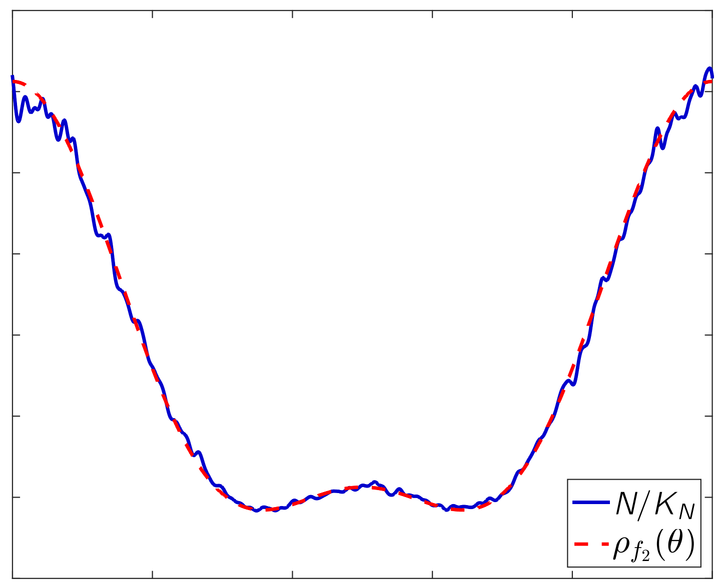


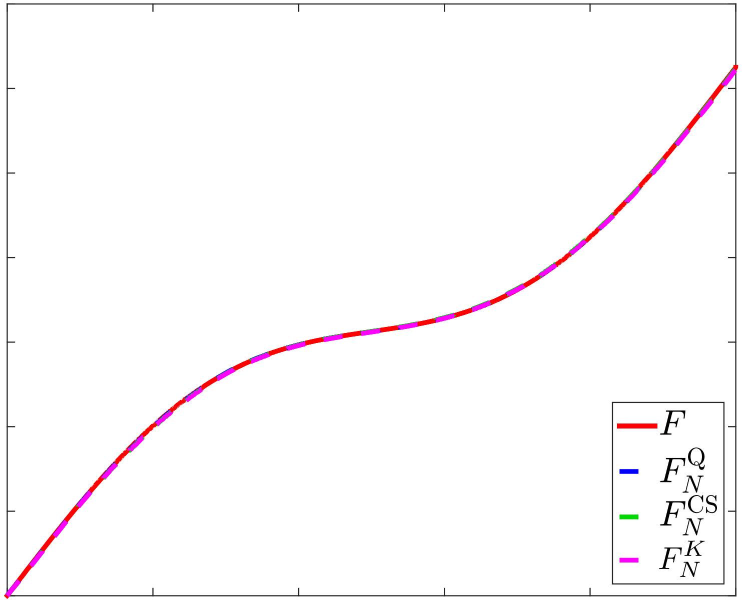


Cat map

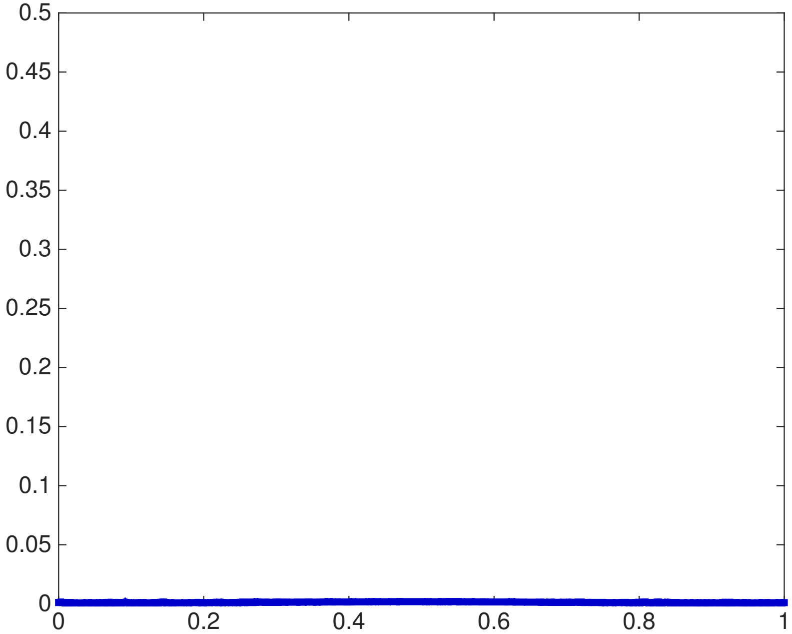
$$
\begin{aligned}
x_1^+ &= 2x_1 + x_2 \mod 1 \\
x_2^+ &= x_1 + x_2 \mod 1
\end{aligned}
$$

$N = 100, \; M = 10^5$

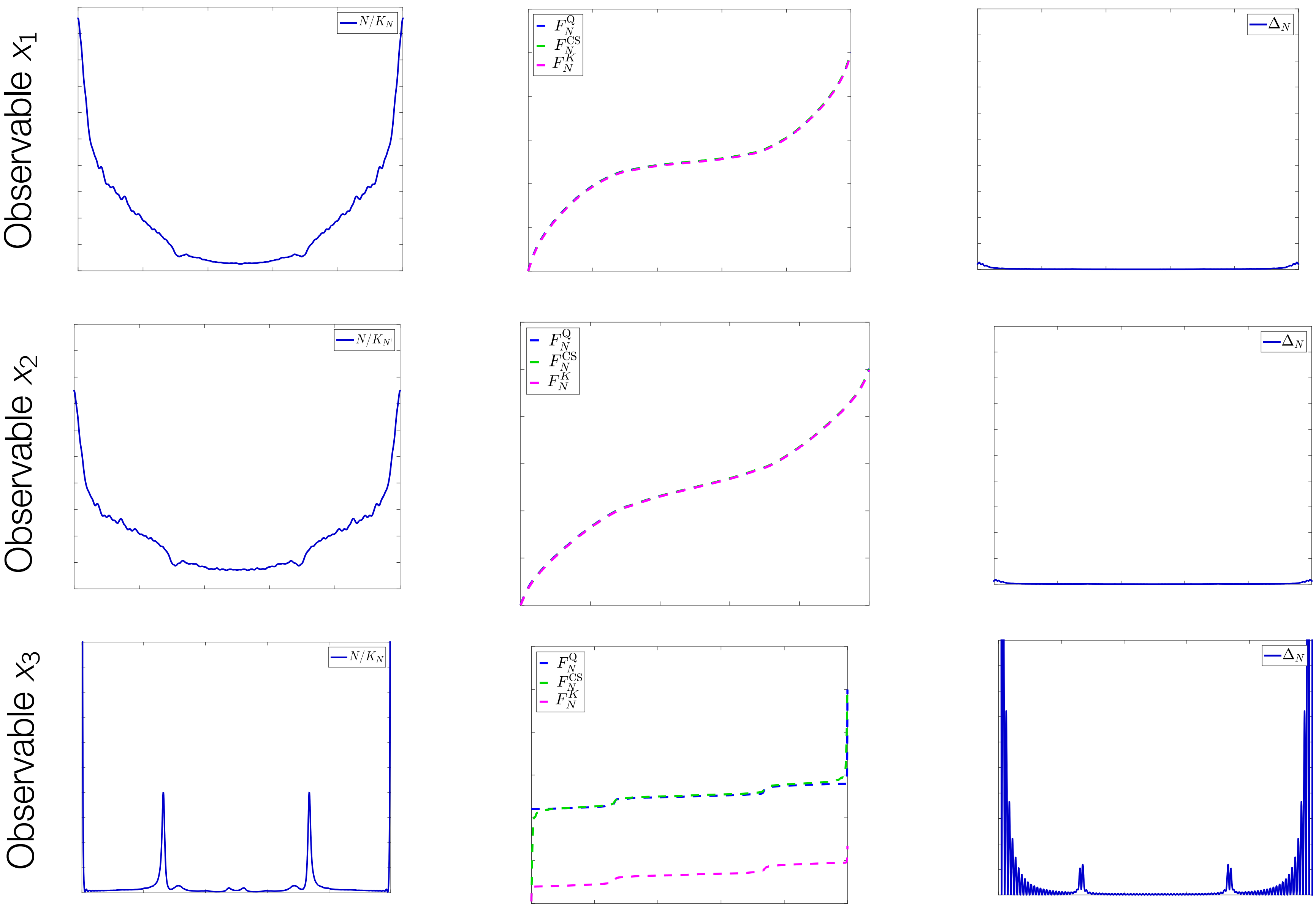
Distribution functions



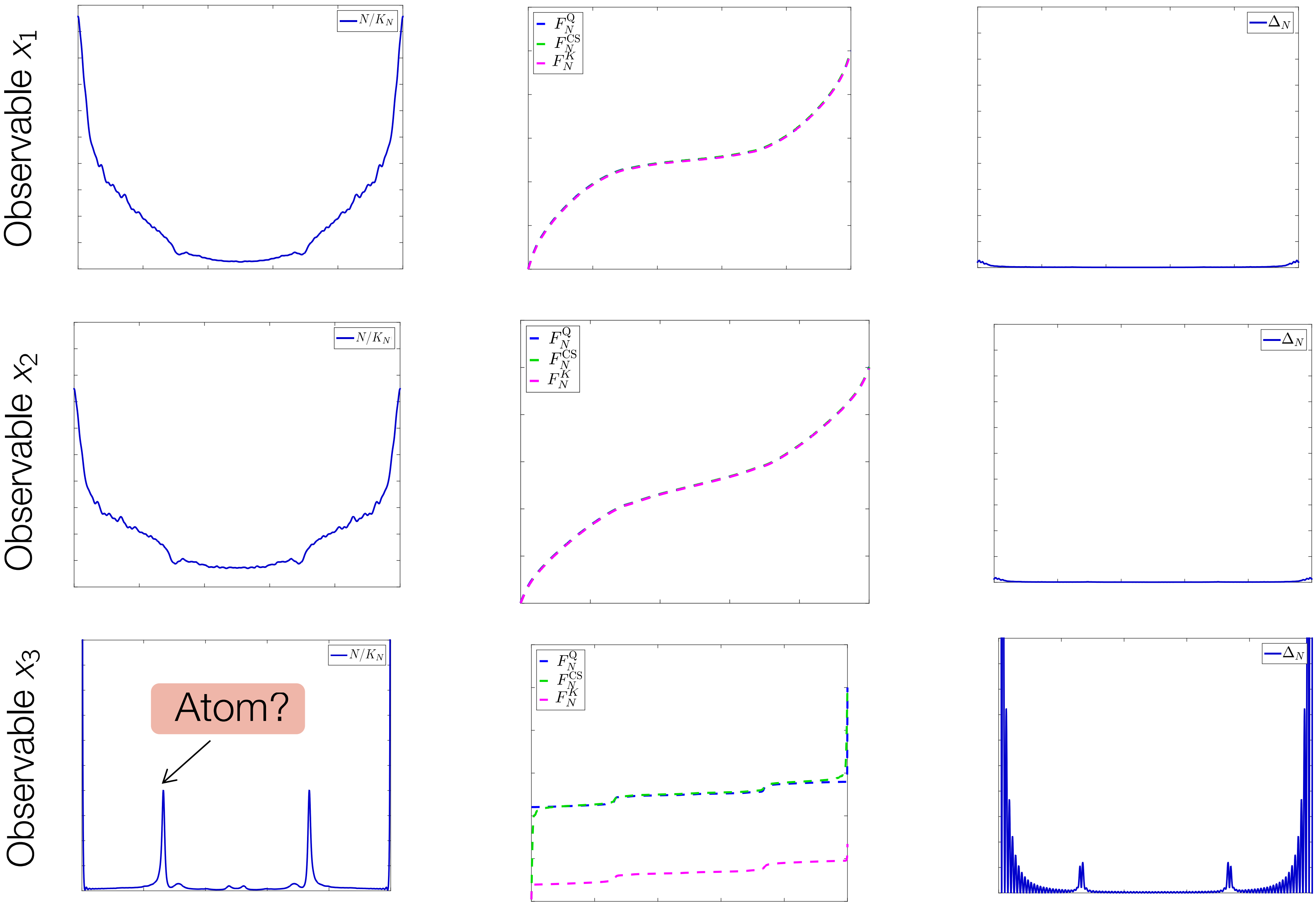
Singularity indicator



Lorenz system

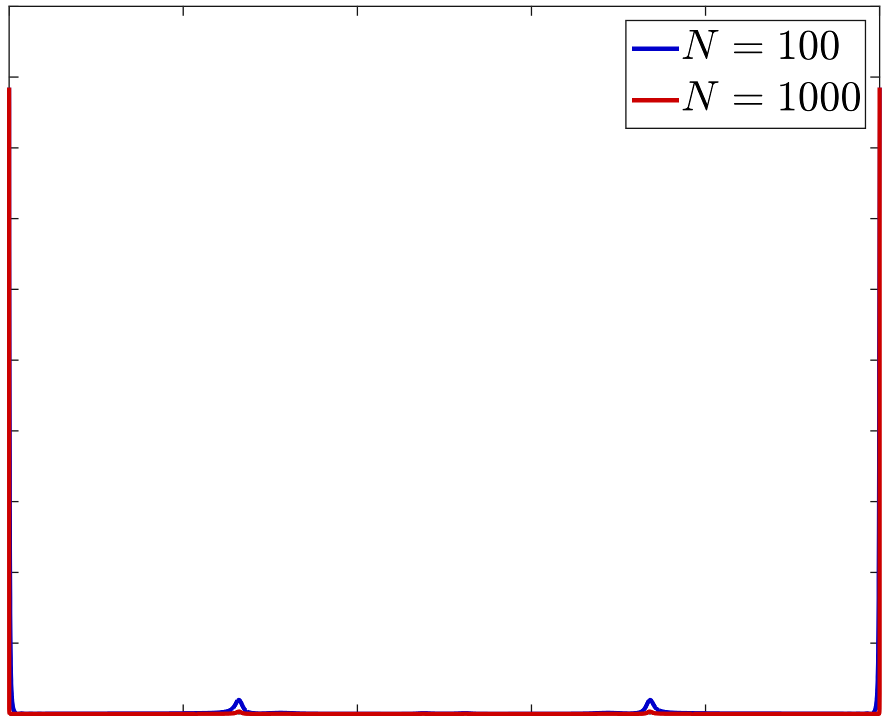


Lorenz system



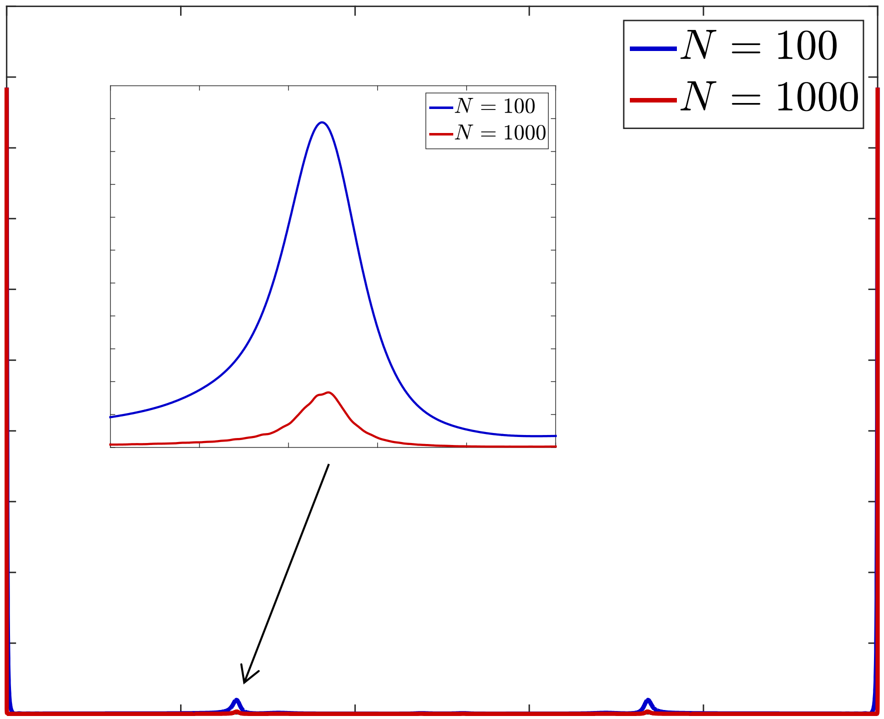
Lorenz system

$1/K_N$ – atomic part



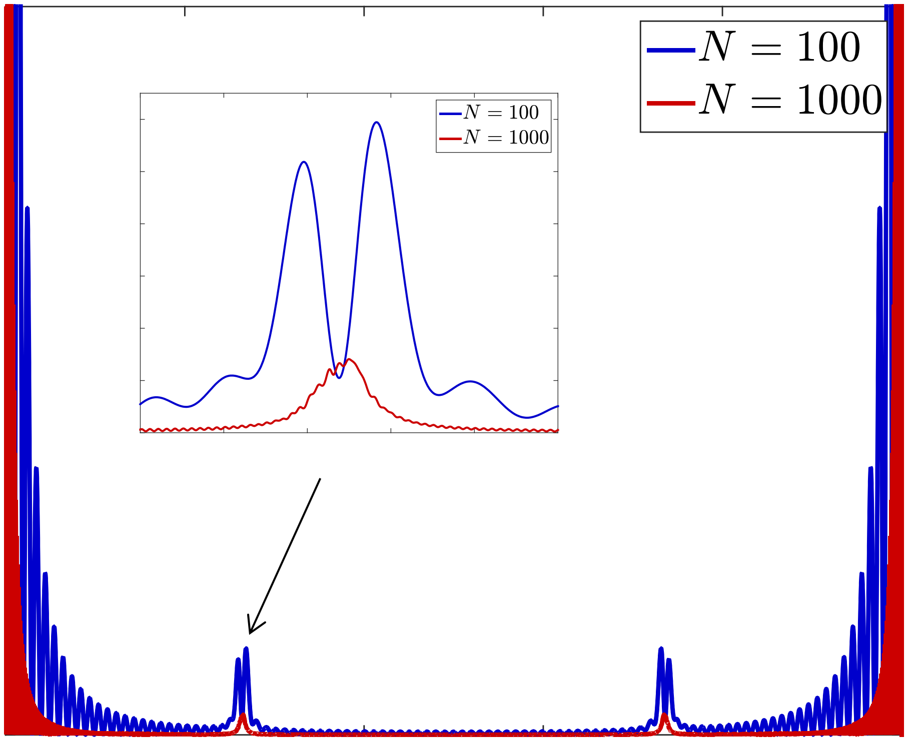
Lorenz system

$1/K_N$ – atomic part



Lorenz system

Singularity indicator



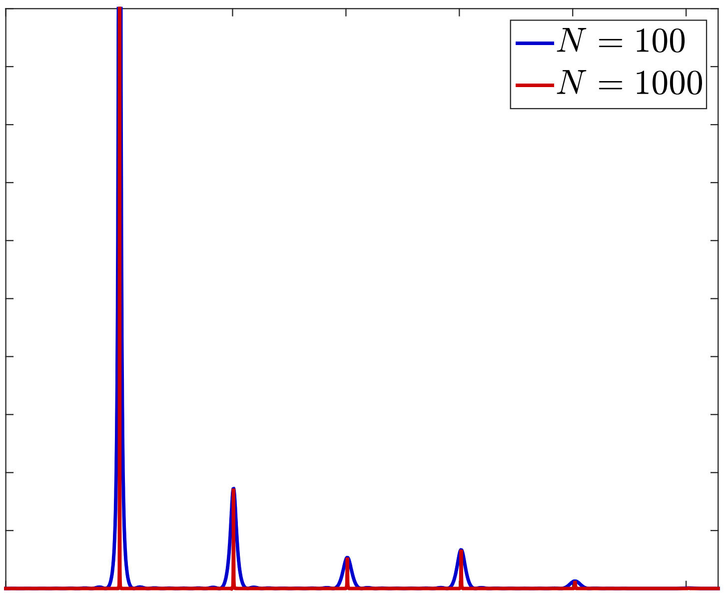
Cavity flow

2-D lid-driven cavity flow

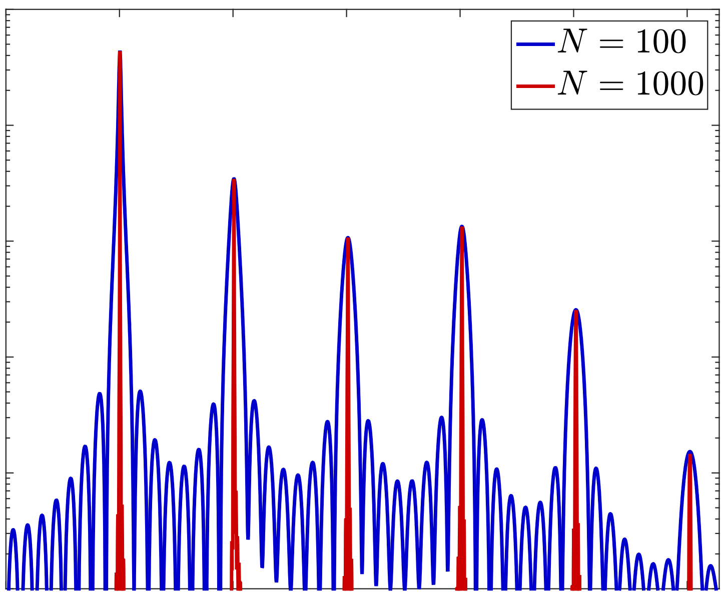
Observable: point measurement of stream function

Re = 13k

$1/K_N$



$1/K_N$ (logscale)



[Data courtesy of H. Arbabi]

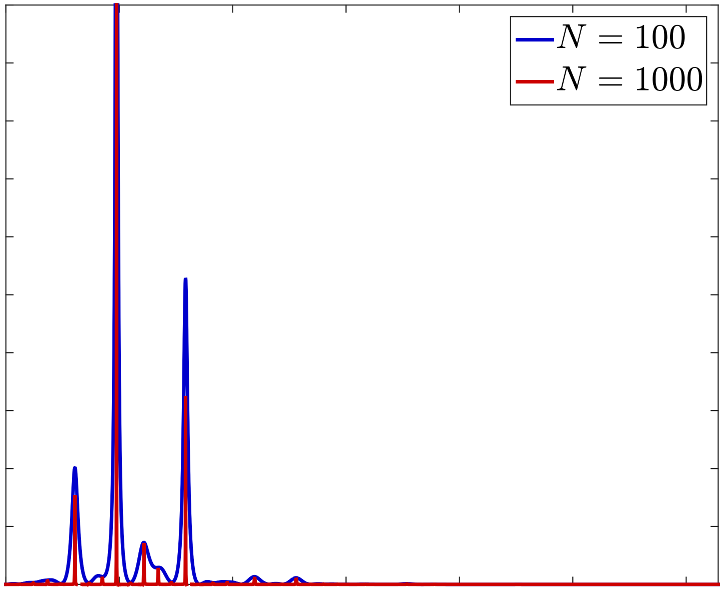
Cavity flow

2-D lid-driven cavity flow

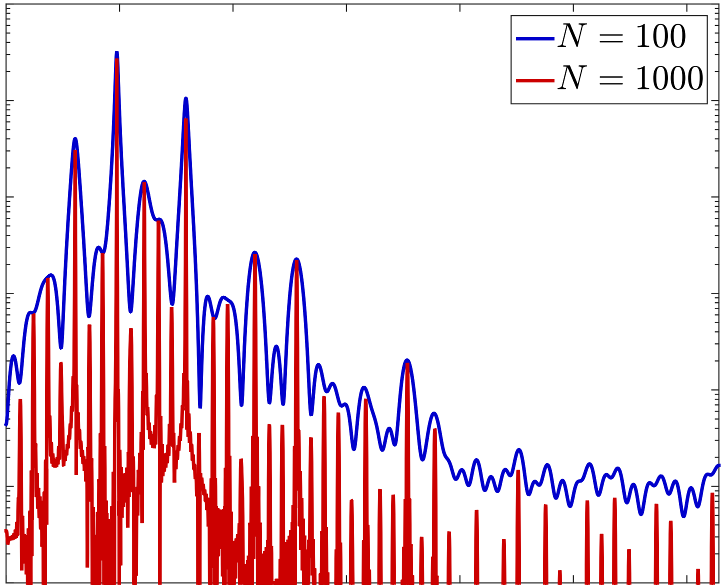
Observable: point measurement of stream function

Re = 16k

$1/K_N$



$1/K_N$ (logscale)



[Data courtesy of H. Arbabi]

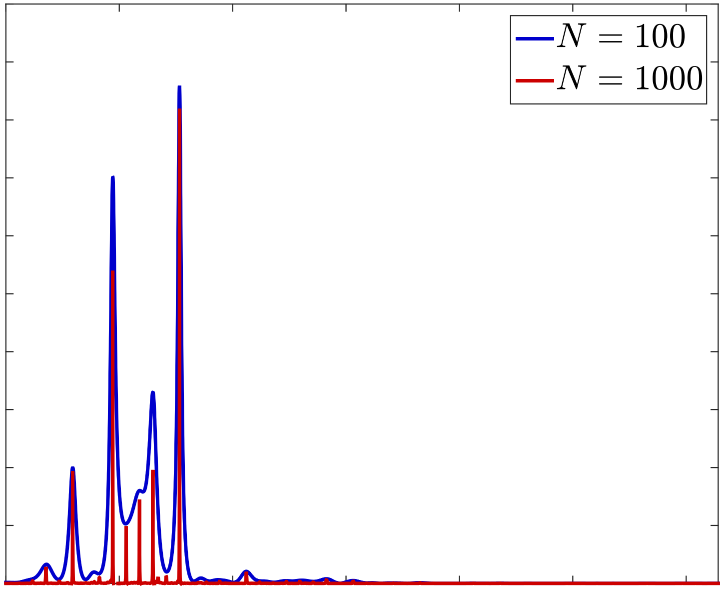
Cavity flow

2-D lid-driven cavity flow

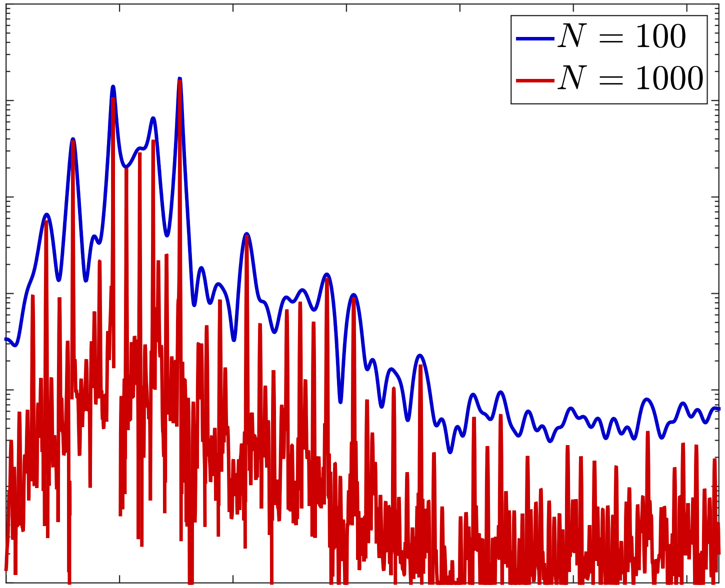
Observable: point measurement of stream function

Re = 19k

$1/K_N$



$1/K_N$ (logscale)



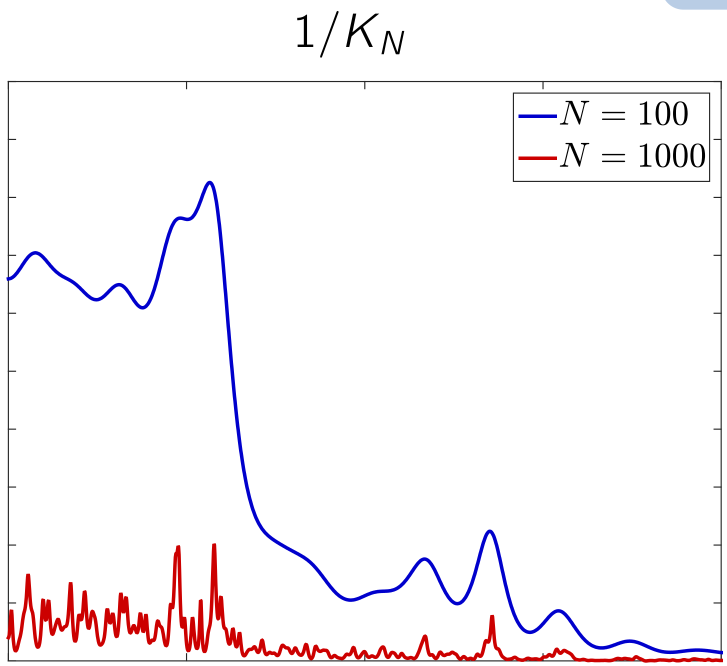
[Data courtesy of H. Arbabi]

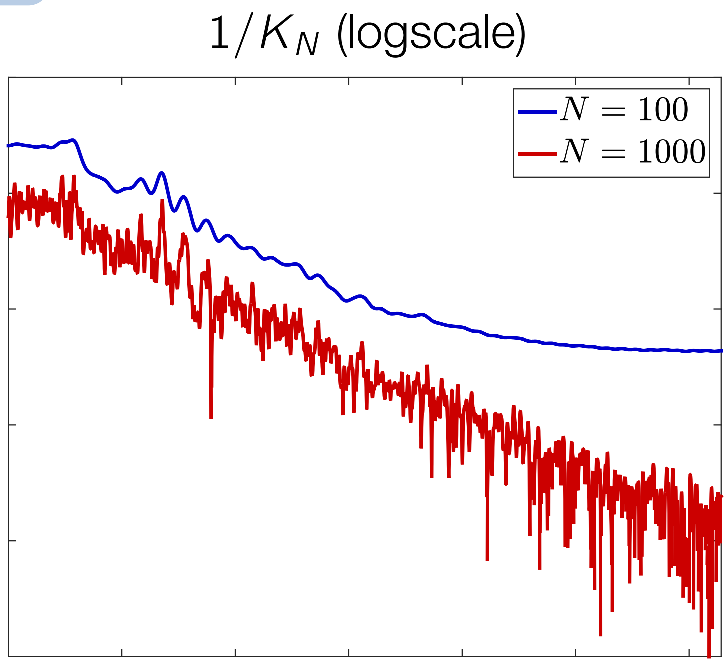
Cavity flow

2-D lid-driven cavity flow

Observable: point measurement of stream function

Re = 30k

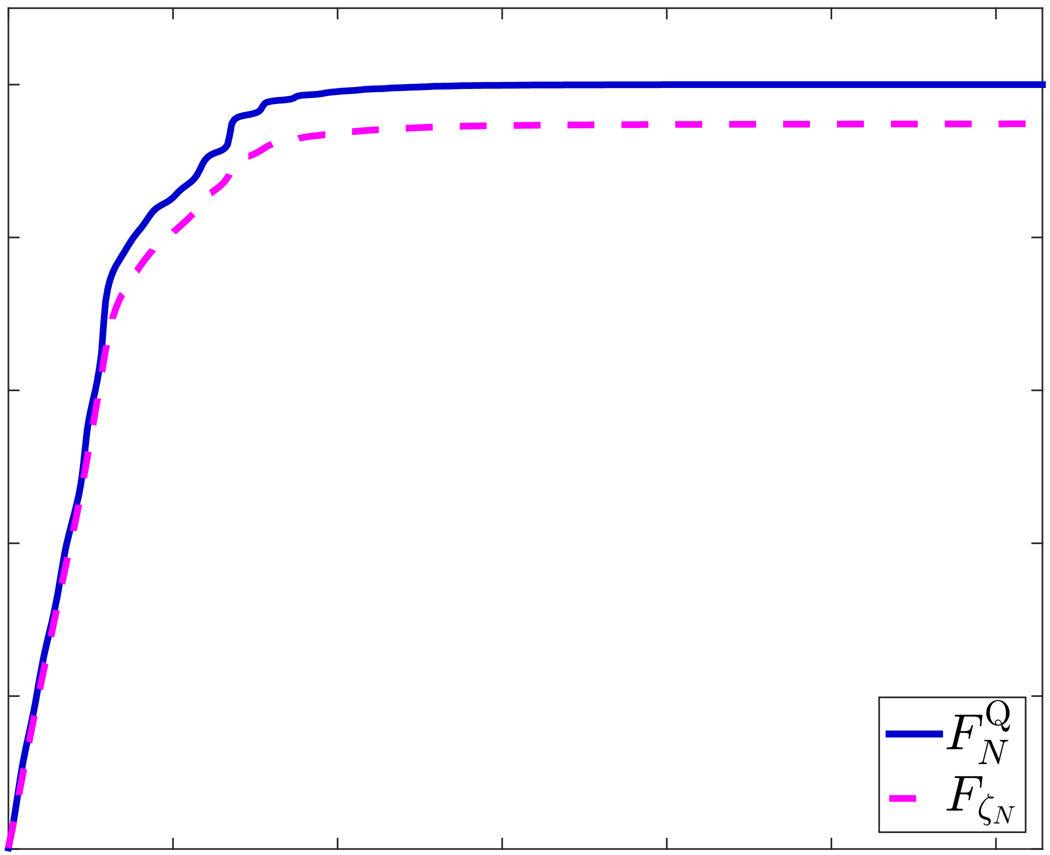




[Data courtesy of H. Arbabi]

Cavity flow – distribution functions

Re = 30k

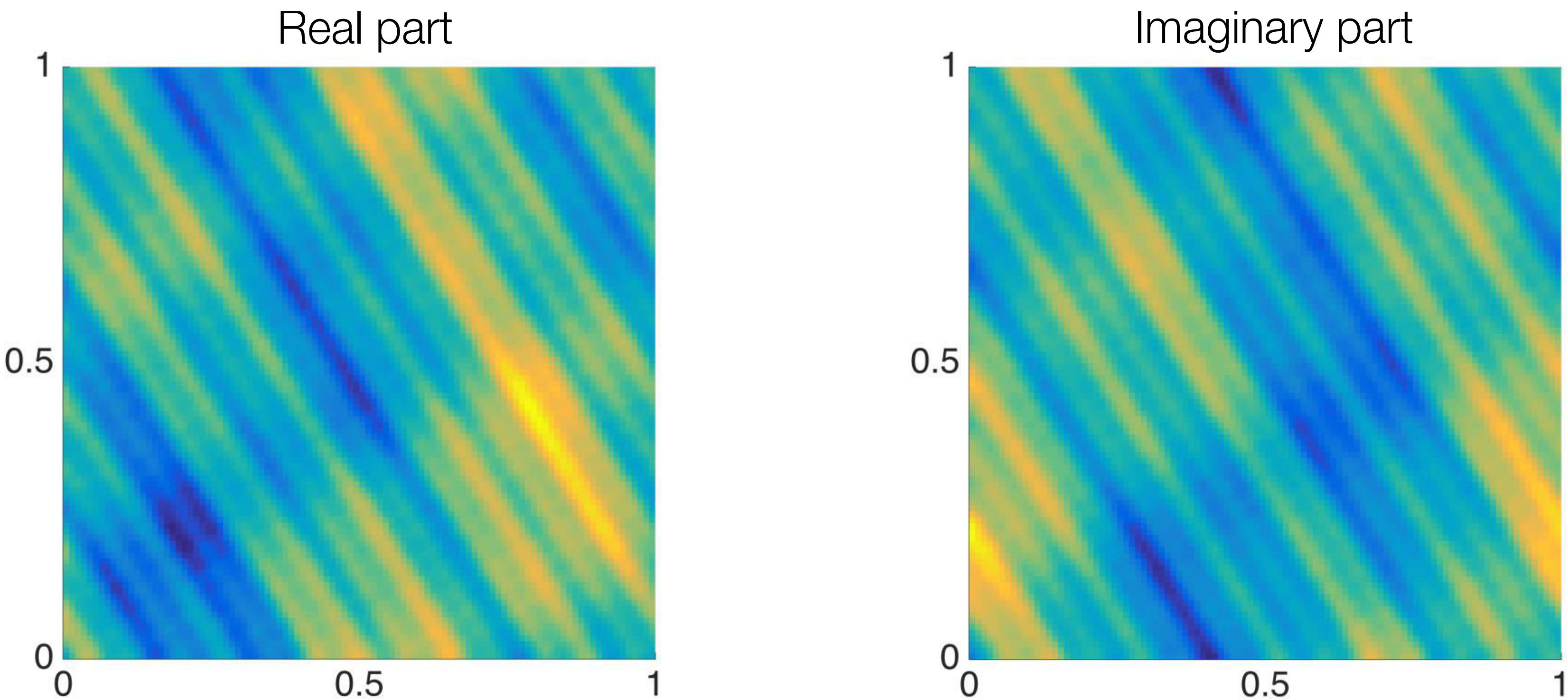


Example – Cat map

$$\begin{aligned} x_1^+ &= 2x_1 + x_2 \mod 1 \\ x_2^+ &= x_1 + x_2 \mod 1 \end{aligned}$$

Data: Single trajectory of length 10^5

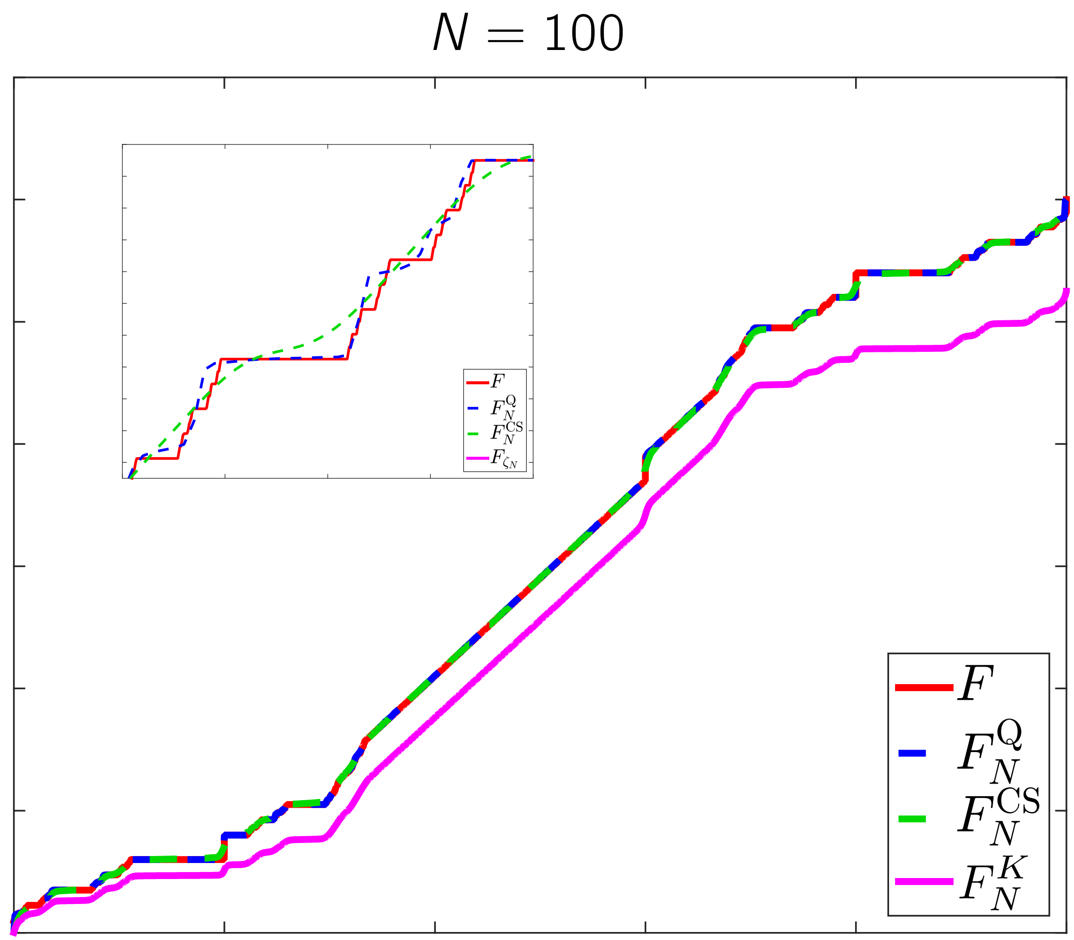
Approximation of P_A with $A = [1/8, 3/8]$ and $N = 100$



cf. *[Govindarajan et al., 2017]*

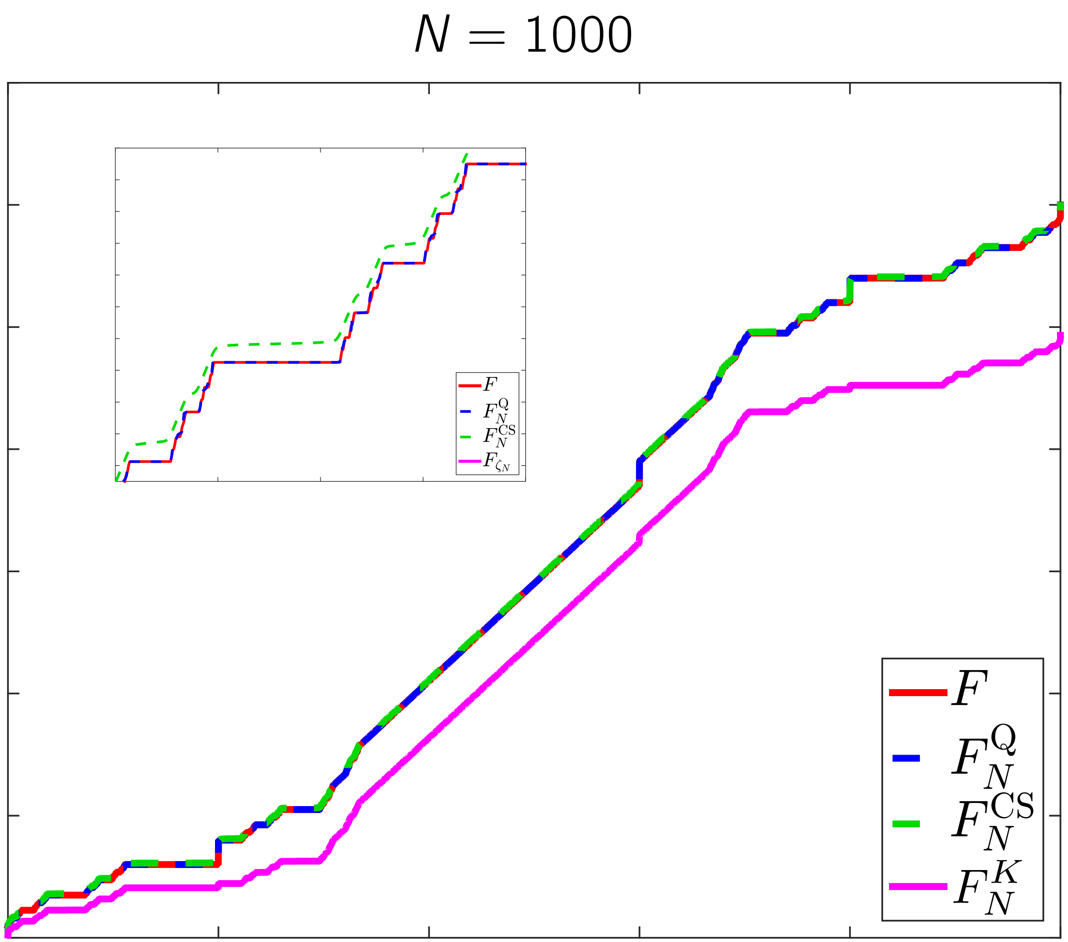
Detecting SC spectrum - CDFs

$\mu = 4I_{[0.3,0.7]}d\theta + 0.1(\delta_0 + \delta_{0.2} + \delta_{0.6} + \delta_{0.8}) +$ Cantor



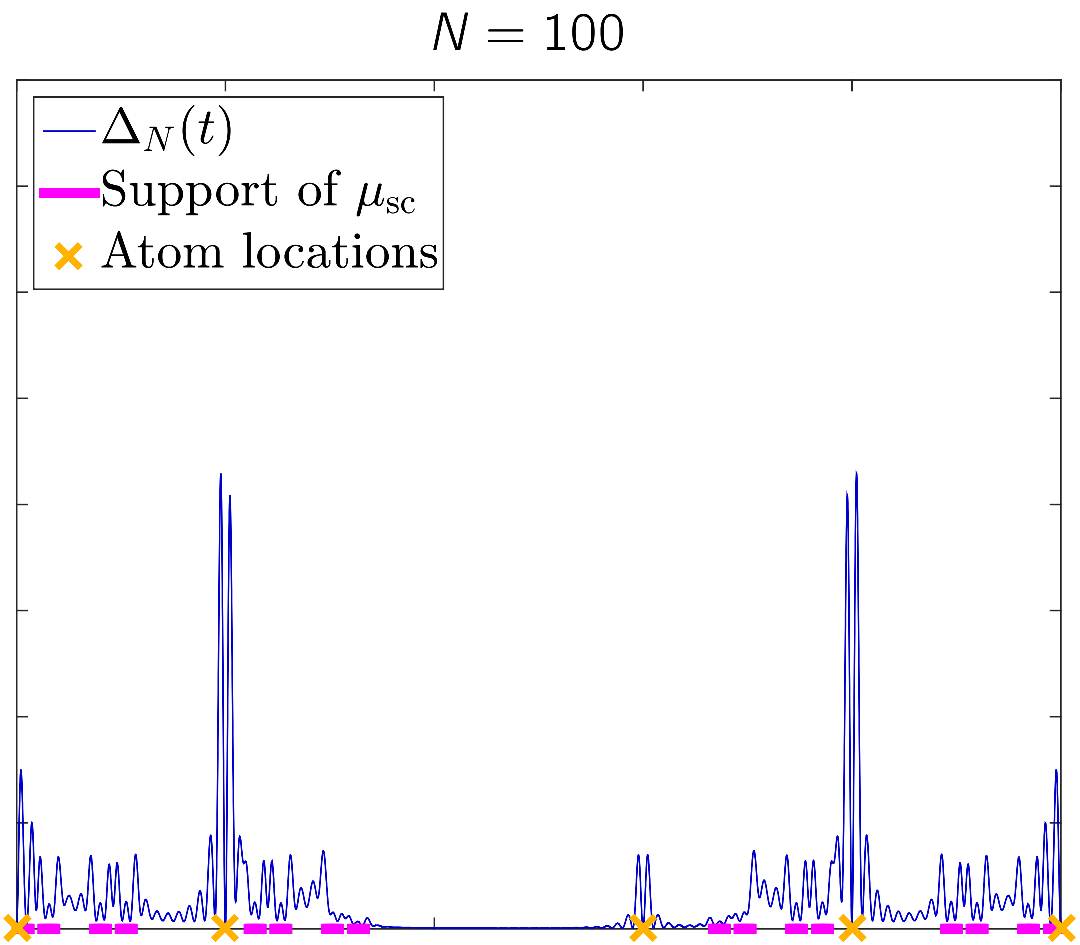
Detecting SC spectrum - CDFs

$\mu = 4I_{[0.3,0.7]}d\theta + 0.1(\delta_0 + \delta_{0.2} + \delta_{0.6} + \delta_{0.8}) +$ Cantor

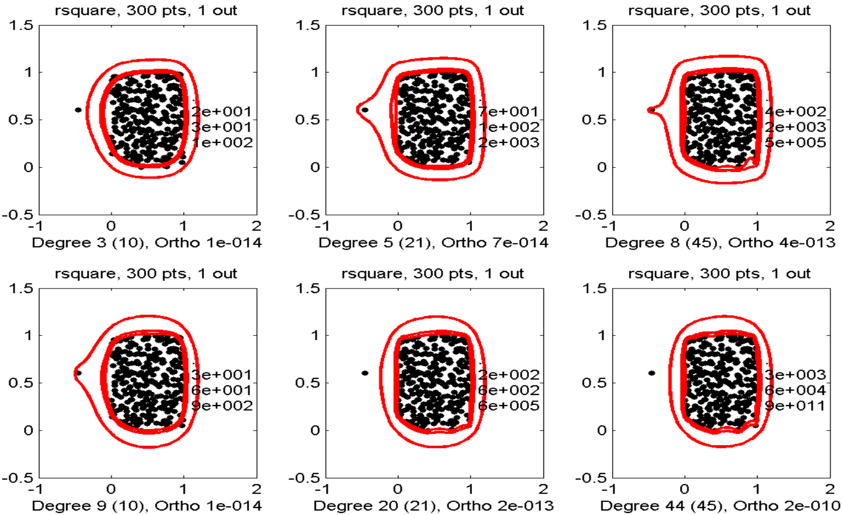


Detecting SC spectrum – Singularity indicator

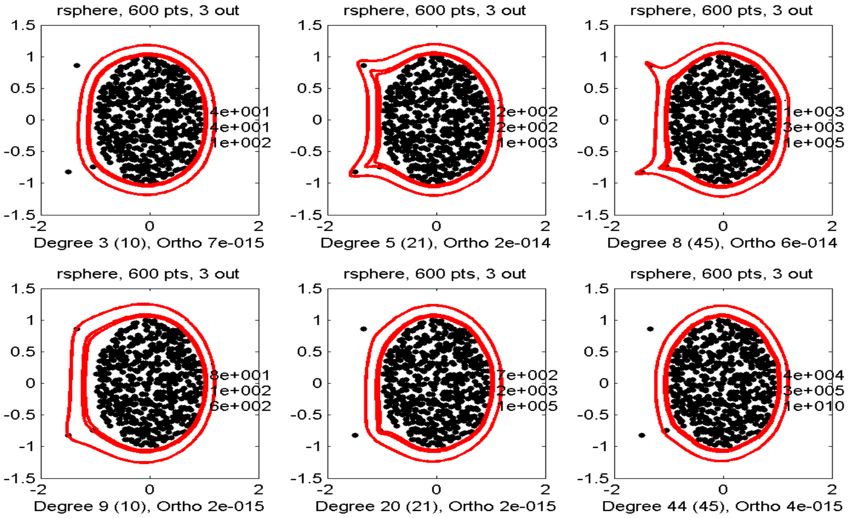
$\mu = 4I_{[0.3,0.7]}d\theta + 0.1(\delta_0 + \delta_{0.2} + \delta_{0.6} + \delta_{0.8}) +$ Cantor



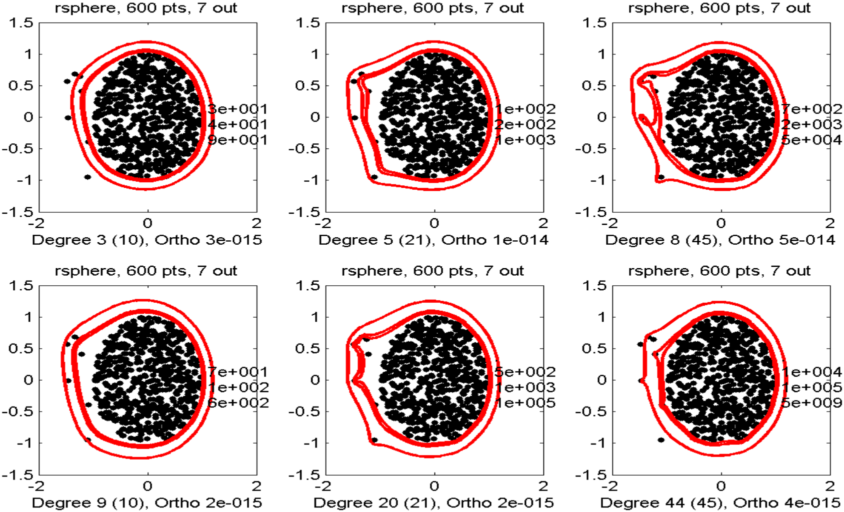
3. Detection of outliers in statistical data



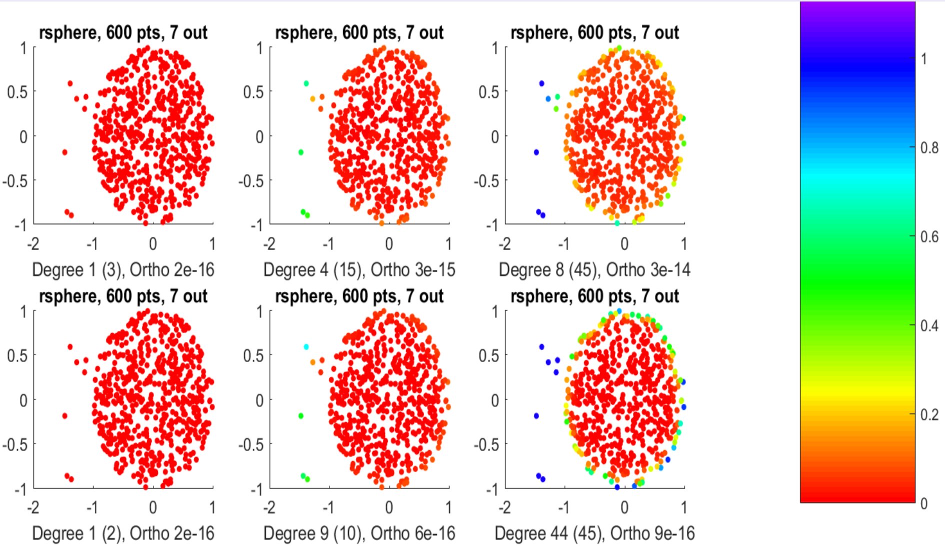
1 outlier, $N = 300$ shape of square



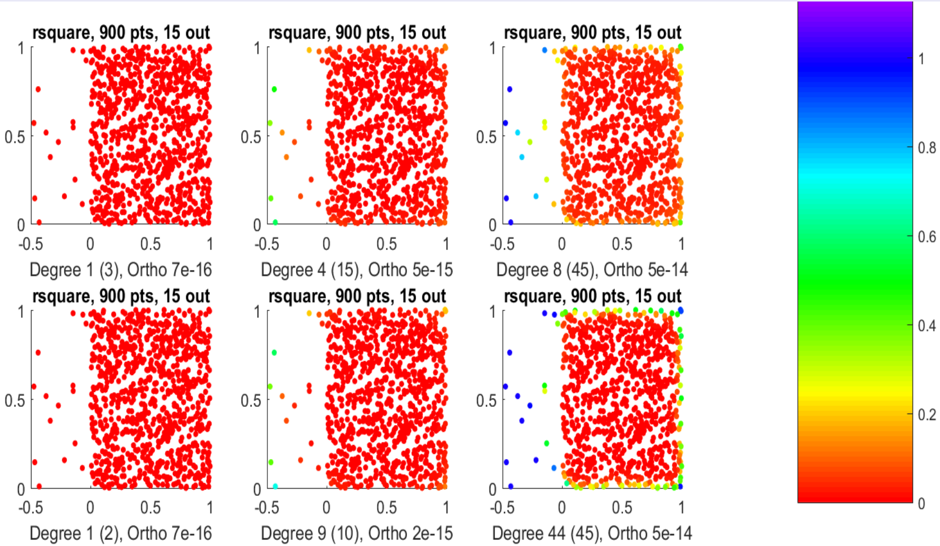
3 outliers, $N = 300$ disk cloud, ”real” level lines do not separate well, ”complex” do



7 outliers, $N = 300$ disk cloud, ”real” level lines do not separate well, ”complex” do



7 outliers, $N = 600$ disk cloud.



15 outliers, $N = 900$ square cloud.

Perturbation of CD kernels

$$\mu_N = \tau_N + \sigma_N, \quad \tau_N = \frac{1}{N}\sum_{j=s+1}^{N} \delta_{z_j} \approx \tau, \quad \sigma_N = \frac{1}{N}\sum_{j=1}^{s} \delta_{z_j}.$$

1. From discrete τ_N to continuous τ:
   $K_n^{\mu_N}(z,z) \approx K_n^{\tau+\sigma_N}(z,z)$?
2. Level lines of $K_n^{\tau}(z,z)$ approach $\mathrm{supp}(\tau)$?
3. Bounds for $K_n^{\mu_N}(z,z)$
   - upper bounds on $\mathrm{supp}(\tau + \sigma_N)$?
   - lower bounds outside $\mathrm{supp}(\tau + \sigma_N)$?

Comparison of measures and kernel (1)

Set $\mathcal{P}_n = \text{span}\{p_0^\mu, ..., p_n^\mu\}$ (independent of μ...), then for all $z \in \mathbb{C}$, $n \geq 0$

$$\frac{1}{K_n^\mu(z,z)} = 1 \Bigg/ \sum_{j=0}^{n} |p_j^\mu(z)|^2 = \min_{p \in \mathcal{P}_n} \left(\frac{\|p\|_{2,\mu}}{|p(z)|} \right)^2.$$

Lem 1: if $\mu \leq \nu$ then for all z

$$K_n^\mu(z,z) \geq K_n^\nu(z,z).$$

Comparison of measures and kernel (2)

Set $\mathcal{P}_n = \text{span}\{p_0^\mu, ..., p_n^\mu\}$ (independent of μ...), then for all $z \in \mathbb{C}$, $n \geq 0$

$$\frac{1}{K_n^\mu(z,z)} = 1 \Big/ \sum_{j=0}^{n} |p_j^\mu(z)|^2 = \min_{p \in \mathcal{P}_n} \left(\frac{\|p\|_{2,\mu}}{|p(z)|} \right)^2.$$

Consider (modified) Grammian

$$M_n(\nu, \mu) = \left(\langle p_j^\mu, p_k^\mu \rangle_{\nu,2} \right)_{j,k=0,...,n}, \quad \left(\frac{\|p\|_{2,\nu}}{\|p\|_{2,\mu}} \right)^2 = \frac{\xi^* M_n(\nu,\mu)\xi}{\xi^*\xi}.$$

Lem 2: if $\text{spec}(M_n(\nu,\mu)) \subset [\frac{1}{2}, \frac{3}{2}]$ then

$$\frac{1}{2} K_n^\nu(z,z) \leq K_n^\mu(z,z) \leq \frac{3}{2} K_n^\nu(z,z).$$

From discrete τ_N to continuous τ

$$\mu_N = \tau_N + \sigma_N, \quad \tau_N = \frac{1}{N}\sum_{j=s+1}^{N} \delta_{z_j} \approx \tau, \quad \sigma_N = \frac{1}{N}\sum_{j=1}^{s} \delta_{z_j}.$$

Cor 1: Suppose (**H**) : $\mathrm{spec}(M_n(\tau_N, \tau)) \subset [\frac{1}{2}, \frac{3}{2}]$. Then for all z

$$\frac{1}{2}K_n^{\tau_N}(z,z) \leq K_n^{\tau}(z,z) \leq \frac{3}{2}K_n^{\tau_N}(z,z),$$
$$\frac{1}{2}K_n^{\mu_N}(z,z) \leq K_n^{\tau+\sigma_N}(z,z) \leq \frac{3}{2}K_n^{\mu_N}(z,z).$$

From discrete τ_N to continuous τ

$$\mu_N = \tau_N + \sigma_N, \quad \tau_N = \frac{1}{N}\sum_{j=s+1}^{N} \delta_{z_j} \approx \tau, \quad \sigma_N = \frac{1}{N}\sum_{j=1}^{s} \delta_{z_j}.$$

Cor 1: Suppose (**H**) : $\mathrm{spec}(M_n(\tau_N, \tau)) \subset [\frac{1}{2}, \frac{3}{2}]$. Then for all z

$$\frac{1}{2} K_n^{\tau_N}(z,z) \le K_n^{\tau}(z,z) \le \frac{3}{2} K_n^{\tau_N}(z,z),$$
$$\frac{1}{2} K_n^{\mu_N}(z,z) \le K_n^{\tau+\sigma_N}(z,z) \le \frac{3}{2} K_n^{\mu_N}(z,z).$$

Recall, e.g., from [A. Chkifa, A. Cohen, G. Migliorati, F. Nobile, and R. Tempone'2015]: for $s = 0$, if $z_1, ..., z_N$ are samplings of i.i.d. random variables with law given by measure τ then

$$\mathrm{Prob}\Big(\textbf{(H) is true} \Big) \ge 1 - 2(n+1)\exp\Big(-\frac{\log(\frac{e}{2})}{2} \frac{N}{\max_{z \in \mathrm{supp}(\tau)} K_n^{\tau}(z,z)} \Big).$$

Level lines of $K_n^\tau(z,z)$ approach $\mathrm{supp}(\tau)$?

Thm 1: [Lasserre & Pauwels 2017]: if τ is area measure on some compact $S \subset \mathbb{R}^d$ with $S =$Clos(Int(S)) then there exist explicit $\gamma_n \in \mathbb{R}$ such that the Hausdorff distance between $S = \mathrm{supp}(\tau)$ and

$$S_n = \{z : K_n^\tau(z,z) \leq \gamma_n\}$$

tends to zero for $n \to \infty$.

Two ingredients of proof (works also for $\mathbb{C}^d$):

- upper bounds for $K_n^\tau(z,z)$ for z in compact subsets of Int(S)
  - through Lem 1 via area measures on small balls
  - classical for $S \subset \mathbb{R}, \mathbb{C}$, recent progress [Totik'2010] in case of $\mathcal{C}^2$ boundary
  - recent progress [Kroó, Lubinsky 2013] in $\mathbb{C}^d, \mathbb{R}^d$
- lower bounds for $K_n^\tau(z,z)$ for z in compact subsets of $\mathbb{C} \setminus S$
  - through peak polynomials following [Kroó, Lubinsky 2013]
  - logarithmic potential theory (Siziak function) in $\mathbb{C}$, pluripotential theory in $\mathbb{C}^d, \mathbb{R}^d$.

But so far no outliers !

Lower bounds for $K_n^{\mu_N}(z,z)$ outside $\text{supp}(\tau+\sigma_N)$

Lem 3: Under hypothesis **(H)** for $z \notin \text{supp}(\tau+\sigma_N)$:

$$\frac{3}{2}\frac{K_n^{\mu_N}(z,z)}{K_n^{\tau}(z,z)} \geq \frac{K_n^{\tau+\sigma_N}(z,z)}{K_n^{\tau}(z,z)}$$

$$\geq \det\Big(C_n^{\tau}(z_j,z_k)\Big)_{j,k=0,\ldots,s} \Big/ \det\Big(C_n^{\tau}(z_j,z_k)\Big)_{j,k=1,\ldots,s}$$

with $z_0 = z$, and $C_n^{\tau}(z,w) = \frac{K_n^{\tau}(z,w)}{\sqrt{K_n^{\tau}(z,z)K_n^{\tau}(w,w)}}$.

Idea of proof: Writing $v_n(z) = \Big(p_0^{\tau}(z), \ldots., p_n^{\tau}(z)\Big)$,

$$K_n^{\tau+\sigma_N}(z,z) = v_n(z) M_n(\tau+\sigma_N,\tau)^{-1} v_n(z)^*,$$

$$M_n(\tau+\sigma_N,\tau) = I + \frac{1}{N} V_n V_n^*, \quad V_n = \Big(v_n(z_1), \ldots, v_n(z_s)\Big).$$

Main Theorem for one complex variable [BB, MP, ES, NS'19]

Consider τ area measure on compact simply connected set $S = \mathrm{supp}(\tau) \subset \mathbb{C}$, and let $n, N \to \infty$ such that **(H)** holds, and

$$\max_{z\in S} K_n^\tau(z,z) \ll N \ll \min_{j=1,\ldots,s} \exp(2ng_S(z_j,\infty)).$$

Then uniformly for $z \in S \supset \mathrm{supp}(\tau_N)$ we have

$$\frac{1}{N} K_n^{\mu_N}(z) = o(1),$$

uniformly on compact subsets of $\mathbb{C} \setminus \mathrm{supp}(\tau + \sigma_N)$

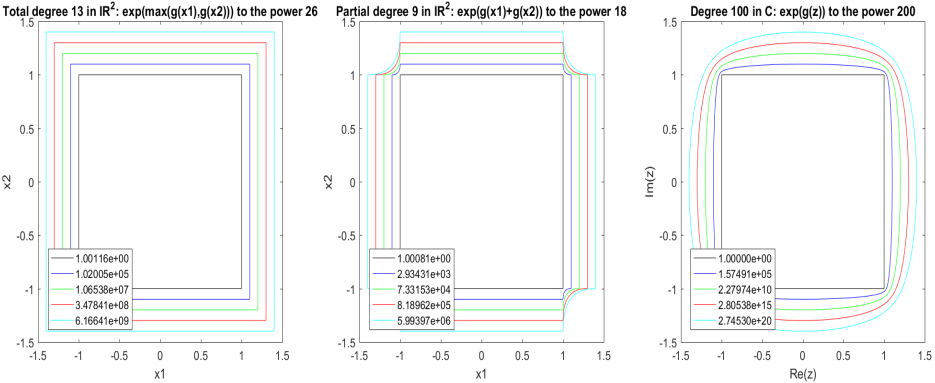
$$\frac{1}{N} K_n^{\mu_N}(z,z) \geq \frac{2}{3N} \exp(2ng_S(z,\infty)) \prod_{j=1}^{s} \exp(-2g_S(z_j,z))(1+o(1)),$$

and for $z_k \in \mathrm{supp}(\sigma_N)$

$$1 - \frac{1}{N} K_n^{\mu_N}(z_k,z_k) \leq \frac{N}{K^{\mu_N - \delta_{z_k}/N}(z_k,z_k)}$$

$$\leq \frac{3N}{2} \exp(-2ng_S(z,\infty)) \prod_{j=1, j\neq k}^{s} \exp(2g_S(z_j,z_k))(1+o(1)).$$

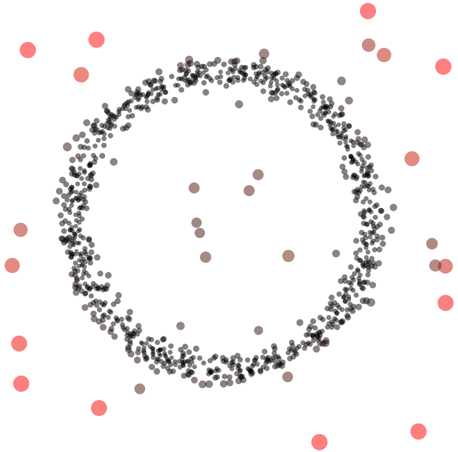
One complex versus 2 real variables



[redacted]

Supports on algebraic varieties

Detecting a circle:



2D torus

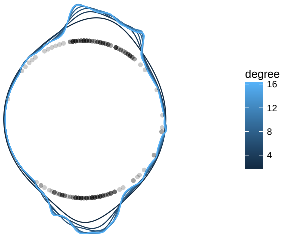


Figure: Dragon fly orientation with respect to the sun, on the torus. The curves represent the empirical Christoffel function for different values of the degree.

2D sphere

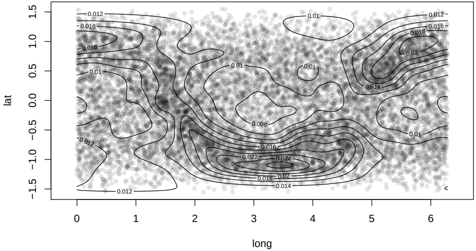


Figure: Each point represent the observation of a double star in the sky. They live on the sphere and are associated to their longitude and latitude. The level sets are those of the empirical Christoffel functions evaluated on the sphere in $\mathbb{R}^3$. The degree is 8. The band which is highlighted by the level sets corresponds to the Milky Way.

2D torus lying on 3D sphere

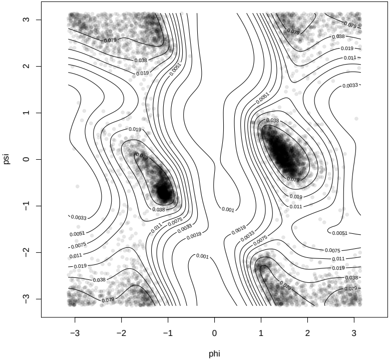


Figure: Each point two dihedral angles for a Glycine amino acid. These angles are used to describe the global three dimensional shape of a protein. They live on the bitorus. The level sets are those of the empirical Christoffel functions evaluated on the sphere in $\mathbb{R}^4$. The degree is 4.

3. Approximation in the mean by complex polynomials

Let μ be a positive, compactly supported measure on $\mathbb{C}$. We denote by $P^2(\mu)$ the closure of complex polynomials in $L^2(\mu)$, and $R^2(F, \mu)$ the closure in $L^2(\mu)$ of rational functions with poles disjoint of F.

In general, the spectral analysis of the multiplier $S_\mu = M_z : P^2(\mu) \longrightarrow P^2(\mu)$ reveals (constructively) the nature of the measure μ. Beyond the spectral theorem for normal operators.

Problem. When is $P^2(\mu) = L^2(\mu)$, or $R^2(F, \mu) = L^2(\mu)$?

Thomson's Theorem, 1991

Let μ be a positive Borel measure, compactly supported on $\mathbb{C}$. There exists a Borel partition $\Delta_0, \Delta_1, \ldots$ of the closed support of μ with the following properties:
1) $P^2(\mu) = L^2(\mu_0) \oplus P^2(\mu_1) \oplus P^2(\mu_2) \oplus \ldots$, where
$\mu_j = \mu|_{\Delta_j},\ j \geq 0$;
2) Every operator $S_{\mu_j} = M_z,\ j \geq 1$, is irreducible with spectral picture:

$$\sigma(S_{\mu_j}) \setminus \sigma_{\text{ess}}(S_{\mu_j}) = G_j, \quad \textit{simply connected},$$

and

$$\text{supp } \mu_j \subset \overline{G_j}, \quad j \geq 1;$$

3) If $\mu_0 = 0$, then any element $f \in P^2(\mu)$ which vanishes $[\mu]$-a.e. on $G = \cup_j G_j$ is identically zero.

Analytic bounded point evaluations

Let μ be a positive Borel measure, compactly supported on $\mathbb{C}$. Then

$$P^2(\mu) \neq L^2(\mu)$$

if and only if there exists an open set U of *analytic bounded point evaluations*

$$|f(a)| \leq C\|f\|_{2,\mu}, \quad f \in \mathbb{C}[z], \quad a \in U,$$

where C does not depend on a.

Brennan's Theorem, 2008

Let μ *be a positive measure without point masses, compactly supported on* $\mathbb{C}$. *Assume the set* F *contains the closed support of the measure* μ, *and the complement* $\mathbb{C} \setminus F$ *does not have components of arbitrarily small diameter.*

Then $R^2(F, \mu) \neq L^2(\mu)$ *if and only if* $R^2(F, \mu)$ *admits analytic bounded point evaluations:*

$$|f(a)| \leq C\|f\|_{2,\mu}, \quad f \in R^2(F, \mu), \quad a \in U \text{ (open set)}.$$

Novel technical ingredient and simplification: X. Tolsa semi-additivity of analytic capacity.

Classification of irreducible subnormal operators...

is a matter of complex hermitian geometry: the kernel of the linear pencil

$$z \mapsto \ker(S^* - \overline{z})$$

is, on the cloud of analytic bounded point evaluations, a hermitian line bundle. Curvature type invariants determine the unitary orbit of the generating operator S.

Density of complex polynomials on the unit circle

Theorem (Kolmogorov, 1941, Krein, 1945) *For a positive measure μ supported on $\mathbb{T}$, one has $P^2(\mu) = L^2(\mu)$ if and only if*

$$\int_{-\pi}^{\pi} |\log \mu'| dt = \infty.$$

Or equivalently, the multiplier M_z is a normal operator. In the opposite case, (Szegő's limit theorem gives even more),

$$\dim \ker(M_z - w)^* = 1, \quad |w| < 1,$$

and hence the Fredholm index of M_z is -1 at every point of the unit disk.

Carry the analysis on affine, algebraic curves

Theorem (S. Biswas-M.P., 2022) *Let X be a rational curve in $\mathbb{C}^n$ and let μ be a positive Borel measure without point masses, supported by a compact subset of X. Then $P^2(\mu) \neq L^2(\mu)$ if and only if there are analytic $P^2(\mu)$-bounded point evaluations.*

Idea of proof

Let $R = (r_1, r_2, \ldots, r_n)$ be an n-tuple of rational functions which properly parametrizes the rational affine curve $X \subset \mathbb{C}^n$. That is, denoting by $S \subset \mathbb{C}$ the poles of R, the holomorphic map

$$R : \mathbb{C} \setminus S \longrightarrow X$$

is one to one, except finitely many points, and it covers X except finitely many points.

Let ρ denote a sufficiently large radius, so that the support of the measure μ is contained in the ball $B(0, \rho)$. The pull-back

$$U = R^{-1} B(0, \rho)$$

is an open subset of $\mathbb{C}$, of finite connectivity, with piece-wise smooth boundary. In particular we can assume that every connected component of the complement of U has positive diameter.

The restricted analytic map

$$R : U \longrightarrow B(0, \rho)$$

has finite fibres, hence it is proper. Grauert's finiteness theorem implies that the direct image sheaf $R_*\mathcal{O}_U$ is coherent and

$$R_*\mathcal{O}_U(B(0,\rho)) = \mathcal{O}(U).$$

The coherence of $R_*\mathcal{O}_U$ and the injectivity of R modulo a finite set imply that

$$\dim \mathcal{O}(U)/R^*\mathcal{O}(B(0,\rho)) < \infty.$$

One can define a pull-back measure ν on U

$$\int \phi \, d\mu = \int \phi \circ R \, d\nu,$$

for every continuous function $\phi : B(0,\rho) \longrightarrow \mathbb{C}$.

Let $R^2(U,\nu)$ denote the closure in $L^2(\nu)$, of rational functions with poles on the complement of U. Runge's approximation theorem implies that $R^2(U,\nu)$ is also the closure of the algebra $\mathcal{O}(U)$ in $L^2(\nu)$. Hence

There exist analytic bounded point evaluations with respect to $P^2(\mu)$ if and only if there exists analytic bounded point evaluations with respect to $R^2(U,\nu)$.

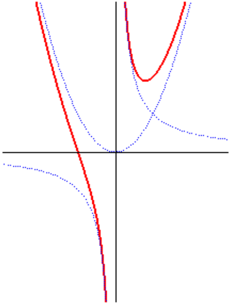
Brennan's Theorem completes the proof.

Invariant subspaces for subnormal tuples

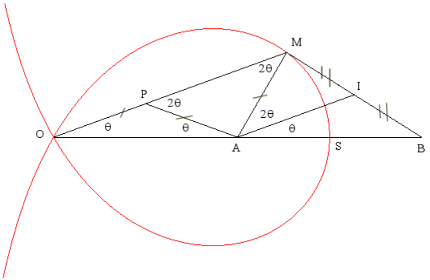
A commuting subnormal tuple with Taylor's joint spectrum contained in a rational curve admits joint invariant subspaces.

As a matter of fact, a continuum of invariant subspace of codimension one exists, parametrized by a relatively open subset of the joint spectrum.

Newton's Trident: $xy = x^3 + 1$



Maclaurin trisectrix: $x(x^2+y^2)=a(3x^2-y^2)$, $x=a\frac{3-t^2}{1+t^2}$, $y=tx$.



The general case

Resolution of singularities and our base change technique lead to the following **Open Question**:

Let X be an open Riemann surface of finite genus, and let μ be a positive Borel measure on X without point masses. Let $U \subset X$ be an open, relatively compact subset of X, with finitely many components of $X \setminus U$, none reduced to a point. Assume the closed support of the measure μ is contained in U. Then analytic functions $\mathcal{O}(U)$ are dense in $L^2(\mu)$ if and only if there are no corresponding bounded analytic point evaluations.

Uniform approximation on Riemann surfaces

Let X be an open Riemann surface. Assume that the measure μ is supported by a piecewise smooth curve $\Gamma \subset X$ with the property that the complement $X \setminus \Gamma$ is connected. Then $P^2(\mu) = L^2(\mu)$.

Proof derived from **Scheinberg's Theorem** (on uniform approximation).

Counterexample to Szegö's condition on the unit circle

Let $Q(w)$ be a polynomial of degree at least three, with simple roots. The hyperelliptic curve

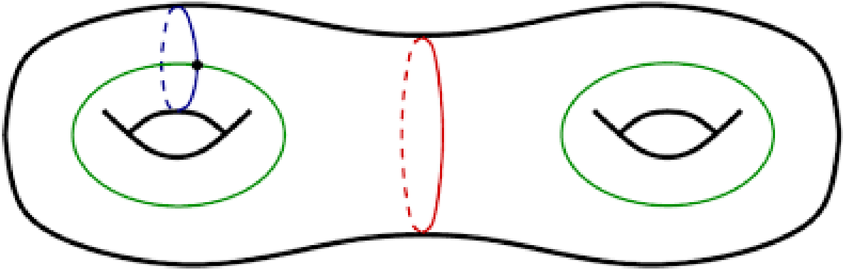
$$X = \{(z, w) \in \mathbb{C}^2; z^2 = Q(w)\}$$

has a single point at infinity, and it is not rational.

Assume X has genus 1. Let $\Gamma \subset X$ be a smooth, simple, closed curve, non homotopically trivial on $\hat{X}$. Let μ be **any** positive measure supported on Γ.

Then $P^2(\mu) = L^2(\mu)$, and indeed μ does not admit ABPE.

genus two



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Constructive approximation of the cloud of ABPE

M.P., *Spectral analysis of 2D outlier layout*, J. Spectral Theory 11(2021), 821-845.

M.P., *Moment estimates of the cloud of a planar measure*, Acta. Appl. Math. 176(2021), 26 pg.

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Géza Freud, Orthogonal Polynomials and Christoffel Functions. A Case Study*

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DEDICATED TO THE MEMORY OF GÉZA FREUD

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Proceedings of Symposia in Pure Mathematics

The Christoffel–Darboux Kernel

Barry Simon*

Abstract. A review of the uses of the CD kernel in the spectral theory of orthogonal polynomials, concentrating on recent results.

Contents

2000 *Mathematics Subject Classification.* 34L40, 47-02, 42C05.

Key words and phrases. Orthogonal polynomials, spectral theory.

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