Two perspectives in number theory: explicit and probabilistic

IISc Eigenfunctions Seminar

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- **[Fundamental Theorem of Arithmetic]** Any natural number $n > 1$ can be factored uniquely as a product of prime powers. This statement appears for the first time in the 1801 textbook "Disquisitones Arithmeticae" by Gauss, and **not** in Euclid's elements.

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- In 1792, Gauss (of age 15) had access to a table of primes up to 3 million.

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It was finally proved in 1896 by Hadamard and de la Vallee Poussin, and is known as the **Prime Number Theorem**.

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Moreover, if the limit

$$\lim_{x \to \infty} \frac{1}{\pi(x)} \int_2^x \frac{dt}{\log t}$$

exists, then it must be equal to 1.

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- This is called the functional equation for $\zeta(s)$.

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- The "non-trivial" zeros of $\zeta(s)$ (that is, zeroes other than even, negative integers) lie in the strip $0 < \mathrm{Re}(s) < 1$.
- **[Riemann hypothesis]** If $\zeta(\sigma + it) = 0$ and $0 < \sigma < 1$, then $\sigma = 1/2$.

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- von Mangoldt's function: Define

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k,\ k \ge 1 \\ 0 & \text{otherwise} \end{cases},\ \psi(x) := \sum_{n \le x} \Lambda(n).$$

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- $\pi(x) \sim \frac{x}{\log x} \iff \theta(x) \sim x \iff \psi(x) \sim x.$

- For $s = \sigma + it$, $\sigma > 1$,

$$\log \zeta(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\sigma} \log\ n} \left\{ \cos(t \log n) - i \sin(t \log n) \right\}.$$

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- Thus, $\zeta(s) \neq 0$ if $\mathrm{Re}(s) = 1$ (letting $\sigma \to 1^+$).

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- Let $c > 1$ and let T be a positive real number. Then,

$$\sum_{n \leq x} \Lambda(n) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} -\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds + E_1(x).$$

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- Thus, $\sum_{n\le x}\Lambda(n) = x+E(x)$, where $E(x)$ is the contribution from integral along horizontal paths and left vertical path and $E_1(x)$.

Explicit methods in number theory

- The prime number theorem is the assertion that

$$\psi(x) := \sum_{n \le x} \Lambda(n) = x + o(x).$$

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- Under certain conditions, $F(s)$ satisfies an Euler product formula, can be analytically (or meromorphically) continued to the complex plane, have zero-free regions, and we have

$$\sum_{n\leq x} f(n)\approx\frac{1}{2\pi i}\int_{c-iT}^{c+iT}F(s)\frac{x^s}{s}ds,\ c>\sigma_0,\ T>0.$$

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$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \ \mathrm{Re}(s) > 1,$$

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- We need regions of analyticity of $-L'(s,\chi)/L(s,\chi)$, that is, zero-free regions of $L(s,\chi)$ of the form

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- Such ZFRs exist for non-quadratic characters χ, where as, for quadratic characters χ, such regions contain an "exceptional" zero of $L(s,\chi)$; so, the analysis to get around this exceptional zero is more delicate.

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$$L(s,\Delta) = \sum_{n=1}^{\infty}\frac{\hat{\tau}(n)}{n^s} = \prod_{p}\left(1-\frac{\hat{\tau}(p)}{p^s}+\frac{1}{p^{2s}}\right)^{-1},\ \mathrm{Re}(s) > 1.$$

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- Wider ZFRs for $L(s, \Delta)$ lead to better and explicit error terms above.

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- $\Delta(z)$ is a special example of a modular cusp form of weight k and level N (in this case $k = 12$, $N = 1$).
- A modular (Hecke) cusp form f of weight k and level N has a Fourier expansion of the form

$$f(z) = \sum_{n \geq 1} n^{(k-1)/2} a_f(n) q^n,\ |a_f(n)| \leq d(n).$$

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- One can define Hecke operators $T_n : S_k(1) \to S_k(1)$, $n \in \mathbb{N}$. $S_k(1)$ has a basis $\mathcal{F}_k(1)$ of Hecke eigenforms, which are eigenfunctions for all the Hecke operators T_n. Each such eigenform has a Fourier expansion

$$f(z) = \sum_{n \geq 1} n^{(k-1)/2} a_f(n) q^n,\ a_f(1) = 1, |a_f(n)| \leq d(n).$$

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- Fourier coefficients of Hecke newforms, in some examples, encode
  - The number of ways of representing an integer by a given quadratic form, for example, a sum of four squares.
  - The number of points on a $\mathbb{Q}$-rational elliptic curve over a finite field with p elements.

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- The study of $\sum_{p \leq x} a_f(p^m) \log p$ enables the proof of the famous Sato-Tate distribution theorem for $\{a_f(p) : \ p \to \infty\} \subset [-2, 2]$, and effective error estimates in this distribution theorem.

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$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p \le x} g\left(a_f(p)\right) = \frac{1}{\pi} \int_{-2}^{2} g(y) \sqrt{1 - \frac{y^2}{4}} dt.$$

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- Note that $X_m(y) = \sum_{j=0}^{\lfloor m/2\rfloor} (-1)^j \binom{m-j}{j} y^{m-2j}$,

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- A wide variety of techniques have been developed to study these questions.
- The "probabilistic" perspective is to interpret, if possible, $f(n)$ as a sum of "random variables".

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Thus, $\frac{1}{x} \sum_{n\le x} \omega(n) \sim \log\log x$ as $x \to \infty$.

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Theorem (Stronger version of Hardy-Ramanujan theorem by Turán, 1934)

$$\frac{1}{x}\sum_{n \le x}(\omega(n) - \log\log x)^2 \ll \log\log x.$$

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Theorem (Erdös-Kac, 1940)

For any integer $r \geq 0$,

$$\lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} \left(\frac{\omega(n) - \log \log x}{\sqrt{\log \log x}} \right)^r$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^r e^{-t^2/2} dt.$$

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- In the words of Kac, "Paul Erdös was in the audience and he immediately perked up. Before the lecture was over, he had proved it, which I could not have done not having been versed in the number theoretic methods, especially those related to the sieve.

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- He was familiar with the viewpoint of Steinhaus, and worked on the possibility of applying the central limit theorem to additive number theoretic functions.
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- In the words of Kac, "Paul Erdös was in the audience and he immediately perked up. Before the lecture was over, he had proved it, which I could not have done not having been versed in the number theoretic methods, especially those related to the sieve. **With his contribution, it became clear that we have had a beginning of a nice chapter of Number Theory, bringing upon it to bear the concepts and methods of Probability Theory.**"

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and perhaps he made even a hint as to the Gaussian distribution of $\omega(n)$. **Though I realized I could settle the above, I found absolutely no interest to do it actually...I had not seen a single possibility of an application of** (1) **(and I do not see this even today, am I right?)"**

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- Murty-Murty (1984) and Murty-Murty-Pujahari (2023): General principles to extend the study of $\omega(n)$ to $\omega(a_n)_{n\in S}$, $S \subset \mathbb{N}$ for integer valued sequences.

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$$f(z) = \sum_{n=1}^{\infty} n^{\frac{k-1}{2}} a_f(n) q^n,$$

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- To obtain Erdös-Kac type theorems for $\sum X_p$, we have to consider the "higher moments",

$$\frac{1}{\mathcal{F}_k(N)} \sum_{f \in \mathcal{F}_k(N)} \left(\sum_{p \le x} a_f(p) \right)^r.$$

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(This was previously proved by Conrey, Duke and Farmer in 1997 as an "average" version of the Sato-Tate conjecture.)

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- $$\frac{1}{|\mathcal{F}_k(N)|} \sum_{f \in \mathcal{F}_k(N)} \left(\sum_{p \leq x} a_f(p) \right)^2 \sim \pi(x).$$

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- *For every integer $r \geq 1$,*

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Theorem (Conrey-Duke-Farmer, 1997, Nagoshi, 2006)

Let I be a fixed interval in $[-2,2]$ and define

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$$\text{Then, } \sum_{p\le x}\mathbb{E}[X_p] = \frac{1}{s_k(N)}\sum_{f\in\mathcal{F}_k(N)}\sum_{p\le x}\chi_I(\mathrm{a_f}(p)) \sim \pi(x)\mu(I)$$

as $x \to \infty$.

Theorem (Prabhu-S, 2019)

Let N be a positive integer. If $(p, N) = 1$, define $X_p : \mathcal{F}_k(N) \to \{0, 1\}$ as $X_p(f) = \chi_I(\mathrm{a_f}(p))$. Let $k = k(x)$ such that

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- The function $G_L(\theta)$ is periodic and has Fourier expansion

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- *For* $\lambda, \omega > 0$, *suppose* $\widehat{g}(t) \ll e^{-\lambda |t|^{\omega}}$, *as* $|t| \to \infty$. *Then, the above asymptotics hold if* $k = k(x) \geq 2$ *satisfies* $\frac{(\log k)}{(\log x)^{1+1/\omega}} \to \infty$ *as* $x \to \infty$.

Motivation and proofs

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- We approximate

$$\sum_{m=0}^{L} S^-(m)a_f(p^{2m}) \le \chi_I(\theta_f(p)) \le \sum_{m=0}^{L} S^+(m)a_f(p^{2m}),$$

and reduce our questions to the evaluation of moments

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- It is natural to consider

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Motivation and proofs

- We approximate
$$\sum_{m=0}^{L} S^{-}(m)a_f(p^{2m}) \le \chi_I(\theta_f(p)) \le \sum_{m=0}^{L} S^{+}(m)a_f(p^{2m}),$$
and reduce our questions to the evaluation of moments
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This motivates us to choose the kind of test functions which ensure appropriate decay of $G(m)$ and the convergence of these moments.

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  - Families of d-regular graphs on n vertices. (Tobias Johnson, 2015.)

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- Question: is there a general theorem from which all the above CLTs can be derived as special examples?