Moment indeterminateness

Mihai Putinar

IISC-2

Contents

First there were numbers

Let $x_0 > x_1 > 0$ be integers. Euclid division:

$$x_0 = b_0 x_1 + x_2$$

$$x_1 = b_1 x_2 + x_3$$

$$\vdots$$

$$x_{n-1} = b_{n-1} x_n$$

with G.C.D. $x_n = (x_0, x_1)$.

Divide and repeat

$$\frac{x_{k-1}}{x_k} = b_{k-1} + \frac{1}{x_k/x_{k+1}}:$$

$$x_0/x_1 = b_0 + \frac{1}{b_1 + \frac{1}{b_2 + \ldots + \frac{1}{b_{n-1}}}} =$$

$$b_0 + \mathbb{K}_{k=1}^{n-1}(\frac{1}{b_k}).$$

Irrationality criteria: the continued fraction does not stop

Hipassus of Metapontum (500 BC): The diagonal x_0 of the square of side x_1 satisfies (via an ingenious geometric recurrence)

$$x_0/x_1 = 1 + \cfrac{1}{2 + \frac{1}{2+\frac{1}{2+\dots}}}.$$

Irrationality criteria: the continued fraction does not stop

Hipassus of Metapontum (500 BC): The diagonal x_0 of the square of side x_1 satisfies (via an ingenious geometric recurrence)

$$x_0/x_1 = 1 + \frac{1}{2 + \frac{1}{2+\frac{1}{2+\dots}}}.$$

More general (Bombelli method, approx. 1560) for N positive integer, not a perfect square:

$$N = a^2 + r, \ \sqrt{a^2 + r} = a + x$$

yields:

$$x = \frac{r}{2a + x},$$

hence

$$\sqrt{N} = a + \frac{r}{2a + \frac{r}{2a+\frac{r}{2a+\dots}}}.$$

The real numbers

For sequences of non-negative integers $b = (b_n)_{n=0}^{J}$, with J finite or not, consider $\mathcal{Z}$ the union of domains

$$\mathcal{D}(b) = \mathbb{Z}_+, \quad b_n > 0, \quad n > 0,$$

or

$$\mathcal{D}(b) = [0, J] \cap \mathbb{Z}_+, \quad J > 0, (b_n > 0, \ n > 0), \ b_J \geq 2,$$

or

$$\mathcal{D}(b) = \{0\}.$$

The real numbers

For sequences of non-negative integers $b = (b_n)_{n=0}^{J}$, with J finite or not, consider $\mathcal{Z}$ the union of domains

$$\mathcal{D}(b) = \mathbb{Z}_+, \quad b_n > 0, \quad n > 0,$$

or

$$\mathcal{D}(b) = [0, J] \cap \mathbb{Z}_+, \quad J > 0, (b_n > 0, \ n > 0), \ b_J \geq 2,$$

or

$$\mathcal{D}(b) = \{0\}.$$

Theorem. The mapping

$$\mathcal{Z} \longrightarrow \mathbb{R}, \quad b \mapsto b_0 + \mathbb{K}_{k=1}^{J}(\frac{1}{b_k})$$

is bijective, and a *homeomorphism* from $\mathcal{Z}$ endowed with pointwise convergence.

Algebra of continued fractions

Recurrence, with $a_j \neq 0$:

$$x_0 = b_0 x_1 + a_1 x_2$$

$$x_1 = b_1 x_2 + a_2 x_3$$

$$\vdots$$

$$x_{n-1} = b_{n-1} x_n + a_n x_{n+1}$$

$$\vdots$$

has partial fractions (no cancellation):

$$\frac{P_n}{Q_n} = b_0 + \mathbb{K}_{k=1}^{n}\left(\frac{a_k}{b_k}\right).$$

Main Theorem: Wallis 1656, Brouncker 1655, Euler 1748

The formal continued fraction, with initial data:

$$P_{-1} = 1,\ P_0 = 0,\ Q_{-1} = 0,\ Q_0 = 1$$

implies

$$P_n = b_n P_{n-1} + a_n P_{n-2},$$

$$Q_n = b_n Q_{n-1} + a_n Q_{n-2},$$

$$P_n Q_{n-1} - P_{n-1} Q_n = (-1)^{n-1} a_1 a_2 \dots a_n,$$

$$\xi = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots + \cfrac{a_n}{b_n + \cfrac{a_{n+1}}{\xi_{n+1}}}}}$$

yields

$$\xi = \frac{\xi_{n+1} P_n + a_{n-1} P_{n-1}}{\xi_{n+1} Q_n + a_{n-1} Q_{n-1}}$$

Enters Positivity

Assume all $a_j, b_j > 0$. Then

$$\frac{P_n}{Q_n} - \frac{P_{n-1}}{Q_{n-1}} = (-1)^{n-1} \frac{a_1 a_2 \dots a_n}{Q_n Q_{n-1}}$$

therefore

$$\frac{P_0}{Q_0} < \dots \frac{P_{2k}}{Q_{2k}} < \frac{P_{2k+1}}{Q_{2k+1}} < \dots < \frac{P_1}{Q_1}.$$

Analytic Theory

Markov's Paradox

Solve

$$z^2 - 2z - 1 = 0$$

Equivalently

$$z = 2 + \frac{1}{z}.$$

The solution should be

$$\xi = 2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \dots}}},$$

that is $\xi = 1 + \sqrt{2}$, because all entries are positive.

Where is the other root $1 - \sqrt{2}$?

The approximants

$$2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ldots + \cfrac{1}{2 + 1 - \sqrt{2}}}}}$$

do not converge.

Koch divergence test

Assume $b_n \in \mathbb{C}$ and $\sum_n |b_n| < \infty$. Then the approximants of $\mathbb{K}_1^\infty(\frac{1}{b_n})$ satisfy:

$$\lim_n P_{2n} = P, \quad \lim_n P_{2n+1} = P',$$

$$\lim_n Q_{2n} = Q, \quad \lim_n Q_{2n+1} = Q',$$

and

$$P'Q - PQ' = 1.$$

Hence clear divergence.

Seidel convergence test

Assume all $b_j > 0$. Then

$$\mathbb{K}_1^\infty\left(\frac{1}{b_n}\right)$$

converges if and only if

$$\sum_n b_n = \infty.$$

The eternal quest: π

From Wallis:

$$\frac{2}{\pi} = \prod_{j=1}^{\infty} \frac{(2j-1)(2j+1)}{(2j)^2}.$$

and Brouncker:

$$\frac{4}{\pi} = 1 + \mathbb{K}_1^{\infty}\left(\frac{(2n-1)^2}{2}\right).$$

straight to the origins of the analytic theory of continued fractions.

Main idea, derived from Wallis infinite product. Consider a function $b(s) > s$ subject to:

$$b(s)b(s+2) = (s+1)^2.$$

And note:

$$b(1) = \frac{2^2}{b(3)} = \frac{2^2}{4^2} b(5) = \frac{2^2}{4^2} \frac{6^2}{b(7)} = \ldots =$$

$$\frac{2^2}{4^2} \frac{6^2}{8^2} \frac{10^2}{12^2} \ldots \frac{(4n-2)^2}{(4n)^2} b(4n+1) = \frac{1^2}{2^2} \frac{3^2}{4^2} \frac{5^2}{6^2} \ldots \frac{(2n-1)^2}{(2n)^2} b(4n+1) =$$

$$\frac{1 \times 3}{2^2} \frac{3 \times 5}{4^2} \ldots \frac{(2n-1)(2n+1)}{(2n)^2} \frac{b(4n+1)}{2n+1}.$$

That is

$$b(1) = \left(\frac{2}{\pi} + o(1)\right) \frac{b(4n+1)}{2n+1}.$$

But $s+2 < b(s+2)$ and the functional equation yields:

$$s < b(s) < \frac{s^2+2s+1}{s+2} = s + \frac{1}{s+2}.$$

Hence $b(1) = \frac{2}{\pi}$.

For the continued fraction we start with the formal series:

$$b(s) = s + c_0 + \frac{c_1}{s} + \frac{c_2}{s^2} + \dots$$

and find the coefficients by applying *Euclid division algorithm to series in* $1/s$. Conclusion

$$b(s) = s + \cfrac{1^2}{2s + \cfrac{3^2}{2s + \cfrac{5^2}{2s + \dots}}}.$$

With convergence derived from the functional equations and elementary inequalities.

Closed form

$$b(s) = 4[\frac{\Gamma(3+s/4)}{\Gamma(1+s/4)}]^2.$$

due to Ramanujan.

The new "continuum"

Laurent series

$$f(z) = \sum_{k \in \mathbb{Z}} \frac{c_k}{z^k},$$

with finitely many $k < 0$ terms. Define

$$[f] = \sum_{k \leq 0} \frac{c_k}{z^k}, \quad \mathrm{Frac}(f) = f - [f].$$

And non-archimedean norm

$$\|f\| = \exp \deg f, \quad \deg(0) = -\infty.$$

The algorithm

Initial $f_0 = f$ produces a *P-fraction*:

$$f = [f_0] + \frac{1}{1/\mathrm{Frac}[f_0]} = [f_0] + \frac{1}{[f_1] + \frac{1}{[f_2]+\dots}}$$

For non-rational $f(z)$ one finds $\deg Q_n \to \infty$ and

$$\|f - \frac{P_n}{Q_n}\| = \exp(-\deg Q_n - \deg Q_{n+1}).$$

Theorem. (Markov, Chebyshev, Gauss) An irreducible rational fraction P/Q is a convergent for the Laurent series f if and only if

$$\deg(f - P/Q) \leq -2 \deg Q - 1.$$

Padé approximation

Specializes to Padé approximation problem: *given* f *and* $n > 0$ *find all polynomials* P, Q, $Q \neq 0, \deg Q \leq n$ *such that*

$$\deg(Qf - P) \leq 2n - 1.$$

A *normal index* is $\deg Q$ for an approximant P/Q.

Constructive aspects

Starting with $f(z) = [f](z) + \sum_{k \geq 1} \frac{c_k}{z^k}$ one defines:

$$H_n(f) = \begin{pmatrix} c_1 & c_2 & \dots c_n \\ c_2 & c_3 & \dots c_{n+1} \\ \vdots & \ddots & \\ c_n & c_{n+1} & \dots c_{2n-1} \end{pmatrix},$$

and

$$J_n(z) = \det \begin{pmatrix} c_1 & c_2 & \dots c_{n+1} & \\ c_2 & c_3 & \dots c_{n+2} & \\ \vdots & & \ddots & \\ c_n & c_{n+1} & \dots c_{2n} & \\ 1 & z & \dots & z^n \end{pmatrix}.$$

Main Theorem.

C. G. J. Jacobi (approx. 1850): An integer $n > 0$ is a normal index for f if and only of $\det H_n(f) \neq 0$. In that case the convergent P/Q with $\deg Q = n$ is, with a constant γ:

$$Q_n(z) = \gamma J_n(z)$$

$$P_n(z) = [J_n(z) f(z)].$$

And $f(z)$ is rational if and only if there exists N with $\det H_n(f) = 0, \quad n \geq N$. (Kronecker)

Abelian integrals

Let $R \in \mathbb{C}[z]$ of degree $\deg R = 2g + 2 \geq 2$ without multiple roots. Pell's type equation:

$$P^2 - Q^2 R = 1$$

has polynomial solutions, $Q \neq 0$ if and only if $\sqrt{R(z)}$ admits a periodic polynomial continued fraction expansion, if and only if there exists $r \in \mathbb{C}[z], \deg r = g$, so that

$$\int \frac{r}{\sqrt{R}} dz$$

can be expressed in elementary functions. (Abel 1826).

References

Perron, Oskar. *Die Lehre von den Kettenbrüchen*, Band I, Band II, Teubner, Stuttgart, 1954, 1957.

Khrushchev, Sergey. *Orthogonal polynomials and continued fractions. From Euler's point of view.* Encyclopedia of Mathematics and its Applications, 122. Cambridge University Press, Cambridge, 2008.

Stieltjes

Stieltjes has accumulated examples and computations concerning semi-convergent series ("the curse of divergent series" according to Abel), leading to a rigorous study of functions of s of the form:

$$b_0 s + c_0 + \mathbb{K}_{k=1}^{\infty}\left(\frac{a_n}{b_n s + c_n}\right),$$

where

$$a_n > 0, b_n \geq 0, \Re c_n \geq 0.$$

Main observation, for $s > 0$:

$$w \mapsto \frac{a_n}{b_n s + c_n + w}$$

preserves $\Re w > 0$.

Complex Markov Convergence Test

If

$$b_0 s + c_0 + \mathbb{K}_{k=1}^{\infty}(\frac{a_n}{b_n s + c_n}),$$

converges to finite values on a subset of $(0, \infty)$ with an accumulation point, then it converges to an analytic function defined on $\Re w > 0$.

Major advance, a la normal family argument, discovered by Stieltjes many years before Vitali. See his letters to Hermite.

Stieltjes Memoir-1894

$$S(z) = \frac{1}{a_1 z + \frac{1}{a_2 + \frac{1}{a_3 z + \frac{1}{a_4 + \dots}}}},$$

with all parameters $a_j \geq 0$. Produces convergents satisfying

$$\lim_n \frac{P_{2n}(z)}{Q_{2n}(z)} = F(z),$$

$$\lim_n \frac{P_{2n+1}(z)}{Q_{2n+1}(z)} = F_1(z),$$

where F, F_1 are analytic functions on $\mathbb{C} \setminus (-\infty, 0]$.

Asymptotic expansion

$$S(z) \sim \frac{c_0}{z} - \frac{c_1}{z^2} + \frac{c_2}{z^3} - \dots.$$

That is

$$\lim_{s\to\infty} s^{n+1}[S(s) - \frac{c_0}{s} + \frac{c_1}{s^2} + \dots + (-1)^n \frac{c_{n-1}}{s^n}] = (-1)^n c_n,$$

for all $n \geq 1$.

Indeterminateness

Assume $\sum_n a_n < \infty$. Then all limits

$$\lim P_{2n}(z) = p(z), \quad \lim Q_{2n}(z) = q(z),$$

$$\lim P_{2n+1}(z) = p_1(z), \quad \lim Q_{2n+1}(z) = q_1(z),$$

exist in $\mathbb{C}$ and

$$\frac{p(z)}{q(z)} = \sum_j \frac{M_j}{z + m_j}$$

with $M_j > 0, m_j \geq 0$. Similarly for $\frac{p_1(z)}{q_1(z)}$, but

$$\frac{p(z)}{q(z)} \neq \frac{p_1(z)}{q_1(z)}.$$

Determinateness

Assume $\sum_n a_n = \infty$. Then the continued fraction converges on $\mathbb{C} \setminus (-\infty, 0]$ to a function of the form

$$S(z) = \int_0^\infty \frac{df(u)}{u+z},$$

where f is monotonically non-decreasing on $[0, \infty)$ and all power moments of $df(u)$ exist. A new notion of integral was developed by Stieltjes for this purpose. Unifying in particular the two cases

$$\sum_j \frac{M_j}{z+m_j} = \int_0^\infty \frac{d\phi(u)}{u+z}.$$

Comparison: old continuum versus the new one

$$\xi = \sum_{j=0}^{\infty} \frac{n_j}{10^j} \in [0, \infty), \quad \xi = b_0 + \mathbb{K}_j\left(\frac{1}{b_j}\right),$$

and

$$\sigma \in \mathrm{Meas}_+[0, \infty), \quad \int_0^{\infty} \frac{d\sigma(u)}{u + z} \sim \frac{1}{a_1 z + \frac{1}{a_2 + \frac{1}{a_3 z + \frac{1}{a_4 + \ldots}}}},$$

In both representations the parameters $b_j \in \mathbb{Z}_+$, respectively $a_j \geq 0$ are *independent*.

Cauchy transforms

There is a *constructive bijective* correspondence between analytic functions $F(z)$ defined in the half-plane $\Im(z) > 0$ and satisfying $\Im F(z) > 0$ there, and positive Borel measures σ of finite mass, defined on $\mathbb{R}$ admitting all power moments:

$$F(z) = az + b + \int_{-\infty}^{\infty} \frac{1+tz}{t-z} d\sigma(t),$$

where $a \geq 0$ and $b \in \mathbb{R}$.
Moreover, the measure σ admits all moments, if and only if

$$\sup_{y \geq 1} |yF(iy)| < \infty$$

and there are real numbers $s_k,\ k \geq 0$, satisfying

$$\lim_{y \to \infty} (iy)^{2n+1} [F(iy) + \frac{s_0}{iy} + \frac{s_1}{(iy)^2} + \ldots + \frac{s_{2n-1}}{(iy)^{2n}}] = -s_{2n}.$$

Recovery: Stieltjes and Perron

If

$$F(z) = \int_{-\infty}^{\infty} \frac{1}{t-z} d\sigma(t), \ \Im z > 0,$$

then

$$\frac{\sigma\{a\} + \sigma\{b\}}{2} + \sigma(a,b) = \lim_{\epsilon \to 0} \frac{1}{\pi} \Im \int_a^b F(x+i\epsilon) dx.$$

Stirling formula

Major application of Stieltjes theory:

$$\log \Gamma(z) = -z + (z - 1/2)\log z + \log(\sqrt{2\pi}) + J(z),$$

where

$$J(z) = \frac{1}{\pi}\int_0^\infty \log \frac{1}{1 - e^{-2\pi u}} \frac{z}{z^2 + u^2} du.$$

The formal power series expansion of J diverges:

$$J(z) \sim \sum_{k=0}^{\infty} (-1)^k \frac{c_k}{z^{2k+1}},$$

but the associated continued fraction **converges**.

Note:

$$c_k = \int_0^\infty u^k d\phi(u), \quad k \geq 0,$$

where

$$\phi(u) = \frac{1}{\pi} \int_0^u \frac{\log(1 - e^{-2\pi\sqrt{t}})}{2\sqrt{t}} dt.$$

Hamburger articles 1919-1921

Real heritor of Stieltjes, author of a detailed study of the moment problem on the real line. All centered on continued fraction expansions of the form:

$$S(z) = \cfrac{a_1}{z + b_1 + \cfrac{a_2}{z+b_2+\cfrac{a_3}{z+b_3+\dots}}},$$

with $a_j \neq 0$ real and b_j complex.

Convergence assured by selection principle of Grommer (à la normal families argument).

Parametrization of all solutions of the truncated problem.

Identification, and recognition of importance, of *Christoffel and Darboux kernel*.

Complemented by R. Nevanlinna function theoretic study (1922) of Stieltjes moment problem.

Carleman

Invaluable references:

Sur les equations integrales singulieres a noyau reel et symmetrique, Uppsala, 1923.

Lecons sur les fonctions quasi-analytiques, Paris 1926.

Main Lemma. Let $0 < \lambda_1 < \lambda_2 < \lambda_3 \ldots \to \infty$ and $0 < \beta_1 < \beta_2 < \beta_3 < \ldots$, so that

$$\sum_1^\infty \frac{\lambda_j - \lambda_{j-1}}{\beta_j} = \infty.$$

An analytic function $f(z)$ defined for $\Im z > 0$ and satisfying:

$$|f(z)| \leq |\frac{\beta_n}{z}|^{\lambda_n}, \quad \Im z > 0, \quad n \geq 1,$$

is identically zero.

Determinateness Criterion

Assume the moment problem

$$\int_{\mathbb{R}} x^n d\sigma_k(x) = s_n, \quad n \geq 0,$$

admits two solutions σ_1, σ_2. The Cauchy transforms are:

$$F_k(z) = \int \frac{d\sigma_k(x)}{x - z}, \ \Im z > 0, \ k = 1, 2.$$

Fix $n > 1$ and remark:

$$|F_1(z) - F_2(z)| = |\frac{1}{z^{2n+1}} \int \frac{x^{2n} d(\sigma_1 - \sigma_2)(x)}{1 - \frac{x}{z}}|.$$

With $\theta = \arg z$ we find

$$|F_1(z) - F_2(z)| \leq \frac{2c_{2n}c_0}{|z^{2n}||\Im z|}.$$

And on the half-plane $\Im z > 1$:

$$|F_1(z) - F_2(z)| \leq \frac{2c_{2n}c_0}{|z^{2n}|}.$$

Main Lemma implies: *If*

$$\sum_n \frac{1}{c_{2n}^{1/2n}} = \infty,$$

then $F_1 = F_2$, *hence* $\sigma_1 = \sigma_2$.

Construction of indeterminate moment sequences

Start with a non-constant $\phi \in C^{(\infty)}[0,1]$ with $\phi^{(k)}(0) = \phi^{(k)}(1) = 0, \quad k \geq 0.$

Then

$$m_p^2 = \int_0^1 [\phi^{(p)}(x)]^2 dx, \quad p \geq 0$$

is a Stieltjes moment sequence.

Indeed:

$$\int_0^1 \phi^{(2p)}(x)\phi^{(2q)}(x)dx = (-1)^{p+q}\int_0^1 [\phi^{(p+q)}(x)]^2 dx,$$

$$\int_0^1 \phi^{(2p+1)}(x)\phi^{(2q+1)}(x)dx = (-1)^{p+q}\int_0^1 [\phi^{(p+q+1)}(x)]^2 dx.$$

Hence

$$\int_0^1 |\sum_{k=0}^{n}(-1)^k c_k f^{(2k)}(x)|^2 dx = \sum m_{p+q}^2 c_p c_q \geq 0,$$

$$\int_0^1 |\sum_{k=0}^{n}(-1)^k c_k f^{(2k+1)}(x)|^2 dx = \sum m_{p+q+1}^2 c_p c_q \geq 0.$$

Fourier transform:

$$\alpha_n - i\beta_n = \int_0^1 \phi(t) e^{i\pi nt} dt,$$

implies

$$\phi(x) = \alpha_0 + 2\sum_{n=0}^{\infty} \alpha_n \cos(\pi nx)$$

and

$$\phi(x) = 2\sum_{n=1}^{\infty} \beta_n \sin(\pi nx).$$

Plancherel theorem yields:

$$m_0^2 = \alpha_0^2 + \sum_1^\infty \alpha_n^2,$$

$$m_p^2 = 2\sum_1^\infty (\pi n)^{2p} \alpha_n^2, \;\; p > 0,$$

and also

$$m_p^2 = 2\sum_1^\infty (\pi n)^{2p} \beta_n^2, \;\; p \geq 0.$$

Distinct solutions

$$\sigma_1 = \alpha_0^2 \delta_0 + \sum_1^\infty 2\alpha_n^2 \delta_{\pi n},$$

and

$$\sigma_2 = \sum_1^\infty 2\beta_n^2 \delta_{\pi n}.$$

Absolutely continuous different solutions

$$A(s) + iB(s) = \int_0^1 e^{ist}\phi(t)dt = \frac{(-1)^p}{(is)^p}\int_0^1 e^{ist}\phi^{(p)}(t)dt,$$

produces

$$m_p^2 = \frac{2}{\pi}\int_0^\infty s^{2p}A(s)^2 ds = \frac{2}{\pi}\int_0^\infty s^{2p}B(s)^2 ds.$$

Quasi-analytic functions

Theorem. (Denjoy) Let $\psi \in C^{(\infty)}[0,1]$ with $\psi^{(n)}(0) = 0, \quad n \geq 0,$ and

$$M_n = \sup_{x \in [0,1]} |\psi^{(n)}(x)|, \; n \geq 0.$$

If

$$\sum_n \frac{1}{M_n^{1/n}} = \infty,$$

then $\psi = 0$.

Let $\phi(t) = \psi(1 - 4(t - \frac{1}{2})^2)$ as before, and note

$$|\phi^{(n)}(x)| \leq K^n M_n, \ \ n \geq 0,$$

with $K > 0$ a constant. Then

$$m_n^2 = \int_0^1 \phi^{(n)}(x)^2 dx \leq M_n^2 K^{2n}.$$

Since $m_n^{1/n} = [m_n^2]^{1/(2n)}$, we infer divergence:

$$\sum \frac{1}{m_n^{1/n}} = \infty,$$

that is the associated moment problem admits a unique solution. This can happen only if $\phi = 0$, that is $\psi = 0$.

The dawn of modern spectral analysis

Account by Carleman of Hilbert and his school (Hellinger, Toeplitz, Grommer):

$$J(x) = \sum_{p=1}^{\infty} (a_p x_p^2 - 2 b_p x_p x_{p+1}),$$

as a specific quadratic form in infinitely many variables. In matrix notation:

$$J(x) - \lambda = \begin{pmatrix} a_1 - \lambda & -b_1 & 0 & 0 \dots & \\ -b_1 & a_2 - \lambda & -b_2 & 0 & \\ 0 & -b_2 & a_3 - \lambda & -b_3 & \dots \\ \vdots & & \ddots & & \vdots \end{pmatrix}$$

is naturally associated, via the recurrence of the partial convergents, to the continued fraction:

$$S(\lambda) = \frac{1}{a_1 - \lambda - \frac{b_1^2}{a_2 - \lambda - \frac{b_2^2}{a_3 - \lambda - \dots}}}.$$

Theorem. Assuming the real entries a_j, b_j uniformly bounded, there exists a positive *spectral* measure ρ with compact support, so that

$$S(\lambda) = \int_m^M \frac{d\rho(t)}{t - \lambda}, \quad \lambda \notin [n, M].$$

Moreover, the power moments (s_j) satisfy

$$S(\lambda) = -\sum_{j=0}^{\infty} \frac{s_j}{z^{j+1}}, \quad |z| >> 1.$$

Even for unbounded $a_j, b_j's$ one can speak of the rational convergents $\frac{Q_n(\lambda)}{P_n(\lambda)}$ of $S(\lambda)$.

All (possible multiple) solutions ρ to the moment problem match asymptotically the moments:

$$\int_{\mathbb{R}} \frac{d\rho(t)}{t-\lambda} \sim -\sum_{j=0}^{\infty} \frac{s_j}{z^{j+1}},$$

in a wedge $0 < \epsilon < \arg z < \pi - \epsilon$.

Finite term relation for orthogonal polynomials

Given a positive, rapidly decreasing at infinity measure μ on $\mathbb{R}$, not reduced to finitely many point masses, the associate orthogonal polynomials P_n satisfy:

$$(\lambda - a_n)P_n(x) = b_{n-1}P_{n-1}(x) + b_nP_{n+1}(x), \quad n \geq 1,$$

with $b_n > 0$ for all $n \geq 0$.

The polynomials P_n are the same and orthogonal with respect to any solution ρ, while the uniqueness is decided by the Christoffel-Darboux kernel: *There exists $\alpha \notin \mathbb{R}$, with*

$$\sum_{j=0}^{\infty} |P_n(\alpha)|^2 = \infty,$$

if and only if the moment problem with data (s_n) has a unique solution. And then any other $\alpha \notin \mathbb{R}$ has the same property.

Marcel Riesz

Linear functionals extension argument

Sur le problème des moments, Troisième Note, Ark. Mat. Fys. 16(1923), 1-52.

Let $(c_n)_{n=0}^{\infty}$ be a moment sequence on the real line. A linear functional

$$\lambda : \mathbb{C}[z] \longrightarrow \mathbb{C}, \quad z^n \mapsto c_n, \quad n \geq 0,$$

is associated, with known positivity condition

$$\lambda(|p(z)|^2) \geq 0, \quad p \in \mathbb{C}[z].$$

For any continuous function $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ of *polynomial growth* we can define by induction a linear extension $\Lambda(\phi)$ satisfying:

$$\Lambda(p) = \lambda(p), \quad p \in \mathbb{C}[z],$$

and

$$\Lambda(\phi) \le \Lambda(\psi), \quad \phi \le \psi.$$

Remarks: 1) Then Λ is a positive, linear functional, hence representable by an integral against a positive measure.
2) A similar non-constructive proof appears in Hahn-Banach Theorem. It came later, and independently!

In particular, for any ϕ continuous (of polynomial growth)

$$\lambda_*(\phi) = \sup_{p \le \phi} \lambda(p) \le \Lambda(\phi) \le \inf_{\phi \le q} \lambda(q) = \lambda^*(\phi),$$

where p, q are polynomials.

Theorem. Let ϕ be a non-polynomial continuous function. If $\lambda_*(\phi) < \lambda^*(\phi)$, then and only then the moment problem is indeterminate.

Or, either $\lambda_*(\phi) = \lambda^*(\phi)$ for all continuous functions of polynomial growth, or for none, different than polynomials.

Proof based on a hard analysis extension of Hamburger work, in the indeterminate case.
Christoffel's function

$$\rho(z) = \frac{1}{\sum_0^\infty |P_n(z)|^2} = \inf_{p \geq 0, p(z)=1} \lambda(p)$$

satisfies

$$\frac{1}{\rho(z)} \leq \gamma e^{\epsilon(|z|)|z|}, \quad z \in \mathbb{C},$$

where $\gamma > 0$ is a constant and $\lim_{r\to\infty} \epsilon(r) = 0$.

Assume (the indeterminate case) that $\lambda_*(\phi) = \lambda^*(\phi)$ for a non-polynomial function ϕ. Then there are sequences of polynomials $p_k \leq \phi \leq q_n$ satisfying

$$\lim_{k,n} \lambda(q_n - p_k) = 0.$$

For a fixed value $\alpha \in \mathbb{C}$ one has

$$|q_k(\alpha) - p_n(\alpha)| \leq \frac{\lambda(q_n - p_k)}{\rho(\alpha)}.$$

Thus q_k and p_n converge to an entire function F, and

$$|F(\alpha)| \leq |p_1(\alpha)| + \frac{K}{\rho(\alpha)}, \ \alpha \in \mathbb{C}.$$

But $F(x) = \phi(x), \quad x \in \mathbb{R}$, has polynomial growth. Phragmen-Lindelöf Theorem implies F is a polynomial, contradiction!

Example

$$\phi(x) = \cos(ax) = 1 - \frac{a^2x^2}{2!} + \frac{a^4x^4}{4!} - \ldots, \quad a > 0.$$

Has clear upper and lower partial sums (polynomials). Hence

$$\lambda^*(\phi) - \lambda_*(\phi) \leq \frac{a^{2n}s_{2n}}{(2n)!}, \quad n \geq 1.$$

Assume, in the indeterminate case $\lambda^*(\phi) - \lambda_*(\phi) \geq \kappa > 0$. Then

$$(\frac{s_{2n}}{(2n)!})^{1/(2n)} \geq \frac{\kappa^{1/(2n)}}{a},$$

or

$$\liminf(\frac{s_{2n}}{(2n)!})^{1/(2n)} \geq \frac{1}{a}.$$

Since a is arbitrary, we find: if

$$\liminf \frac{s_{2n}^{1/(2n)}}{n} < \infty,$$

then the moment problem is determinate.

Laguerre divergent series

$$\sum_{k=0}^{\infty}(-1)^k k! w^k$$

has radius of convergence $R = 0$. Yet, the continued fraction expansion of

$$\sum_{k=0}^{\infty}(-1)^k \frac{k!}{z^{k+1}}$$

is easily computable:

$$F(z) = \cfrac{1}{z + \cfrac{1}{1+\cfrac{1}{z+\cfrac{2}{1+\cfrac{2}{z+\cfrac{3}{1+\cfrac{3}{z+\dots}}}}}}},$$

and **convergent**.

Stieltjes summation method

The limit is

$$F(z) = \int_0^\infty \frac{e^{-t}}{t+z} dt,$$

and we know that asymptotically

$$F(z) \sim \sum_{k=0}^\infty (-1)^k \frac{k!}{z^{k+1}}$$

for $z \to \infty$ along the imaginary axis (or a wedge with vertex at $z = 0$).

Higher dimensions

Still very challenging and incomplete when compared to the classical 1D situation.

Polynomially parametrized algebraic, affine curves

$$\mathbb{R} \mapsto X \subset \mathbb{R}^d$$

Theorem (Kimsey-P.) *Let μ be a positive measure on X, fast decaying at infinity. Then μ is X-indeterminate if and only if*

$$\Lambda^{\mu}(\lambda) > 0,$$

locally, in a neighbourhood of at least one point of $X_{\mathbb{C}} \setminus X$.

Proof derived from monotonic approximation techniques inspired by M. Riesz.

Rational parametrizations are not welcome

Push forward via rational parametrizations may alter moment indeterminateness. Example:

$$F(t) = \left(t, \frac{1}{1+t^2}\right), \ t \in \mathbb{R}.$$

Any admissible measure in $\mathbb{R}^2$ supported by the image curve Γ of equation $y(1+x^2) = 1$ is determinate, due to the fact that bounded polynomials on Γ are separating points.

Quantitative side: Christoffel function criterion

Let $X \subset \mathbb{R}^d$ be a polynomially parametrized, affine algebraic curve. For a point $\lambda \in X_{\mathbb{C}} \setminus X$ the bound on point evaluations is encoded in

$$\Lambda^{\mu}(\lambda) = \inf_{p(\lambda)=1} \|p\|^2_{2,\mu}, \quad p \in \mathbb{C}[X_{\mathbb{C}}] = \mathbb{C}[z]/I(X_{\mathbb{C}}), \ [p] \neq 0.$$

In general restricted by the degree filtration: $\deg p \leq N$.

Plaumann-Scheiderer curves

Let (q) be a non-trivial, real principal ideal in $\mathbb{R}[x, y]$. Then

$$(q) + \Sigma^2 = \{p \in \mathbb{R}[x, y] : p(x) \geq 0 \text{ for all } x \in V(q)\}$$

if and only if the following conditions hold:

(i) All real singularities of $V(q)$ are ordinary multiple points with independent tangents.

(ii) All intersection points of $V(q)$ are real.

(iii) All irreducible components of $V(q)'$ (i.e., the union of all irreducible components of $V(q)$ that do not admit any non-constant bounded polynomial functions) are non-singular and rational.

(iv) The configuration of all irreducible components of $V(q)'$ contains no loops.

Indeterminateness criterion

Theorem (Kimsey-P.) *Let $X \subset \mathbb{R}^d$ be a real algebraic curve on which all positive polynomials are sums of squares. Let μ be an admissible measure supported by X which admits analytic bounded point evaluations on $X_{\mathbb{C}}$. Then the measure μ is indeterminate.*

Examples of polynomially parametrized curves

Abhyankar and Moh Theorem: *a rational curve* $X \subset \mathbb{C}^2$ *admits a polynomial parametrization if and only if it can be transformed into a straight line by invertible linear transforms and automorphisms of the form*

$$x_1 = x + h(y), \quad y_1 = y, \quad h \in \mathbb{C}[y].$$

Zaindenberg and Lin Theorem: *any simply connected, irreducible polynomial curve in* $\mathbb{C}^2$ *is equivalent, in the above sense, to a basic cusp curve:*

$$x^k = y^\ell,$$

where k, ℓ *are relatively prime positive integers.*

Abhyankar and Moh Theorem: *A rational curve $X \subset \mathbb{C}^2$ admits a polynomial parametrization if and only if its compactification in projective space contains a single place at infinity.*
That means that the polynomial equation describing the curve

$$X = \{(x, y) \in \mathbb{C}^2;\ F(x, y) = 0\}$$

starts with an exact power of a linear function, plus a reminder:

$$F(x, y) = (ax + by)^d + G(x, y), \quad |a| + |b| > 0, \quad \deg G < d.$$

Low degree examples

A *cubic with a nodal singular point* admits a polynomial parametrization:

$$y^2 = x^2(x+1); \quad x = t^2 - 1, \; y = t^3 - t.$$

Among 2D cuartics, the *Kampyle of Eudoxus* is a polynomial curve:

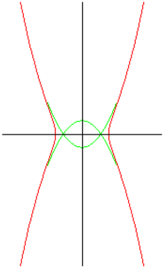
$$x^4 = a^2(x^2 + y^2), \quad a > 0,$$

or better, in polar coordinates

$$\rho = \frac{a}{\cos^2 \theta},$$

can be rationally parametrized as function of $t = tan\frac{\theta}{2}$.

Kampyle



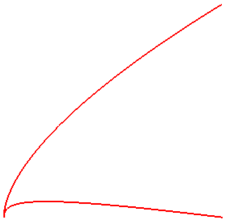
Another quartic in two dimensions exhibits a *Ramphoid cusp* (that is both branches at the singular point are tangent to the same semi-axis):

$$y^4 - 2axy^2 - 4ax^2y - ax^3 + a^2x^2 = 0, \quad a > 0,$$

with parametrization

$$x = at^4, \; y = a(t^2 + t^3).$$

Ramphoid cusp



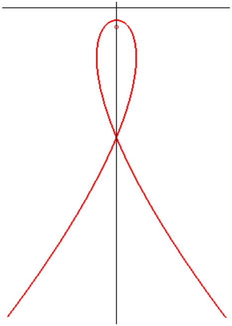
L'Hospital quintic

$$64y^5 = a(25x^2 + 20y^2 - 20ay + 4a^2)^2, \quad a > 0,$$

with parametrization

$$x = \frac{a}{2}(t - \frac{t^5}{5}), \quad y = \frac{a}{4}(1 + t^2)^2.$$

L'Hospital Quintic



The touch of Fantappiè transform

$\Gamma \subseteq \mathbb{R}^d$ is an acute, convex and solid cone and
$\Gamma^* = \{\eta \in \mathbb{R}^d : \eta \cdot x \geq 0 \quad \text{for all} \quad x \in \Gamma\}$ be the dual cone of Γ.
Let μ be an admissible measure supported on Γ^*. In this case the Fantappié's transform

$$F_\mu(z, w) = \int_{\Gamma^*} \frac{d\mu(x)}{w \cdot x - z}$$

admits a complex analytic extension to the domain

$$\operatorname{Re} w \in \Gamma \quad \text{and} \quad \operatorname{Re} z < 0$$

and *determines* μ.

In particular the range of real values $w \in \Gamma, \ z < 0$, is a uniqueness set for the complex analytic function F_μ defined on the tube domain over this convex set. The values

$$F_\mu(-1, a) = \int_{\Gamma^*} \frac{d\mu(x)}{a \cdot x + 1}, \quad a \in \Gamma,$$

determine the measure μ.

Theorem (Kimsey-P.) *Let $\Gamma \subseteq \mathbb{R}^d$ be an acute, convex and solid cone and let μ be an admissible measure supported by the dual cone Γ^*. There exists a different, admissible measure supported on Γ^* and moment equivalent to μ if and only if there exists $a \in \mathrm{int}\Gamma$, such that*

$$\sup_{\substack{p(x) \leq \frac{1}{a\cdot x+1} \\ x\in\Gamma^*}} \int p d\mu < \inf_{\substack{q(x) \geq \frac{1}{a\cdot x+1} \\ x\in\Gamma^*}} \int q d\mu,$$

where p, q are polynomials functions defined on Γ^*.
Moreover, the range of values of a above is an open, everywhere dense subset of $\mathrm{int}\,\Gamma$.

References

D. Kimsey, M.P., Multivariate indeterminateness: Marcel Riesz variational principle, arXiv:2307.16018

M. P., K. Schmüdgen, Multivariate determinateness. Indiana Univ. Math. J. 57 (2008), no. 6, 2931–2968.

K. Schmüdgen, *The moment problem*. Graduate Texts in Mathematics, 277. Springer, Cham, 2017. xii+535 pp.

Indeterminate probability distributions

Theorem. (Berg, 1988) *X random variable with normal density $\frac{1}{\sqrt{\pi}} e^{-x^2}$. Then*

X^{2n+1} is indeterminate for $n \geq 1$ integer;

$|X|^\alpha$ is indeterminate for $\alpha > 4$ and determinate for $0 < \alpha \leq 4$.

The later in both Stieltjes and Hamburger sense.

Log-normal distribution

Of density $d(x) = \frac{1}{\sqrt{2\pi}}\frac{1}{x}e^{-\frac{(\log x)^2}{2}}$ defined on the semi-axis is indeterminate as showed by Stieltjes.

Specifically, the densities

$$d(x)[1 + r\sin(2\pi\log x)],\ \ r \in [-1, 1]$$

have the same power moments.

References

T. S. Chihara, On determinate Hamburger moment problems. Pacific J. Math. 27 (1968), 475–484.

C. Berg, The cube of a normal distribution is indeterminate. Ann. Probab. 16 (1988), no. 2, 910–913.

J. Stoyanov, L. Tolmatz, Method for constructing Stieltjes classes for M-indeterminate probability distributions. Appl. Math. Comput. 165 (2005), no. 3, 669–685.