The Fantappiè transform

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Contents

Profit function

Let $n \geq 1$, denote $\mathbb{R}_+ = [0, \infty)$ and consider a positive measure μ supported by $\mathbb{R}_+^n$. The integral transform

$$\Pi(p, p_0) = \int_{\mathbb{R}_+^n} (p_0 - p \cdot x)_+ \mu(dx),$$

can be regarded as the profit function, with $p_0 \geq 0$ price per unit output and $p = (p_1, p_2, \ldots, p_n)$ are the prices of production factors of technologies characterized by a vector $x \in \mathbb{R}_+^n$ and distributed by μ.

Some basic questions

Is the distribution unique with a given profit function?

Characterization of all possible profit functions.

Can the distribution be reconstructed from the profit function?

Relation to Radon transform

Assume first $\mu(dx) = \xi(x)dx$, where ξ is a smooth function, rapidly decaying at infinity. Then

$$\frac{\partial \Pi(p, p_0)}{\partial p_0} = \int_{\{x \in \mathbb{R}^n_+;\ p \cdot x \leq p_0\}} \xi(x)dx,$$

and

$$\frac{\partial^2 \Pi(p, p_0)}{\partial p_0^2} = \int_{\mathbb{R}^n_+} \xi(x)\delta(p_0 - p \cdot x)dx.$$

Link to well studied transforms

$$\mathcal{L}(\mu)(p) = \int_0^\infty e^{-\tau} \frac{\partial^2 \Pi(p,\tau)}{\partial \tau^2} d\tau = \int_{\mathbb{R}_+^n} e^{-p\cdot x} \mu(dx),$$

and

$$\mathcal{F}(\mu)(p,p_0) = \int_0^\infty \frac{1}{\tau + p_0} \frac{\partial^2 \Pi(p,\tau)}{\partial \tau^2} d\tau = \int_{\mathbb{R}_+^n} \frac{1}{p\cdot x + p_0} \mu(dx).$$

Uniqueness

Follows from the uniqueness of Laplace transform: if the positive measure μ satisfies

$$\int_{\mathbb{R}^n_+} e^{-A|x|}\mu(dx) < \infty,$$

for some $A > 0$, then $\Pi(p, p_0),\ |p| > A$, determines μ.

The inversion is more challenging, because the observable data are only for $p \in \mathbb{R}^n_+$.

The Fantappiè transform

Let $\Gamma \subset \mathbb{R}^n$ be an acute, convex, solid cone and let
$\Gamma^* = \{y \in \mathbb{R}^n;\ x \cdot y \geq 0,\ x \in \Gamma\}$.
Let μ be a positive measure of finite mass and supported by Γ^*.
The *Fantappiè transform* of μ is the function

$$\Phi(p, p_0) = \int_{\Gamma^*} \frac{\mu(dx)}{p_0 + p \cdot x}, \quad p \in \Gamma, p_0 > 0.$$

Much aligned with the 1D Stieltjes transform.

Hand in hand with Laplace transform

$$\int_{\Gamma^*} \frac{\mu(dx)}{p_0 + p \cdot x} = \int_0^\infty e^{-p_0 \tau} \mathcal{L}(\mu)(\tau p) d\tau.$$

Suggesting the efficient inversion approach.

Analytic continuation

The Fantappiè integral transform extends analytically to the complex domain:

$$\Phi(u_0, u) \triangleq \int_{\Gamma^*} \frac{d\mu(x)}{u_0 + u \cdot x}, \quad \Re u \in \mathrm{int}\Gamma, \Re u_0 > 0,$$

retaining, by definition, homogeneity of degree -1 in the complex argument $(u_0, u) \in \mathbb{C} \times \mathbb{C}^d$. Let $\Omega = (0, \infty) \times \mathrm{int}\Gamma \subset \mathbb{R}^{d+1}$ be the interior of the domain of continuity of the real Fantappiè transform.

Herglotz-Nevanlinna type property

For **any** finite mass, positive measure μ supported on Γ^*,

$$\Re\Phi(u_0, u) = \int_{\Gamma^*} \frac{(\Re u_0 + (\Re u)\cdot x)d\mu(x)}{|u_0 + u\cdot x|^2} \geq 0,$$

and the tube domain $\Omega \times i\mathbb{R}^{n+1}$ is a homogeneous space, with a well known harmonic analysis.

Characterization of Fantappiè transforms

Theorem (Bernstein, Bochner, Gilbert, Henkin, Shananin):

Let $\Gamma \subset \mathbb{R}^n$ *be an acute, convex, solid cone and let* $F \in C^\infty(\mathrm{int}(\Gamma \times \mathbb{R}_+))$ *continuous on* $\Gamma \times \mathbb{R}_+$. *The function* F *is the Fantappiè transform of a positive measure supported by the dual cone* Γ^* *if and only if* $F(p, p_0)$ *is homogeneous of degree* -1 *and it is completely monotonic, that is*

$$(-1)^{|\alpha|} D^\alpha F \geq 0, \quad \text{in} \quad \mathrm{int}(\Gamma \times \mathbb{R}_+),$$

for all multi-indices $\alpha \in \mathbb{N}^{n+1}$.

Separate monotonicity suffices

A function $F(p_0, p_1, \ldots, p_n)$ is completely monotone in $\mathbb{R}_+^{n+1}$ if and only if its is completely monotone in each variable separately, and

$$(-1)^{|\alpha|}(D^\alpha F)(r(k)) \geq 0, \quad \alpha \in \mathbb{N}^{n+1},$$

where the sequence of points $r(k) \in \mathrm{int}\mathbb{R}_+^{n+1}$ satisfies

$$\lim_k \min_{j=0}^n |r(k)_j| = \infty.$$

Bonus: characterization of transforms of compactly supported measures, closed form of extremal rays,...

Inversion

As defined,

$$\mathcal{F}(\mu)(p, p_0) = \int_{\Gamma^*} \frac{\mu(dx)}{p_0 + p \cdot x},$$

or

$$\mathcal{L}(\mu)(p) = \int_{\Gamma^*} e^{-px} \mu(dx),$$

are known only for $p \in \Gamma, p_0 > 0$.
Real inversion, alternative to Stieltjes complex variable inversion was proposed by Widder (in $1D$):

$$\mu([0, x]) = \lim_{p \to \infty} \sum_{k=0}^{[xp]} \frac{(-p)^k}{k!} \mathcal{L}(\mu)^{(k)}(p).$$

A higher dimensional analog exists for tempered measures.

References

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Analytic duality in complex affine space

Given a measure μ with compact support on $\mathbb{C}^n$, its *Fantappiè transform* is

$$(\mathcal{F}\mu)(z) = \int \frac{d\mu(w)}{1 - w^* z},$$

defined and analytic for every $z \in \mathbb{C}$, such that $\langle z, \mathrm{supp}(\mu)\rangle = z \cdot \mathrm{supp}(\mu)^* \neq 1$.

Again, related to the Laplace-Fourier transform:

$$(\mathcal{F}\mu)(z) = \int_0^\infty (\int e^{tzw^*} d\mu(w)) e^{-t} dt.$$

Fanttapiè's credo

Let $K \subset \mathbb{C}$ be a compact set and $L : \mathcal{O}(K) \longrightarrow \mathbb{C}$ a linear, continuous functional. Denote

$$g(z) = L[\frac{1}{2\pi i}\frac{1}{z - \cdot}], \quad z \notin K,$$

the indicatrix of L, as an analytic function defined on $\mathbb{C} \setminus K$, normalized by $g(\infty) = 0$.

Let $f \in \mathcal{O}(K)$ and Γ a collection of closed curves in the domain of f, surrounding K. For every $z \in K$

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(w)dw}{w - z}.$$

The continuity of the functional L implies

$$L(f) = \frac{1}{2\pi i}\int_\Gamma f(w)L(\frac{1}{w-\cdot})dw = \int_\Gamma f(w)g(w)dw.$$

And vice-versa, any such holomorphic function g produces a continuous linear functional L, with the astonishing conclusion (Grothendieck-Köthe duality):

$$\mathcal{O}(K)' = \{g \in \mathcal{O}(\mathbb{C}\setminus K);\ g(\infty) = 0\}.$$

Martineau-Aizenberg duality theorem

For a bounded convex domain $\Omega \subset \mathbb{C}^n$, *the Fantappiè transform establishes a continuous bijection between the space of analytic germs on* $\overline{\Omega}$ *and the space of analytic functions on the "dual" domain* $\Omega^\circ = \{z \in \mathbb{C};\ \langle z, \Omega\rangle \neq 1.\}$*:*

Towards the proof:
Assume Ω_k is convex, relatively compact in B, so that $\overline{B} \subset \Omega_k^\circ$.
Let $\mathcal{H}(\Omega_k^\circ) = \mathcal{F}(L^2_a(\Omega_k))$, endowed with the norm which makes

$$\mathcal{F} : L^2_a(\Omega_k) \longrightarrow \mathcal{H}(\Omega_k^\circ)$$

unitary.

Assume that Ω_k is a decreasing sequence of domains,

$$\overline{\Omega} = \cap_{k=1}^{\infty} \Omega_k, \quad \Omega^\circ = \cup_{k=1}^{\infty} \Omega_k^\circ.$$

Then the non-degenerate pairing between $L_a^2(\Omega_k)$ and $\mathcal{H}(\Omega_k^\circ)$

$$(f, \mathcal{F}g) = \langle f, g \rangle_{2,\Omega}$$

carries over to a non-degenerate pairing between

$$\mathcal{O}(\overline{\Omega}) = \lim_{\rightarrow} L_a^2(\Omega_k)$$

and

$$\mathcal{O}(\Omega^\circ) = \lim_{\leftarrow} \mathcal{O}(\Omega_k^\circ).$$

On power series, carrying their domains of convergence each, the pairing becomes

$$(\sum_{\alpha} a_\alpha z^\alpha, \sum_{\beta} b_\beta z^\beta) = \sum_{\alpha} a_\alpha \overline{b_\alpha} \frac{\alpha!}{|\alpha|!}.$$

Define for later use, the sesquilinear form

$$Q(f, g) = (f, g) + f(0)\overline{g(0)} = \sum_{\alpha \neq 0} a_\alpha \overline{b_\alpha} \frac{\alpha!}{|\alpha|!} + 2a_0 \overline{b_0}.$$

Lost uniqueness

Let C be a big circle on the sphere ∂B and let σ be the normalized arc length measure on C. Then

$$\int_C \frac{\sigma(d\zeta)}{1-\zeta^* z} = 1, \quad z \in B.$$

Dual of Fantappiè transforms

A closer look at analytic functions in the ball with non-negative real part. Notation

$$\mathcal{O}^+(B) = \{f \in \mathcal{O}(B) : \quad \Re f \geq 0\},$$

$$\mathcal{M}^+(B) = \{f = 2\mathcal{F}\mu - \mu(S) + it; \quad \mu \geq 0 \text{ on } S, \ t \in \mathbb{R}\}$$

that is

$$f(z) = \int_S \frac{1 + w^* z}{1 - w^* z} d\mu(w) + it.$$

Clearly $\mathcal{M}^+(B) \subset \mathcal{O}^+(B)$ because $\Re \frac{1+w^*z}{1-w^*z} \geq 0$. Hence *all* positive measures μ are allowed!

Dual cones

Denote

$$A^\dagger = \{f : \Re Q(f,g) \geq 0, \ \forall g \in A\}.$$

Theorem (JMcCarthy-P)

$\mathcal{O}^+(B)^\dagger = \mathcal{M}^+(B)$ and

$\mathcal{M}^+(B)^\dagger = \mathcal{O}^+(B)$.

"Transfer function" realization

For every $f \in \mathcal{M}^+(B)$ there exists a commutative spherical isometry, i.e. a n-tuple of operators $T = (T_1, ..., T_n)$ acting on a Hilbert space H satisfying $T_1^* T_1 + ... + T_n^* T_n = I$ and a vector $\xi \in H$, such that

$$f(z) = \langle [2(I - z_1 T_1 - ... - z_n T_n)^{-1} - 1]\xi, \xi\rangle + it, \quad t \in \mathbb{R}.$$

And vice-versa.
Moreover, if $S = (S_1, \cdots, S_n)$ is a commutative n-tuple of operators satisfying the quantized unit ball inequality:

$$S_1^* S_1 + S_2^* S_2 + \cdots + S_n^* S_n \leq I$$

and $f \in \mathcal{M}^+(B)$, then von-Neumann inequality $\Re f(S) \geq 0$ holds true.

Drury-Arveson space

Also known as the *symmetric Fock space*:
$H^2_n(B)$ with reproducing kernel $\frac{1}{1-w^*z}$.

$$\mathrm{Mult}\ H^2_n(B) \subset H^\infty(B)$$

is the natural framework for bounded analytic interpolation in higher dimension.

Therorem. (Agler-McCarthy) $H^2_n(B)$ *is universal among all reproducing Hilbert spaces with the Nevanlinna-Pick property.*

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Multiplicative Theory

Markov's Problem

Moment problem for measures of the form $g(t)dt$ with a Lebesgue measurable function $g : [-1, 1] \longrightarrow [0, 1]$.

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$$\sum_{k=0}^{\infty} \frac{s_k(g)}{z^{k+1}} = -\int_{-1}^{1} \frac{g(t)dt}{t-z}, \quad |z| > 1.$$

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The key property of these rather special generating series was discovered by A. Markov, via the formal exponential transform:

$$\exp[-\sum_{k=0}^{\infty} \frac{s_k(g)}{z^{k+1}}] = 1 - \sum_{k=0}^{\infty} \frac{t_k(g)}{z^{k+1}}.$$

A. Markoff transform-Math. Ann. 1896

Nouvelles applications des fractions continues.

Par

ANDRÉ MARKOFF à St. Pétersbourg.

Dans ma thèse „Sur quelques applications des fractions continues algébriques", publiée en russe (année 1884), je m'occupais à la détermination des valeurs limites des certaines intégrales dépendantes d'une fonction $f(y)$, laquelle n'est assujettie qu'à des conditions suivantes:

1) $f(y) > 0$ pour $a < y < b$;

2) les intégrales

$$\int_a^b f(y)\,dy,\ \int_a^b y f(y)\,dy, \ldots, \int_a^b y^{i-1} f(y)\,dy$$

doivent avoir des valeurs données.

La question sur ces valeurs limites est soulevée par Tchebychef dans sa note*) „Sur les valeurs limites des intégrales."

Nous devons aussi à Tchebychef les inégalités importantes, qui demontrées et généralisées**) par moi sont la base des recherches sur les questions de ce genre.

Maintenant nous allons considérer les questions semblables aux précédentes, en remplaçant seulement l'inégalité

$$f(y) > 0$$

par les deux suivantes:

$$L > f(y) > 0.$$

*) Journal de Liouville, 2 série. XIX.

**) Mathematische Annalen, Band XXIV, p. 172. Voir aussi: C. Possé, Sur quelques applications des fractions continues algébriques, St. Pétersbourg, 1886.

The measure $g(t)dt$ is determined by finitely many of its moments if and only if there exists an integer d, such that

$$\det[t_{j+\ell}(g)]_{j,\ell=0}^{d} = 0,$$

in which case we already know that g is the sublevel set of a polynomial function, that is a **finite collection of intervals**.

The exponential transform

Théorème I.

Si la fonction $F(y)$ satisfait à la condition

$$L > F(y) > 0,$$

*les développements suivants**)

$$(8)\qquad e^{\frac{1}{L}\int_a^b \frac{F(y)\,dy}{z-y}} = \cfrac{1}{1-\cfrac{c_1}{z-a-\cfrac{c_2}{1-\cfrac{c_3}{z-a-\ddots}}}},$$

$$(9)\qquad e^{\frac{1}{L}\int_a^b \frac{F(y)\,dy}{z-y}} = 1+\cfrac{\gamma_1}{z-b+\cfrac{\gamma_2}{1+\cfrac{\gamma_3}{z-b+\cfrac{\gamma_4}{1+\ddots}}}},$$

$$(10)\qquad \frac{z-b}{z-a}\,e^{\frac{1}{L}\int_a^b \frac{F(y)\,dy}{z-y}} = 1-\cfrac{\partial_1}{z-a-\cfrac{\partial_2}{1-\cfrac{\partial_3}{z-a-\cfrac{\partial_4}{1-\ddots}}}},$$

---

*) Ce genre des fractions continues était employé déjà par Stieltjes dans ses „Recherches sur les fractions continues“.

Moreover, in this case the exponential transform is a rational function

$$\exp[-\sum_{k=0}^{\infty} \frac{s_k(g)}{z^{k+1}}] = \frac{Q(z)}{P(z)},$$

with $\deg Q + 1 = \deg P \leq d$. To determine the polynomials Q and P one only needs the truncated transform $\exp[\sum_{k=0}^{d} \frac{s_k(g)}{z^{k+1}}]$ and a well known *Padé approximation scheme*.

Verification

$$\exp \int_a^b \frac{dt}{t-z} = \exp \log \frac{b-z}{a-z} =$$

$$\frac{b-z}{a-z} = 1 + \frac{b-a}{a-z} = 1 + \int \frac{(b-a)\delta_a(t)}{t-z},$$

with $\Im z > 0$ (to begin with).

Recognition

Achyeser, N.; Krein, M.

Das Momentenproblem bei der zusẗzlichen Bedingung von A. Markoff. (German) Commun. Soc. Math. Kharkoff et Inst. Sci. Math. et Mécan., Univ. Kharkoff, IV. Sér. 12, 13-33 (1935).

Über eine Transformation der reellen Toeplitzschen Formen und das Momentenproblem in einem endlichen Intervalle. (German) Commun. Soc. Math. Kharkoff et Inst. Sci. Math. et Mécan., Univ. Kharkoff, IV. Sér. 11, 21-26 (1935).

Das Momentenproblem bei der zusätzlichen Bedingung von A. Markoff. (German) Communications Kharkoff (4) 12, 13-35 (1935).

Bemerkung zur Arbeit "Über Fouriersche Reihen beschränkter summierbarer Funktionen und ein neues Extremumproblem". (German) Communications Kharkoff (4) 12, 37-40 (1935).

Über Fouriersche Reihen beschränkter summierbarer Funktionen und ein neues Extremumproblem. II. (German) Commun. Soc. Math. Kharkoff et Inst. Sci. Math. et Mécan., Univ. Kharkoff, IV. Sér. 10, 3-32 (1934).

Über Fouriersche Reihen beschränkter summierbarer Funktionen und ein neues Extremumproblem. I. (German) Commun. Soc. Math. Kharkoff et Inst. Sci. Math. et Mecan., Univ. Kharkoff, IV. Sér. 9, 9-28 (1934).

More recent book

M. G. Krein and A. A. Nudelman.
The Markov moment problem and extremal problems. American Mathematical Society, Providence, R.I., 1977. Ideas and problems of P. L. Čebyšev and A. A. Markov and their further development, Translated from the Russian by D. Louvish, Translations of Mathematical Monographs, Vol. 50.

Abstract framework: can work with more general collections of functions (Chebyshev systems), or with trigonometric polynomials on a torus.

Shade degree reconstruction

Let μ denote a positive Borel measure on $\mathbb{R}^n$, rapidly decreasing at infinity.

Assume

$$(\int |p| d\mu = 0,\ p \in \mathbb{R}[x]) \Rightarrow (p = 0).$$

Then

$$\mathbb{R}[x] \subset L^1(\mu).$$

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Typical example: Lebesgue measure on a square, or a disk.

L-problem of moments

Fix a positive integer N and a positive L.

Problem. *Reconstruct, or approximate, a shade function* $g \in L^1(\mu),\ -L \le g \le L,\ \mu - a.e.$, *from a finite section of its power moments:*

$$s_\alpha(g) = \int x^\alpha g d\mu, \quad |\alpha| \le N.$$

Convexity in action

The collection of moments $s(g) = (s_\alpha(g))_{|\alpha| \leq N}$ fills, with varying g, a convex set K in a finite dimensional euclidean space V.

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Every linear functional Φ defined on V is given by a polynomial $p \in \mathbb{R}[x]$ of degree less than or equal to N. To be more precise, for $p(x) = \sum_{|\alpha|\leq N} p_\alpha x^\alpha$,

$$\Phi(s(g)) = \sum_\alpha p_\alpha s_\alpha(g) = \int pgd\mu.$$

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Whence

$$\Phi(s(g)) = \int pg d\mu \leq \|g\|_\infty \|p\|_1 \leq L\|p\|_1,$$

where the infinity norm is taken on the support of the measure μ and $\|\cdot\|_1 = \|\cdot\|_{1,\mu}$.

If the point $s(g)$ lies in the interior of the convex set K, or the above inequality is strict for at least one linear functional, then the original shade function g is not determined by its measurements $s(g)$.

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On the contrary, if

$$\int hgd\mu = L\|h\|_1, \quad \deg(h) \leq N,$$

then necessarily $g(x) = L\,\mathrm{sgn}(h)$.

Conclusion

Thus only **black and white "pictures"** (by ad-hoc convention L is black and $-L$ is white), delimited by a *single* algebraic equation $h(x) = 0$ are determined by the power moments of degree up to N. And vice-versa.

Minor renormalization: can start with shade functions $g \in L^1(\mu)$ subject to the bounds $0 \leq g \leq 1$. Then we infer that g is determined by its power moments $(s_\alpha(g))_{|\alpha| \leq N}$ if and only if $g = \chi_E$ is the characteristic function of the non-negativity set

$$E = \{x \in \mathbb{R}^n, \; h(x) \geq 0\}$$

associated to a polynomial h of degree at most N

Example

$$E_1 = \{(x, y) \in \mathbb{R}^2,\ x \geq 0, y \geq 0\},$$

cannot be defined by a single polynomial inequality, while the union of two opposed orthants

$$E_2 = \{(x, y) \in \mathbb{R}^2,\ xy \geq 0\}$$

does.

Excercise: prove it!

Non-trivial consequence

Returning to our orthant example, we can take μ to be Lebesgue area measure restricted to the unit disk.

For any $N \geq 1$, there exists a measurable function f, $0 \leq f \leq 1$, different than χ_{E_1}, such that

$$\int_{E_1} x^\alpha d\mu = \int f x^\alpha d\mu, \quad |\alpha| \leq N,$$

and then there are infinitely many such f's.
On the other hand, if for a measurable function $g, 0 \leq g \leq 1$, one has

$$\int_{E_2} x^\alpha d\mu = \int g x^\alpha d\mu, \quad |\alpha| \leq 2,$$

then, quite unexpectedly, $g = \chi_{E_2}$, μ-a.e. .

Matrix perturbation

Let A, B be self-adjoint, $d \times d$ complex matrices. Assume

$$B - A = \xi\langle\cdot, \xi\rangle = \xi \otimes \xi.$$

The min-max principle implies:

$$\lambda_1(A) \leq \lambda_1(B) \leq \lambda_2(A) \leq \lambda_2(B) \leq \ldots \leq \lambda_d(A) \leq \lambda_d(B).$$

Denote

$$g = \sum_{j=1}^{n} \chi_{[\lambda_j(A), \lambda_j(B)]}.$$

Perturbation determinant

Then

$$\det(B-z)(A-z)^{-1} = \prod_{j=1}^{d} \frac{\lambda_j(B)-z}{\lambda_j(A)-z} = \exp \int \frac{g(t)dt}{t-z}.$$

On the other hand

$$\det(B-z)(A-z)^{-1} = \det[I + (A-z)^{-1}\xi \otimes \xi] =$$

$$1 + \langle (A-z)^{-1}\xi, \xi \rangle = 1 + \int \frac{d\mu(t)}{t-z},$$

in view of the spectral theorem.

The phase shift

For any polynomial $p \in \mathbb{C}[X]$ one has

$$\mathrm{trace}[p(B) - p(A)] = \int p'(t)g(t)dt.$$

Note that $g(t)$ is any extremal solution to the L-problem of moments on the real line.

Geometric interlacing phenomenon: Rayleigh Theorem

The principal semi-axes of an ellipsoid in Euclidean space are interlaced to the principal semi-axes of its intersection with a hyperplane.

Multivariable exponential representation

Theorem (Budisič, P.) *Let $\Gamma \subset \mathbb{R}^n$ be a closed, solid and acute convex cone and let μ be a finite mass positive measure supported by the polar cone Γ^*. The Fantappiè transform of the measure μ admits the exponential representation:*

$$\int_{\Gamma^*} \frac{d\mu(x)}{z_0 + z \cdot x} = -\exp[iF(z_0, z)], \quad (z_0, z) \in T_\Omega,$$

where $\Omega = (0, \infty) \times \mathrm{int}\Gamma$. In its turn, the analytic function F admits the integral representation

$$F(\zeta) = iC + \int_{\mathbb{R}^{d+1}} H(\zeta, \sigma; \alpha)\phi(\sigma)d\sigma, \quad \zeta = (z_0, z) \in T_\Omega,$$

where $\phi : \mathbb{R}^{n+1} \longrightarrow [0, \pi]$ is a measurable function and $C \in \mathbb{R}$ is a real constant.

Hardy space over the tube domain

$T_\Omega = \mathbb{R}^{n+1} + i\Omega$, where $\Omega = \mathrm{int}\Gamma$.

The reproducing kernel of the Hardy space $H^2(T_\Omega)$, also known as *Szegö's kernel* is

$$S(z, w) = \frac{1}{(2\pi)^d} \int_{\Omega^*} e^{i(z-\overline{w})\cdot x} dx.$$

$$H(z, w; \alpha) = 2\frac{S(z, w)S(w, \alpha)}{S(z, \alpha)} - \frac{|S(w, \alpha)|^2}{S(\alpha, \alpha)},$$

the *Herglotz kernel* associated to the tube domain T_Ω, with $z, w \in T_\Omega$, $\alpha \in i\Omega$.

Example: the orthant

Let $\Omega = (0, \infty)^d$ be the open positive orthant in $\mathbb{R}^d$, self-dual in the sense $\Omega^* = \overline{\Omega}$, the closure of itself.
Szegö's kernel of the tube domain over Ω is

$$S(z, w) = \frac{1}{(2\pi)^d} \int_{\Omega^*} e^{i(z-\overline{w})\cdot u} du =$$

$$\frac{1}{(2\pi i)^d} \prod_{k=1}^{d} \frac{1}{\overline{w_k} - z_k}, \quad z, w \in T_\Omega = \mathbb{R}^d + i\Omega.$$

With the selection of the reference point $\alpha = (i, i, ..., i) \in i\Omega$, Herglotz kernel becomes

$$\begin{aligned} H(z, u; \alpha) &= 2\frac{S(z,u)S(u,\alpha)}{S(z,\alpha)} - \frac{|S(u,\alpha)|^2}{S(\alpha,\alpha)} \\ &= \frac{1}{(2\pi)^d}\left[2\prod_{k=1}^{d}\frac{1 - iz_k}{(u_k - z_k)(u_k + i)} - \prod_{k=1}^{d}\frac{2}{1 + u_k^2}\right] \\ &= 2\prod_{k=1}^{d}\frac{1}{2\pi i}\left(\frac{1}{u_k - z_k} - \frac{1}{u_k + i}\right) - \\ &\prod_{k=1}^{d}\frac{1}{2\pi i}\left(\frac{1}{u_k - i} - \frac{1}{u_k + i}\right). \end{aligned}$$

In particular, for $d = 1$ we recover the familiar Szegö kernel $S(z, w) = \frac{1}{2\pi i} \frac{1}{\overline{w} - z}$ of the upper half plane, and Herglotz kernel becomes

$$H(z, u; i) = \frac{1}{\pi i} \left(\frac{1}{u - z} - \frac{u}{u^2 + 1} \right).$$

Verification

Let $\Phi(z)$ be an analytic function, mapping the open upper half-plane into itself and satisfying $\lim_{s\to\infty}\Phi(is)=0$. Then $\ln\Phi(z)$ is well defined, with $\Im\ln\Phi(z)\in(0,\pi)$.

$$\Phi(z)=\exp[iF(z)],\quad \Im z>0,$$

where $F(z)$ is analytic, $\Re F(z)\in(0,\pi)$ and

$$F(z)=i\Im F(i)+\int_{\mathbb{R}}H(z,u:i)\phi(u)du,$$

and $\phi\in L^\infty(\mathbb{R}),\ 0\le\phi\le\pi$. In conclusion, we obtain the representation

$$\Phi(z)=e^{\Im F(i)}\exp\int_{\mathbb{R}}\left(\frac{1}{u-z}-\frac{u}{u^2+1}\right)\frac{\phi(u)}{\pi}du.$$

Dirichlet measure

Let Δ^n be the simplex $(x_0, x_1, \dots, x_n) \in \mathbb{R}_+^{n+1}$, $x_0 + x_1 + \dots + x_n = 1$. Dirichlet's measure with parameters $\tau_0, \tau_1, \dots, \tau_n > 0,\ \tau_0 + \tau_1 + \dots + \tau_n = 1$, supported by Δ^n is

$$\mu(dx) = \frac{x_0^{\tau_0 - 1} x_1^{\tau_1 - 1} \dots x_n^{\tau_n - 1}}{\Gamma(\tau_0)\Gamma(\tau_0)\dots\Gamma(\tau_n)} dx.$$

Then

$$\int_{\Delta^n} \frac{\mu(dx)}{1 - x_0 z_0 - x_1 z_1 - \dots - x_n z_n} = \prod_{j=0}^{n} (1 - z_j)^{-\tau_j}.$$

Markov-Krein correspondence

Better

$$\int_{\Delta^n} \frac{\mu(dx)}{1 - xz} = \exp \int \log \frac{1}{1 - xz} \sigma(dx), \;\; z \in i\mathbb{R}^{n+1},$$

where

$$\sigma(dx) = \sum_{j=0}^{n} \tau_j \delta_{e_j},$$

$$xz = x_0 z_0 + x_1 z_1 + \ldots + x_n z_n.$$

Functoriality

Theorem (Kerov-Tsilevich) *Let μ, σ be probability measures on $\mathbb{R}^{n+1}$ entering into Markov-Krein correspondence:*

$$\int \frac{\mu(dx)}{1-xz} = \exp \int \log \frac{1}{1-xz}\sigma(dx), \ z \in i\mathbb{R}^{n+1},$$

and let $f : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^k$ be an affine map. Then

$$\int \frac{f_*\mu(dx)}{1-xz} = \exp \int \log \frac{1}{1-xz} f_*\sigma(dx), \ z \in i\mathbb{R}^{n+1}.$$

Surjectivity

Fort every probability measure σ of compact support on $\mathbb{R}^{n+1}$ there exists a probability measure μ satisfying

$$\int \frac{\mu(dx)}{1 - xz} = \exp \int \log \frac{1}{1 - xz} \sigma(dx), \; z \in i\mathbb{R}^{n+1},$$

Relevant for Dirichlet processes and representation theory of the infinite symmetric group.

Probabilistic interpretation

Let $0 < x_0 < x_1 < \ldots < x_n$ be fixed. On the simplex Δ_n with coordinates $(p_0, p_1, \ldots, p_n)$ consider the Dirichlet distribution

$$\mu(dp) = \frac{p_0^{\tau_0 - 1} p_1^{\tau_1 - 1} \ldots p_n^{\tau_n - 1}}{\Gamma(\tau_0)\Gamma(\tau_0) \ldots \Gamma(\tau_n)} dx.$$

Let M a discrete probability measure with weight p_k supported by $x_k, 0 \leq k \leq n$, where $(p_0, p_1, \ldots, p_n)$ is distributed according to μ. The distribution of the mean value $X = \sum_{k=0}^n x_k p_k$ of the random measure M is the Markov-Krein transform of $\tau = \sum_{k=0}^n \tau_k \delta_{x_k}$.

Cifarelli and Regazzini formula

Let τ be a probability distribution on $\mathbb{R}$ satisfying

$$\int_{\mathbb{R}} \log(1+|x|)\tau(dx) < \infty.$$

Then the measure μ in the Krein-Markov transform has density:

$$\frac{\mu(dx)}{dx}(a) = \frac{\sin(\tau(a,\infty)\pi)}{\pi} \exp \int \log \frac{1}{|t-a|}\tau(dt).$$

Simplex in complex space

Let $a_0, a_1, \ldots, a_k \in \mathbb{C}^n$ and Δ_k the real simplex. The measure

$$\mu(f) = \int_{\Delta_k} f(\sum_0^k t_j a_j) dt_0 dt_1 \ldots dt_k, \quad f \in \mathcal{O}(\mathbb{C}^n),$$

defines the analytic functional

$$\mu_{a_0,a_1,\ldots,a_k} = \prod_{j=1}^k (j - \sum_{k=1}^n z_k \frac{\partial}{\partial z_k})\mu.$$

Then

$$\mu_{a_0,a_1,\ldots,a_k}(\frac{1}{1 - * \cdot z}) = \prod_{j=0}^k \frac{1}{1 - a_j \cdot z}.$$

1D splines

Consider $n \geq 3$ and $\nu = \sum_{j=1}^{n} \delta_{a_j}$, where $a_1 < a_2 < \ldots < a_n$. Then

$$\int_{\mathbb{R}} \frac{h(x)dx}{1 - xz} = \exp\Big(\int_{\mathbb{R}} \log \frac{1}{1 - uz} d\nu(u)\Big) = \prod_{j=1}^{n} \frac{1}{1 - a_j z}.$$

The weight is a spline distribution

$$h(x) = (n-1) \sum_{a_j > x} c_j (a_j - x)^{n-2},$$

where $c_j = \prod_{j \neq k} \frac{1}{a_k - a_j}$.

Cauchy's measure is left invariant

$$\frac{1}{\pi}\int_{\mathbb{R}} \frac{1}{1-xz}\frac{dx}{1+x^2} = \exp \frac{1}{\pi}\int_{\mathbb{R}} [\log \frac{1}{1-uz}]\frac{dx}{1+x^2}$$

assuming $\Im z > 0$.

The Gaussian

Let

$$D_{-1}(z) = e^{z^2/4} \int_z^\infty e^{-x^2/2} dx.$$

Then

$$\frac{d\tau(u)}{du} = \frac{1}{\sqrt{2pi}} \frac{1}{|D_{-1}(iu)|^2}$$

is the distribution of a probability measure.

Simplest continued fraction expansion

$$\int_{\mathbb{R}} \frac{d\tau(u)}{z-u} = \cfrac{1}{z - \cfrac{2}{z - \cfrac{3}{z - \ddots}}},$$

Markov-Krein transform

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\frac{e^{-x^2/2}}{z-x}dx = \exp\left(-\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\frac{\log(z-u)du}{|D_{-1}(iu)|^2}\right).$$

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Faraut, Jacques; Fourati, Faiza. Markov-Krein transform. Colloq. Math. 144 (2016), no. 1, 137–156.

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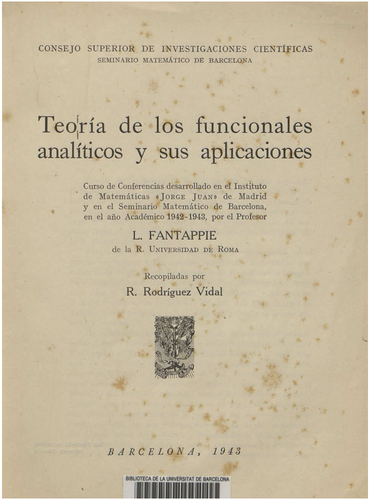
Fantappiè's book 1943



CONSEJO SUPERIOR DE INVESTIGACIONES CIENTÍFICAS
SEMINARIO MATEMÁTICO DE BARCELONA

Teoría de los funcionales analíticos y sus aplicaciones

Curso de Conferencias desarrollado en el Instituto de Matemáticas «JORGE JUAN» de Madrid y en el Seminario Matemático de Barcelona, en el año Académico 1942-1943, por el Profesor

L. FANTAPPIE
de la R. UNIVERSIDAD DE ROMA

Recopiladas por
R. Rodríguez Vidal

BARCELONA, 1943

Fantappie works

Para redactar el presente volumen hemos tenido en cuenta diversas observaciones que tuvo la amabilidad de hacernos particularmente el profesor Fantappié. Así, por ejemplo, algunos puntos (especialmente del cap. VIII de la primera parte y el II de la segunda) que convenía exponer con más desarrollo del que fué posible darles en el curso, han sido redactados a la vista de varias memorias del ilustre conferenciante, oportunamente citadas.

La primera exposición de los temas tratados en estas lecciones se encuentra fundamentalmente en:

- *I Funzionali Analitici.* Memorie della R. Accademia Nazionali dei Lincei, serie sesta, vol. III, fasc. II, 1930. (F. A.)
- *I Funzionali delle funzioni di due variabili.* Mem. de la R. Acc. d'Italia, vol. II, 1931. (D. V.)
- *Nuovi Fondamenti della teoría dei funzionali analitici,* Atti della R. Acc., vol. XII, 1941. (N. F.)
- *Integrazione con quadrature dei sistemi a derivate parziali lineari.* Rend. del Circ. Mat. di Palermo. Tomo LVII, 1933.
- *Integrazioni in termini finiti di ogni sistema di equazioni a derivate parziali, lineare e a coeficiente constanti, d'ordine qualunque.* Reale Acc. d'Italia. Memorie della classe di scienze fisiche..., vol. VIII, 1937.
- *Risoluzioni in termini finiti del problema di Cauchy, con dati iniziale su una ipersuperficie qualunque.* Memorie della R. Acc. d'Italia, fasc. 12, serie VII, vol. II, 1941.

que deberá consultar quien desee profundizar en la teoría. Las iniciales del tipo F. A. son usadas a veces en el texto para abreviar las citas.

El modesto trabajo de esta redacción no hubiese sido posible sin el apoyo del Consejo Superior de Investigaciones Científicas, y muy especialmente del Profesor don José M.ª Orts, Director del Seminario Matemático

The projective indicatrix

es definida y ultrarregular fuera del *hiperplano* I $(\alpha_1, \ldots, \alpha_n)$ de ecuación

$$1 + \alpha_1 t_1 + \ldots + \alpha_n t_n = 0 \qquad (37,2)$$

variedad de forma lo más simple posible de puntos singulares (todos polos de primer orden).

Aplicando el funcional *analítico lineal* F a las funciones de la variedad (37,1) que penetre en su campo de definición, éste se reduce a una función analítica y regular

$$F\left[\frac{1}{1 + \alpha_1 t_1 + \ldots + \alpha_n t_n}\right] = p(\alpha_1, \ldots, \alpha_n) \qquad (37,3)$$

a la cual llamaremos *indicatriz proyectiva* del referido funcional F.

Para $n = 1$ se tiene

Further reading

what is "syntropy"?
https://frontiersmagazine.org/luigi-fantappie-and-the-physics-of-life/

Luigi Fantappiè: mathematical analysis, education, and fascism in Brazil (1934–1939)
https://link.springer.com/article/10.1007/s40329-018-0235-3

Revista Brasileira de Historia da Ciencia, Rio de Janeiro, v. 10, n. 2, p. 222-232/ 2017

Luigi Fantappiè and his students at the FFCL: autonomy and professionalization of mathematics in São Paulo

The summary of his activities is here analyzed in comparison with interviews given by his students years after the establishment of the FFCL, and with official documents from the University of Sao Paulo in the 1930s. The general picture gained from this analysis is that the professionalization of mathematics and its autonomy from engineering was not achieved in an abrupt way, when the University was established, but was the result of a long and conflicting process of which Fantappiè was a key figure.