Fourier uncertainty and bandlimited functions

Emanuel Carneiro

ICTP - Trieste

Analysis Seminar - IIS Bangalore
Sep 2024

Our operator

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$$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x\xi}\, f(x)\,\mathrm{d}x.$$

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Fourier uncertainty: "one cannot have an unrestricted control of a function and its Fourier transform simultaneously."


EVERY SAGA HAS A BEGINNING
STAR WARS
EPISODE I
THE PHANTOM MENACE
STAR WARS
EPISODE II
ATTACK OF THE CLONES
STAR WARS
EPISODE III
REVENGE OF THE SITH
A long time ago in a galaxy far, far away...
STAR WARS
THE STAR WARS SAGA CONTINUES
THE EMPIRE STRIKES BACK
MARK HAMILL · HARRISON FORD · CARRIE FISHER
BILLY DEE WILLIAMS · ANTHONY DANIELS
STAR WARS
RETURN OF THE JEDI
STAR WARS
THE FORCE AWAKENS
DECEMBER 18
STAR WARS
THE LAST JEDI
DECEMBER 15

Alternative titles

1. Why should you care about the function

$$\frac{(x^2-1)\coth\left(\frac{\pi\sqrt{3}}{2}\right)\cos\pi x-\sqrt{3}\,x\sin\pi x}{x^6-1}\ ?$$

Alternative titles

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2. The moving company project

This is a story about many mathematical and non-mathematical values that I particularly like...

Part I: Prelude

Prelude

Banff, Canada, 2015

The Geometry, Algebra and Analysis of Algebraic Numbers

Our basic principle

Theorem (Paley-Wiener)

For $f \in L^2(\mathbb{R})$, the following are equivalent:

(i) $\mathrm{supp}(\widehat{f}) \subset [-\Delta, \Delta]$.

(ii) *f can be extended to an entire function of order 1 and*

$$|f(z)| \leq C_\varepsilon e^{(2\pi\Delta+\varepsilon)|z|} \quad \text{for all } \varepsilon > 0.$$

Raymond Paley (1907 - 1933)

Norbert Wiener (1894 - 1964)

Monotone extremal functions

There I gave a talk about a work with F. Littmann.

Theorem

Let $F : \mathbb{C} \to \mathbb{C}$ be a real entire function such that:

(i) *F has exponential type at most 2π;*

(ii) *$F(x) \geq \operatorname{sgn}(x)$ for all $x \in \mathbb{R}$;*

(iii) *F is non-decreasing on $(-\infty, 0)$ and non-increasing on $(0, \infty)$.*

Then

$$\int_{-\infty}^{\infty} \{F(x) - \operatorname{sgn}(x)\}\, \mathrm{d}x \geq 2.$$

The unique extremizer is:

$$M(x) = -2 \int_{-\infty}^{x} \frac{\sin^2 \pi s}{\pi^2 s\,(s+1)^2}\, \mathrm{d}s - 1.$$

Hilbert's inequality

Theorem (around 1908)

If $a_1, \ldots, a_N \in \mathbb{C}$ *then*

$$\left| \sum_{m \neq n} \frac{a_m \, \overline{a_n}}{m - n} \right| \leq \pi \sum_n |a_n|^2$$

Weighted Hilbert's inequality

Theorem (Montgomery and Vaughan - 1974)

Let $N \in \mathbb{N}$. Let $\lambda_1, \ldots, \lambda_N$ be a set of distinct real numbers and define $\delta_n := \min\{|\lambda_n - \lambda_m| : m \neq n\}$. If $a_1, \ldots, a_N \in \mathbb{C}$ then

$$\left| \sum_{m \neq n} \frac{a_m \, \overline{a_n}}{(\lambda_m - \lambda_n)} \right| \leq C \sum_n \frac{|a_n|^2}{\delta_n}.$$

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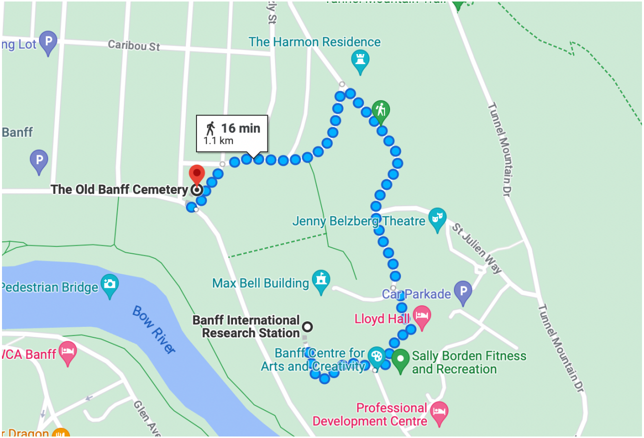
- Preissmann (1984): $C = (1.31...)\pi$.
- Our proof is for $C = 2\pi$ (via Fourier analysis).
- Conjecture $C = \pi$.

A nice memory

A stroll to the cemetery



A nice memory

A stroll in the cemetery

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A stroll in the cemetery

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A nice memory

A stroll in the cemetery

- We discussed the following "monotone one-delta problem":

  Find $F : \mathbb{R}^d \to \mathbb{R}$, radial and non-negative, of exponential type 2π, such that $F(0) \geq 1$ and $\int_{\mathbb{R}^d} F(\mathbf{x})\, \mathrm{d}\mathbf{x}$ is minimal.

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- This boils down to the following problem (in dimension 1): given $g : \mathbb{C} \to \mathbb{C}$ of exponential type π such that

$$\int_{\mathbb{R}} |g(x)|^2 |x| \, \mathrm{d}x = 1,$$

  find the minimal value of

$$\int_{\mathbb{R}} |g(x)|^2 |x|^{d+1} \, \mathrm{d}x.$$

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- At the moment I could only solve this for $d = 2$.

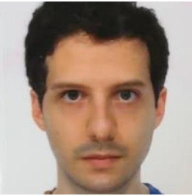
Part II: Seven years later...

Fast forward to 2022 at ICTP

Our analysis group became bigger

Cristian G.-Riquelme

Andrea Olivo



Antonio Pedro Ramos

Sheldy Ombrosi

Lucas Oliveira

Mateus Sousa

A toy model problem

- For $f \in L^2(\mathbb{R})$ with $\mathrm{supp}(\widehat{f}) \subset [-\frac{1}{2}, \frac{1}{2}]$, find the sharp inequality:

$$\int_{\mathbb{R}} |f(x)|^2 \, \mathrm{d}x \leq C \int_{\mathbb{R}} |f(x)|^2 \, x^2 \, \mathrm{d}x.$$

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- <u>Solution:</u>

$$\begin{aligned} \|f\|_2^2 = \sum_{n\in\mathbb{Z}} \left|f(n+\tfrac{1}{2})\right|^2 &= 4 \sum_{n\in\mathbb{Z}} \frac{1}{4} \left|f(n+\tfrac{1}{2})\right|^2 \\ &\leq 4 \sum_{n\in\mathbb{Z}} (n+\tfrac{1}{2})^2 \left|f(n+\tfrac{1}{2})\right|^2 \\ &= 4\|x\, f\|_2^2. \end{aligned}$$

A toy model problem

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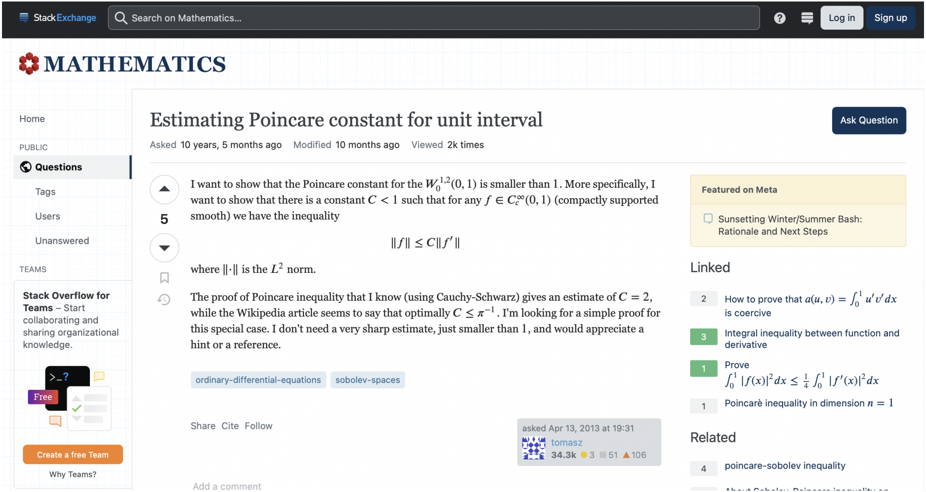
- Solution:

$$\begin{aligned} \|f\|_2^2 = \sum_{n\in\mathbb{Z}} \left|f(n+\tfrac{1}{2})\right|^2 &= 4\sum_{n\in\mathbb{Z}} \frac{1}{4}\left|f(n+\tfrac{1}{2})\right|^2 \\ &\leq 4\sum_{n\in\mathbb{Z}} (n+\tfrac{1}{2})^2 \left|f(n+\tfrac{1}{2})\right|^2 \\ &= 4\|x\,f\|_2^2. \end{aligned}$$

Recall $f(x) = \sum_{n\in\mathbb{Z}} \frac{\sin \pi(x-n-\frac{1}{2})}{\pi(x-n-\frac{1}{2})} \, f(n+\frac{1}{2})$. Equality if and only if

$$f(x) = c\,\frac{\cos \pi x}{x^2 - \frac{1}{4}}.$$

Poincaré inequalities



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Estimating Poincare constant for unit interval

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Asked 10 years, 5 months ago Modified 10 months ago Viewed 2k times

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I want to show that the Poincare constant for the $W_0^{1,2}(0,1)$ is smaller than 1. More specifically, I want to show that there is a constant $C < 1$ such that for any $f \in C_c^\infty(0,1)$ (compactly supported smooth) we have the inequality

$$\|f\| \le C\|f'\|$$

where $\|\cdot\|$ is the L^2 norm.

The proof of Poincare inequality that I know (using Cauchy-Schwarz) gives an estimate of $C = 2$, while the Wikipedia article seems to say that optimally $C \le \pi^{-1}$. I'm looking for a simple proof for this special case. I don't need a very sharp estimate, just smaller than 1, and would appreciate a hint or a reference.

ordinary-differential-equations sobolev-spaces

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asked Apr 13, 2013 at 19:31
tomasz
34.3k ● 3 ■ 51 ▲ 106

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1 Poincarè inequality in dimension $n = 1$

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The constant you are looking for is the following:

$$\frac{1}{C^2} = \inf\left\{ \int_0^1 (f')^2\,dx \ :\ \int_0^1 (f)^2\,dx = 1 \right\}. \qquad (1)$$

Since

$$\int_0^1 (f')^2\,dx = \langle -f'', f\rangle,$$

you are in fact looking for the first eigenvalue of the following Sturm-Liouville problem:

$$\begin{cases} -\frac{d^2 f}{dx^2} = \lambda f, \\ f(0) = f(1) = 0. \end{cases} \qquad (2)$$

Indeed, we can now rewrite the minimization in (1) as

$$\inf_{f\neq 0} \frac{\langle -f'', f\rangle}{\langle f, f\rangle},$$

and this equals exactly the smallest eigenvalue of the problem (2), just like in ordinary linear algebra; see Wikipedia on the Rayleigh quotient.

The problem (2) can be integrated explicitly and you find that the first eigenvalue is π^2 with eigenfunction $\sin(\pi x)$ (and scalar multiples of it). Therefore

$$C = \frac{1}{\pi} < 1.$$

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edited Nov 4, 2022 at 13:18

answered Apr 13, 2013 at 19:49

Giuseppe Negro

31.4k ● 6 ■ 64 ▲ 219

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2 Proving Poincare in One Dimension

3 Bound gradient in $H_0^2(\Omega)$ by Laplacian

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Indium and Gallium Toxicity: Part 1

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One may ask...

- For $f \in L^2(\mathbb{R})$ with $\mathrm{supp}(\widehat{f}) \subset [-\frac{1}{2}, \frac{1}{2}]$, find the sharp inequalities:

$$\int_{\mathbb{R}} |f(x)|^2 \, \mathrm{d}x \leq C \int_{\mathbb{R}} |f(x)|^2 \, x^4 \, \mathrm{d}x.$$

$$\int_{\mathbb{R}} |f(x)|^2 \, \mathrm{d}x \leq C \int_{\mathbb{R}} |f(x)|^2 \, x^6 \, \mathrm{d}x.$$

$$\int_{\mathbb{R}} |f(x)|^2 \, \mathrm{d}x \leq C \int_{\mathbb{R}} |f(x)|^2 \, x^8 \, \mathrm{d}x.$$

and so on...

A sharp uncertainty principle

For example, for $f \in L^2(\mathbb{R})$ with $\mathrm{supp}(\widehat{f}) \subset [-\frac{1}{2}, \frac{1}{2}]$,

$$\int_{\mathbb{R}} |f(x)|^2 \,\mathrm{d}x \leq \int_{\mathbb{R}} |f(x)|^2 \, x^6 \,\mathrm{d}x.$$

A sharp uncertainty principle

Theorem

For $f \in L^2(\mathbb{R})$ with $\operatorname{supp}(\widehat{f}) \subset [-\frac{1}{2}, \frac{1}{2}]$, we have

$$\int_{\mathbb{R}} |f(x)|^2 \, \mathrm{d}x \leq \int_{\mathbb{R}} |f(x)|^2 \, x^6 \, \mathrm{d}x.$$

This inequality is sharp and the unique extremizer is

$$f(x) = \frac{(x^2 - 1) \coth\left(\frac{\pi\sqrt{3}}{2}\right) \cos \pi x - \sqrt{3}\, x \sin \pi x}{x^6 - 1}.$$

Moving

Moving

Setup

- For each $\alpha > -1$ and $\delta > 0$, let $\mathcal{H}_\alpha(d\,;\delta)$ be the Hilbert space of entire functions $F : \mathbb{C}^d \to \mathbb{C}$ of exponential type at most δ with

$$\|F\|_{\mathcal{H}_\alpha(d;\delta)} := \left(\int_{\mathbb{R}^d} |F(\mathbf{x})|^2 \, |\mathbf{x}|^{2\alpha+2-d} \, \mathrm{d}\mathbf{x} \right)^{1/2} < \infty.$$

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- For $\alpha \geq \beta > -1$ note that $\mathcal{H}_\alpha(d\,;\delta) \subset \mathcal{H}_\beta(d\,;\delta)$.

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- For $\alpha \geq \beta > -1$ note that $\mathcal{H}_\alpha(d\,;\delta) \subset \mathcal{H}_\beta(d\,;\delta)$.

- *Extremal Problem* (EP): For $\alpha \geq \beta > -1$ and $\delta > 0$ real parameters, and $d \in \mathbb{N}$, find the value of

$$(\mathbb{EP})(\alpha,\beta\,;d\,;\delta) := \inf_{0 \neq F \in \mathcal{H}_\alpha(d;\delta)} \frac{\int_{\mathbb{R}^d} |F(\mathbf{x})|^2\, |\mathbf{x}|^{2\alpha+2-d}\,\mathrm{d}\mathbf{x}}{\int_{\mathbb{R}^d} |F(\mathbf{x})|^2\, |\mathbf{x}|^{2\beta+2-d}\,\mathrm{d}\mathbf{x}}.$$

Part I - Qualitative properties

A change of variables yields:

$$(\mathbb{EP})(\alpha, \beta\,;d\,;\delta) = \delta^{2\beta-2\alpha}\,(\mathbb{EP})(\alpha, \beta\,;d\,;1).$$

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- Proof involves a suitable radial symmetrization mechanism and an auxiliary extremal problem.

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Theorem (Continuity)

The function $(\alpha, \beta, \delta) \mapsto (\mathbb{EP})(\alpha, \beta\, ; d\, ; \delta)$ *is continuous in the range* $\alpha \geq \beta > -1$ *and* $\delta > 0$.

Part II - Asymptotics

Recall we are looking at:

$$(\mathbb{EP})(\alpha, \beta\,; 1\,; 1) := \inf_{0 \neq f \in \mathcal{H}_\alpha(1;1)} \frac{\int_{\mathbb{R}} |f(x)|^2 \, |x|^{2\alpha+1} \, \mathrm{d}x}{\int_{\mathbb{R}} |f(x)|^2 \, |x|^{2\beta+1} \, \mathrm{d}x}.$$

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Theorem (Asymptotics)

For $\alpha \geq \beta > -1$ we have

$$\log\left((\mathbb{EP})(\alpha, \beta\,; 1\,; 1)\right) = 2(\alpha-\beta)\log(\alpha+2) + \log\left(\frac{\beta+1}{\alpha+1}\right) + O\left(\left(\frac{(\alpha-\beta)(\alpha+2)}{(\alpha+1)}\right)\log\left(\frac{2(\alpha+1)(\alpha-\beta+1)}{(\alpha-\beta)(\alpha+2)}\right)\right),$$

where the implied constant is universal.

Part III - Sharp constants

Let $A_\nu(z) = z^{-\nu} J_\nu(z)$ and $B_\nu(z) = z^{-\nu} J_{\nu+1}(z)$. Set

$$C_\nu(z) = \frac{B_\nu(z)}{A_\nu(z)} = \frac{J_{\nu+1}(z)}{J_\nu(z)}.$$

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Theorem

Let $\beta > -1$, let $k \in \mathbb{N}$ and set $\lambda_0 := \big((\mathbb{EP})(\beta + k, \beta\,; 1\,; 1)\big)^{1/2k}$.

(i) *If $k = 1$ we have $\lambda_0 = \mathrm{j}_{\beta,1}$.*

(ii) *If $k \geq 2$, set $\ell := \lfloor k/2 \rfloor$. Then λ_0 is the smallest positive solution of*

$$A_\beta(\lambda) \det \mathcal{V}_\beta(\lambda) = 0\,,$$

where $\mathcal{V}_\beta(\lambda)$ is the $\ell \times \ell$ matrix with entries (set $\omega := e^{\pi i/k}$)

$$\big(\mathcal{V}_\beta(\lambda)\big)_{mj} = \sum_{r=0}^{k-1} \omega^{r(4\ell-2m-2j+3)}\, C_\beta\big(\omega^r \lambda\big).$$

- For instance, when $k = 3$ we have

$$\lambda \mapsto A_\beta(\lambda)\big(C_\beta(\lambda) - C_\beta(\omega\lambda) + C_\beta(\omega^2\lambda)\big).$$

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- When $\beta = -\frac{1}{2}$, this reduces to

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  and one can see that the smallest root is π.
- This leads to

$$\int_{\mathbb{R}} |f(x)|^2\,\mathrm{d}x \le \int_{\mathbb{R}} |f(x)|^2\,x^6\,\mathrm{d}x$$

  when $\mathrm{supp}(\widehat{f}) \subset [-\frac{1}{2}, \frac{1}{2}]$.

Application I - Sharp Poincaré inequalities

Corollary

$$\int_I |f^{(n)}(x)|^2 \,\mathrm{d}x \leq C \int_I |f^{(m)}(x)|^2 \,\mathrm{d}x$$

for $f \in H_0^{(n+k)}(I))$

Application I - Sharp Poincaré inequalities

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- Steklov (1896): $(m, n) = (1, 0), (2, 1)$
- Janet (1931): $(n + 1, n)$, $n \geq 2$.
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$$\int_{B_r} \left|\nabla^{n_1}\left(\Delta^n g\right)(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x} \leq C \int_{B_r} \left|\nabla^{m_1}\left(\Delta^m g\right)(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x}$$

for $g \in W_0^{2m+m_1,2}(B_r)$

Poincaré inequalities vs. Fourier uncertainty

Application II - Monotone one delta problem (even d)

<u>Problem:</u> Find $F : \mathbb{R}^d \to \mathbb{R}$, radial and non-negative, of exponential type 2π, such that $F(0) \geq 1$ and $\int_{\mathbb{R}^d} F(\mathbf{x})\, \mathrm{d}\mathbf{x}$ is minimal.

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Remark: The case $d = 1$ was numerically studied by A. Chirre, D. Dimitrov, E. Quesada-Herrera and M. Sousa (PAMS '24). They arrived a very precise approximation of the answer (1.27...).

A generalization (de Branges spaces)

- Let $E : \mathbb{C} \to \mathbb{C}$ be a Hermite-Biehler function, i.e. $|E^*(z)| < |E(z)|$ for $z \in \mathcal{U}$ (here $E^*(z) := \overline{E(\overline{z})}$).

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- Let $\mathcal{H}(E)$ be the space of entire functions F such that

$$\|F\|^2_{\mathcal{H}(E)} := \int_{\mathbb{R}} |F(x)|^2 |E(x)|^{-2}\,\mathrm{d}x < \infty$$

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- The problem is

$$(\mathbb{EP}2)(E;k) := \inf_{0 \neq f \in \mathcal{H}(E)} \frac{\|z^k F\|^2_{\mathcal{H}(E)}}{\|F\|^2_{\mathcal{H}(E)}}.$$

Write $E = A - iB$ with A and B real entire.

Glimpse of the strategy

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- If f is even (set $\xi_n = \pi(n - \frac{1}{2})$).

$$f(z) = \sum_{n=1}^{\infty} 2\,\xi_n f(\xi_n) \frac{\cos z}{(z^2 - \xi_n^2)}.$$

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- Hence one arrives at the following problem. Find

$$\lambda_0^{2k} = \inf_{\{a_n\} \in \mathcal{A}} \frac{\sum_{n=1}^{\infty} a_n^2\, \xi_n^{2k}}{\sum_{n=1}^{\infty} a_n^2}.$$

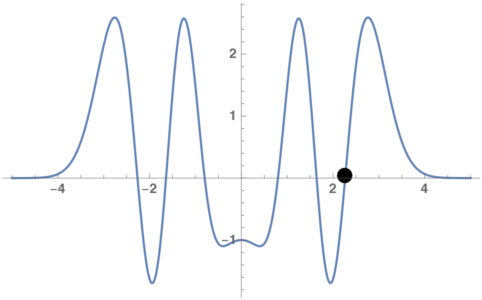
Part III: Two more years later...

Sign uncertainty principle

Bourgain, Kahane and Clozel, 2010

- A continuous $f : \mathbb{R}^d \to \mathbb{R}$ is eventually non-negative if $f(x) \geq 0$ for sufficiently large $|x|$, and we define

$$r(f) := \inf\{r > 0 \ : \ f(x) \geq 0 \text{ for all } |x| \geq r\}.$$



$$f(x) = (x^{10} - 8x^8 + 15x^6 - x^4 - 2x^2 - 1)e^{-x^2}$$

Sign uncertainty principle

Bourgain, Kahane and Clozel, 2010

- Consider the family:

$$\mathcal{A}(d) = \left\{ \begin{array}{l} f \in L^1(\mathbb{R}^d) \setminus \{0\} \text{ continuous, even, real-valued; } \widehat{f} \in L^1(\mathbb{R}^d); \\ f(0) = \int_{\mathbb{R}^d} \widehat{f} \leq 0 \ \ ; \ \ \widehat{f}(0) = \int_{\mathbb{R}^d} f \leq 0; \\ f \text{ and } \widehat{f} \text{ are eventually non-negative.} \end{array} \right\}.$$

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- Define

$$\mathbb{A}(d) := \inf_{f \in \mathcal{A}(d)} \sqrt{r(f)\, r(\widehat{f})}.$$

(note that this is invariant under dilations $f_\delta(x) := f(\delta x)$).

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- They show:

$$\sqrt{\frac{d+2}{2\pi}} \geq \mathbb{A}(d) \geq \sqrt{\frac{d}{2\pi e}}.$$

A related problem

Given a locally finite, even and non-negative Borel measure μ on $\mathbb{R}$, $\delta > 0$, find

$$\mathbb{E}(\mu;\delta) := \inf_{0 \neq F \in \mathcal{E}(\mu;\delta)} r(F)\,,$$

where the infimum is taken over the class of functions

$$\mathcal{E}(\mu;\delta) := \left\{ \begin{array}{l} F \text{ real entire of exp. type at most } \delta; \\ \\ F \in L^1(\mathbb{R},\mu) \ \text{ and } \ \int_{\mathbb{R}} F(x)\,\mathrm{d}\mu(x) \leq 0; \\ \\ F \text{ is eventually non-negative.} \end{array} \right\}.$$

Zeros of L-functions

- Katz and Sarnak conjectured that for each family $\{L(s,f),\ f \in \mathcal{F}\}$

$$\lim_{Q\to\infty} \frac{1}{|\mathcal{F}(Q)|} \sum_{f\in\mathcal{F}(Q)} \sum_{\gamma_f} \phi\left(\gamma_f \frac{\log c_f}{2\pi}\right) = \int_{\mathbb{R}} \phi(x)\, W(x)\, \mathrm{d}x,$$

for $\phi : \mathbb{R} \to \mathbb{R}$ is an even, Schwartz with $\widehat{\phi}$ compactly supported.

$$W(x) \;=\; 1 \pm a\,\frac{\sin 2\pi x}{2\pi x} + b\,\delta(x).$$

- To find bounds for γ_f: let
  - $\phi \geq 0$ for $|x| \geq r$
  - $\int_{\mathbb{R}} \phi(x)\, W(x)\, \mathrm{d}x < 0$.

A glimpse on our results

Theorem (with A. P. Ramos and T. Ismoilov)

If $\mathrm{d}\mu(x) = |E(x)|^{-2}\,\mathrm{d}x$ *(in an integral sense), letting* ξ_1 *be the first positive zero of* $A(z) = E(z) + \overline{E(\overline{z})}$*,*

$$\mathbb{E}(\mu;\delta) = \xi_1 .$$

Unique extremizers are

$$F(z) = \frac{A(z)^2}{(z^2 - \xi_1^2)}.$$

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3. $$\int F\,\mathrm{d}\mu \leq 0 \iff r^2 \geq \frac{\int x^2\,|U(x)|^2\,\mathrm{d}\mu}{\int |U(x)|^2\,\mathrm{d}\mu} .$$

Many thanks!

Zeros of L-functions

- Let $\mathcal{F}$ be a set of number theoretical objects. For $f \in \mathcal{F}$ associate

$$L(s, f) = \sum_{n=1}^{\infty} \lambda_f(n)\, n^{-s}\,,$$

Conductor c_f and assume GRH. Denote the non-trivial zeros by $\rho_f = \frac{1}{2} + i\gamma_f$. Let $\mathcal{F}(Q) := \{f \in \mathcal{F} \,:\, c_f = Q\}$.

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For the five symmetry groups, Katz and Sarnak determined the density functions:

$$\begin{aligned} W_{\mathrm{U}}(x) &= 1\,; \\ W_{\mathrm{Sp}}(x) &= 1 - \frac{\sin 2\pi x}{2\pi x}\,; \\ W_{\mathrm{O}}(x) &= 1 + \tfrac{1}{2}\delta(x)\,; \\ W_{\mathrm{SO(even)}}(x) &= 1 + \frac{\sin 2\pi x}{2\pi x}\,; \\ W_{\mathrm{SO(odd)}}(x) &= 1 - \frac{\sin 2\pi x}{2\pi x} + \delta(x). \end{aligned}$$

Analysis question

- Assume the validity of

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for even Schwartz functions $\phi : \mathbb{R} \to \mathbb{R}$ with $\mathrm{supp}\left(\widehat{\phi}\right) \subset [-\Delta, \Delta]$, with $\Delta > 0$ fixed; what is the sharpest upper bound that one can get for the height of the first zero in the family as $Q \to \infty$?

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- Note that the blue condition is equivalent to

$$\frac{\int_{\mathbb{R}} x^2 |g(x)|^2\, W_G(x)\, \mathrm{d}x}{\int_{\mathbb{R}} |g(x)|^2\, W_G(x)\, \mathrm{d}x} < a^2.$$

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$$= \sum_{m,n=1}^{N} a_m \bar{a}_n \widehat{\psi}_{\delta_{\min(m,n)}}(\lambda_m - \lambda_n) = -\sum_{m,n=1}^{N} \frac{a_m \bar{a}_n}{\pi i(\lambda_m - \lambda_n)} + \widehat{\psi}(0) \sum_{n=1}^{N} \frac{|a_n|^2}{\delta_n}.$$