DIRECTIONAL SINGULAR INTEGRALS IN CODIMENSION ONE

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&
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Report on joint work with

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APRG seminar
Indian Institute of Science
Bangalore, India

25/09/2024

I. DIRECTIONAL AVERAGES AND DIRECTIONAL SINGULAR INTEGRALS

▶ **averages** Fix $V \subseteq \mathbb{S}^{n-1}$ and a scale $r > 0$:

$$A_{v,r} f(x) := \frac{1}{2r} \int_{-r}^{r} f(x + vt)\, dt, \quad v \in V, \quad x \in \mathbb{R}^n$$

$$M_{V,r} f(x) := \sup_{v \in V} |A_{v,r} f(x)|, \qquad M_V f(x) := \sup_{r>0} M_{V,r} f(x)$$

NORM ESTIMATE IS SCALE INVARIANT; TAKE $r = 1$.

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- **singular integrals** The prototypical example

$$H_v f(x) := \text{p.v.} \int_{\mathbb{R}} f(x+vt) \frac{dt}{t} \simeq \int_{\mathbb{R}^n} \hat{f}(\xi)\, \operatorname{sgn}(\xi\cdot v)\, e^{2\pi i \xi\cdot x}\, d\xi$$

up to a linear combination with the identity.

$$\sim \int_{\mathbb{R}^n} \hat{f}(\xi)\, \mathbb{1}_{[0,\infty)}(\xi\cdot v)\, e^{2\pi i \xi \cdot x}\, d\xi, \quad x \in \mathbb{R}^n.$$

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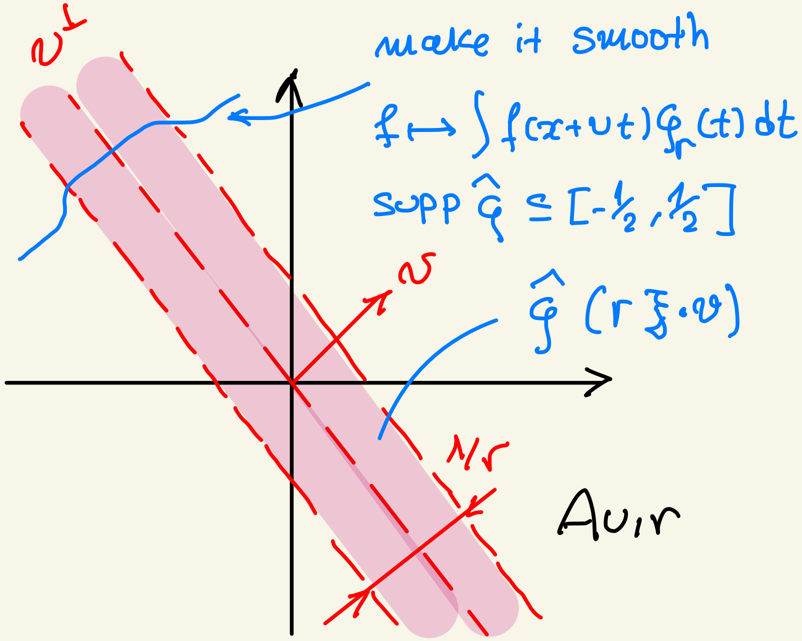
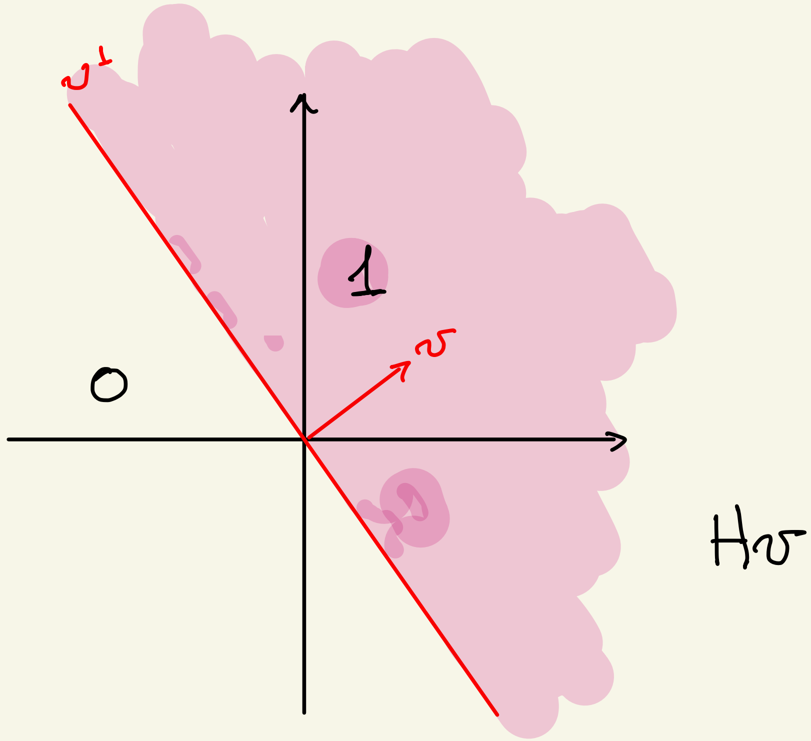
- **singular integrals** More generally, if $m \in HM(\mathbb{R})$, $x \in \mathbb{R}^n$:

$$T_{m,v} f(x) := \int_{\mathbb{R}^n} \hat{f}(\xi)\, m(\xi \cdot v)\, e^{2\pi i \xi \cdot x}\, d\xi, \quad T_{m,V} f(x) := \sup_{v \in V} |T_{m,v} f(x)|$$

$$\| m \|_{HM(\mathbb{R}^d)} := \sup_{0 \le |\alpha| \le M} \sup_{\xi \in \mathbb{R}^d \setminus \{0\}} |\xi|^{|\alpha|} |D^\alpha f(\xi)| < \infty.$$

DESCRIPTION IN FREQUENCY





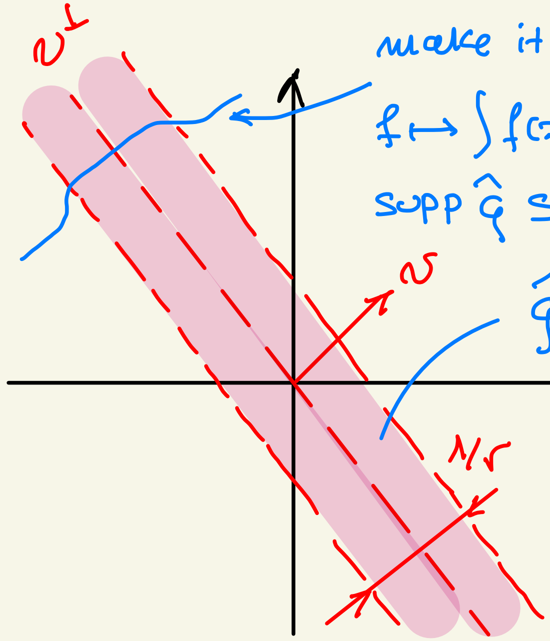
DESCRIPTION IN FREQUENCY

make it smooth

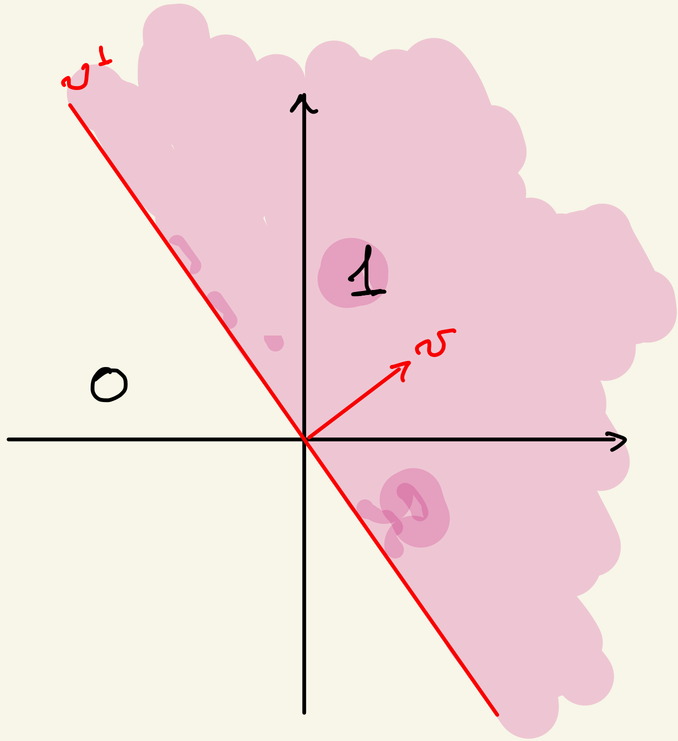
$f \mapsto \int f(x+vt)\varphi_r(t)\,dt$

$\operatorname{supp}\hat{\varphi} \subseteq [-\tfrac12, \tfrac12]$

$\hat{\varphi}(r\xi\cdot v)$



$A_{v,r}$



H_v

LINEARIZATION OF THE MAXIMAL MULTIPLIERS

$$M_V f(x) := \sup_{r>0} \fint_{-r}^{r} f(x+v(x)t)\,dt$$

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$v : \mathbb{R}^n \longrightarrow V \subseteq S^{n-1}$

a measurable vector-field of directions.

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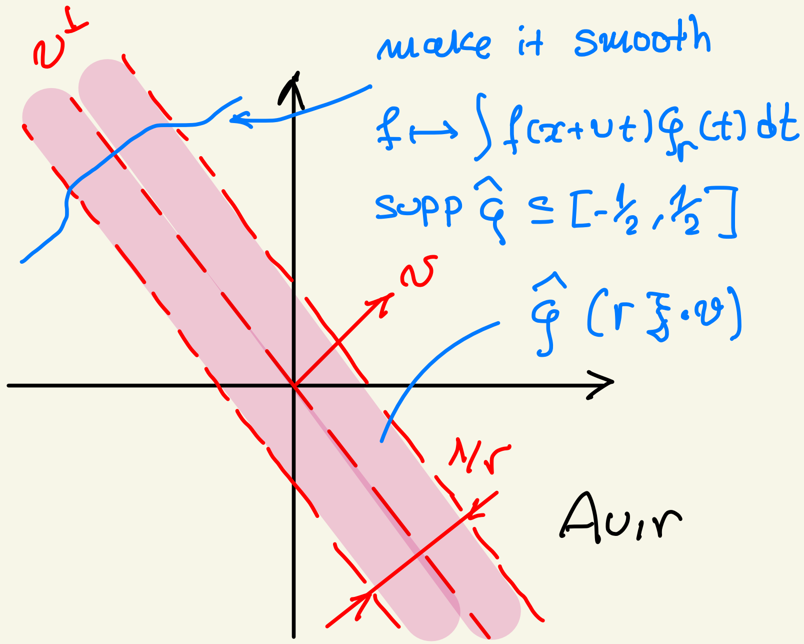

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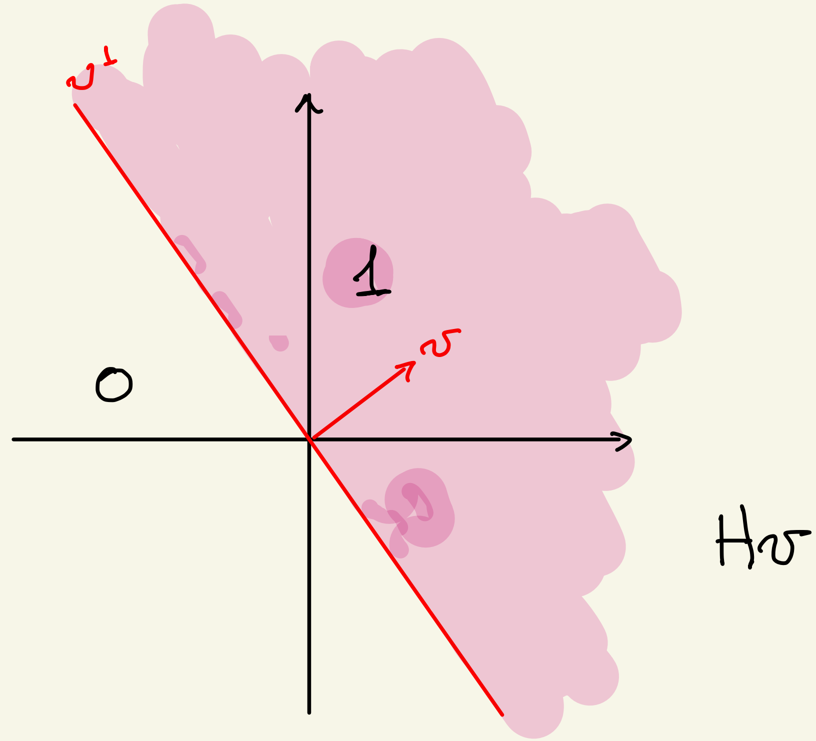

LINEARIZATION OF THE MAXIMAL MULTIPLIERS

$$M_{v,s} f(x) := \sup_{0<r<s} \int_{-r}^{r} f(x+v(x)t)\,dt$$

$$H_{v,s} f(x) := \text{p.v.} \int_{-s}^{s} f(x+v(x)t)\,\frac{dt}{t}$$

$s>0$

some fixed truncation of scales.

KAKEYA COUNTEREXAMPLES AND OTHER OBSTRUCTIONS

- All the (maximal) operators above have Kakeya counterexamples if no restriction is imposed on $V \subseteq \mathbb{S}^{n-1}$

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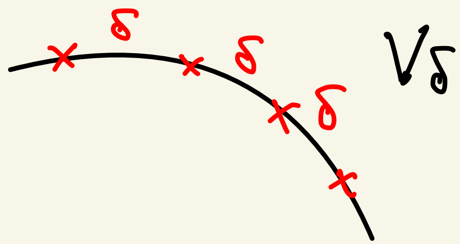

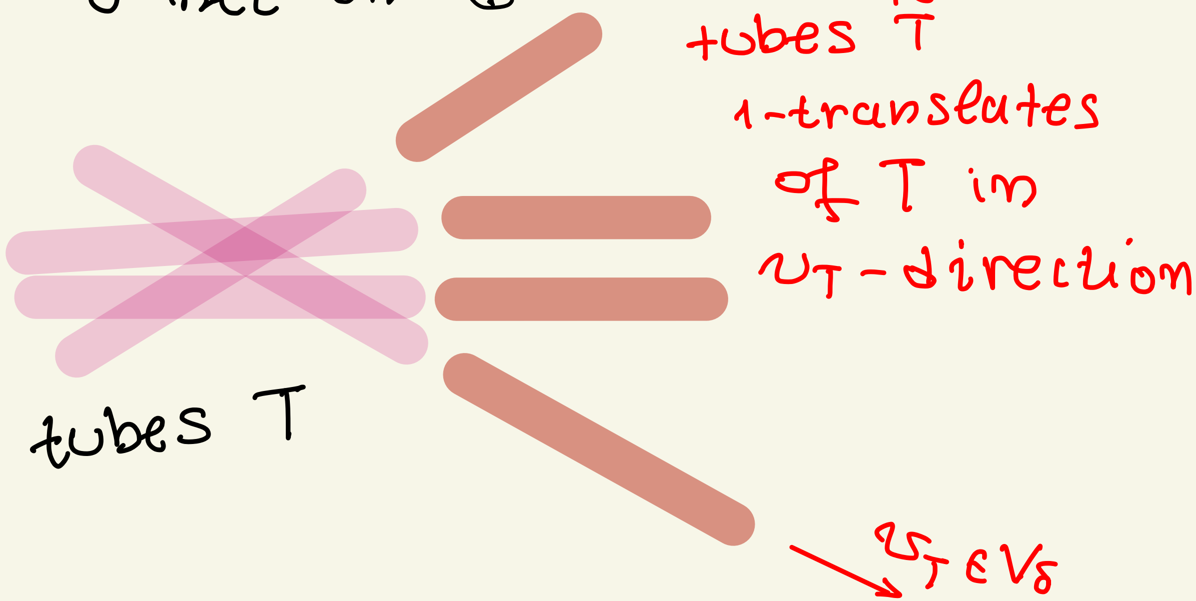

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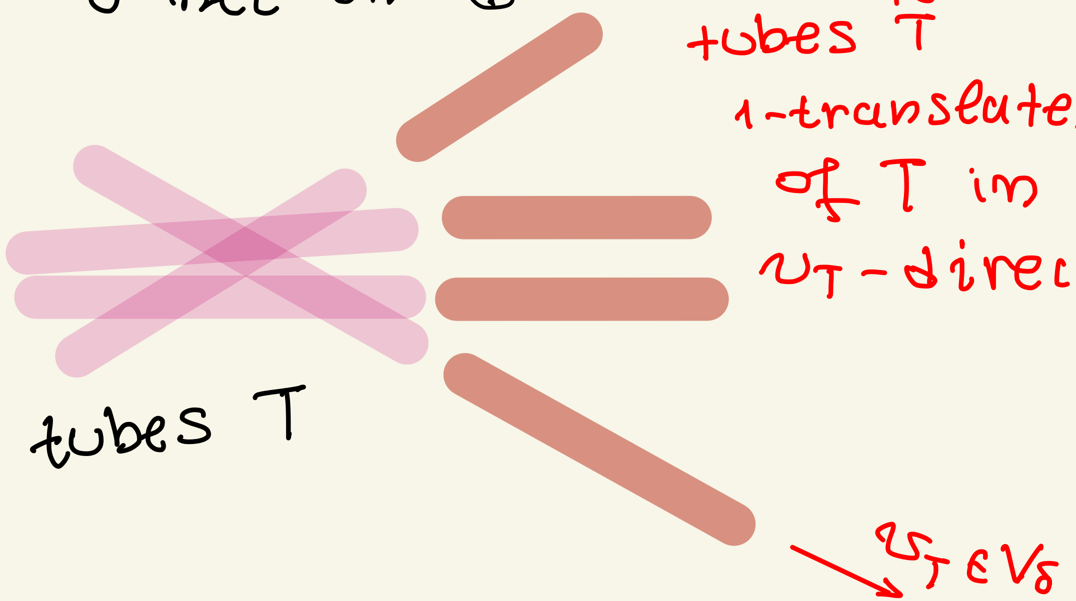

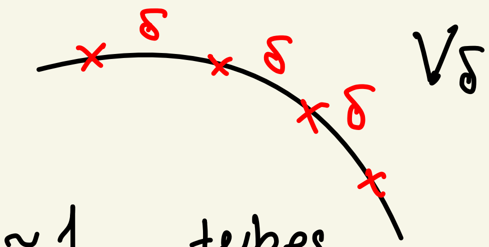

$|\cup \tilde{T}| \sim 1$, tubes $\tilde{T}$ are disjoint and there are $\sim 1/\delta$ of them

$|\cup T| \sim (\log 1/\delta)^{-1}$ Kakeya tubes

maximum compression

Schoenberg (62), Besicovitch (20)
Córdoba (77)

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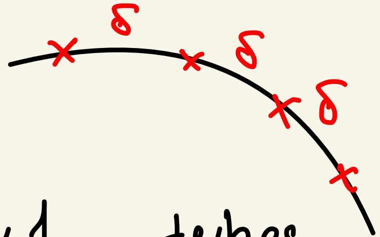
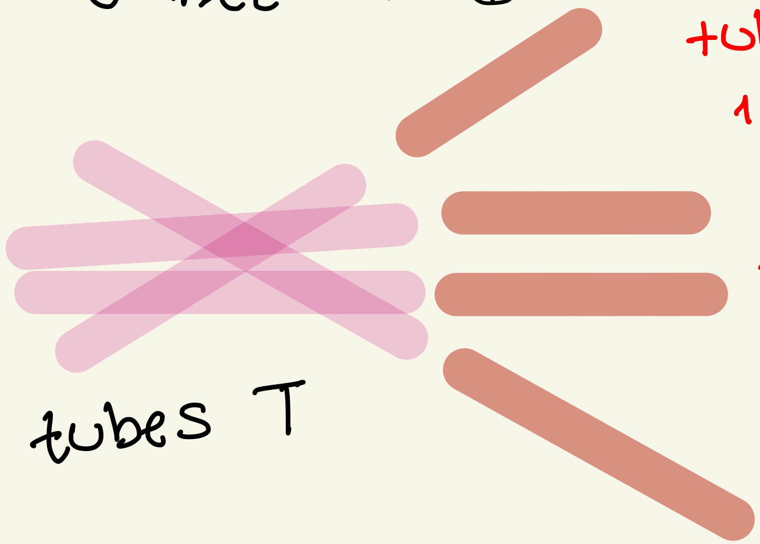


V_δ



tubes $\tilde{T}$
1-translates of T in v_T-direction

tubes T

$v_T \in V_\delta$

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$$\Rightarrow \quad \|H_{V_\delta}\|_p, \ \|M_{V_\delta}\|_p \gtrsim (\log 1/\delta)^{1/p}, \quad p \geq 2.$$

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e.g. $V = \left\{ \frac{(1, 2^{-k})}{|(1, 2^{-k})|} : k \in \mathbb{N} \right\}$, $M_V : L^p(\mathbb{R}^n) \xrightarrow{\text{bdd}} L^p(\mathbb{R}^n)$

$1 < p \leq \infty$

Córdoba-Fefferman '77, Nagel-Stein-Wainger '78
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Question: Does $M_V : L^p \xrightarrow{\text{bdd}} L^p$ imply V lacunary?

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- H_V also has non-Kakeya counterexamples!

for any $V \subseteq \mathbb{S}^{n-1}$

$$\|H_V\|_{L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)} \gtrsim \sqrt{\log \# V}$$

Karagulyan '07
Łaba-Marinelli-Pramanik '17

OPTIMAL CARDINALITY ($\#V$) BOUNDS FOR M_V

- If $V \subseteq \mathbb{S}^{n-1}$ is arbitrary then $M_{V,1}, M_V, H_V$ are all unbounded but one can (try to) quantify the failure in terms of $\#V \to \infty$.

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- THM [Di Plinio - I.P. '21] Let $Z_m \subseteq \mathbb{S}^{n-1}$ be an algebraic variety of dimension $1 \leq m \leq n-1$. Then

$$\sup_{\substack{V \subseteq Z_m \\ \#V \leq N}} \|M_{V,1}\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \sim_\varepsilon N^{\frac{m-1}{2m} + \varepsilon} \quad \forall \varepsilon > 0$$

$n=3, m=2$, Demeter '12, $m=1$, any n, Córdoba '82

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- If V is a subset of a lacunary set

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- In higher dimensions, if $V\subseteq S^{n-1}$ arbitrary
  $$\sup_{\#V\le N} \|H_V\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \simeq N^{\frac{n-2}{2(n-1)}}.$$
  (Kim–Pramanik '22)

II. THE ZYGMUND AND STEIN CONJECTURES

- Remember the linearized versions $(n=2)$

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- α-Hölder with $\alpha<1$ still allows Kakeya.
- Zygmund suggested tha $M_{v(\cdot),s}$ with $v(\cdot)$ Lipschitz and $s \simeq \|v\|_{LIP}^{-1}$ should be weak-type $(2,2)$; Stein for $H_{v(\cdot),s}$.

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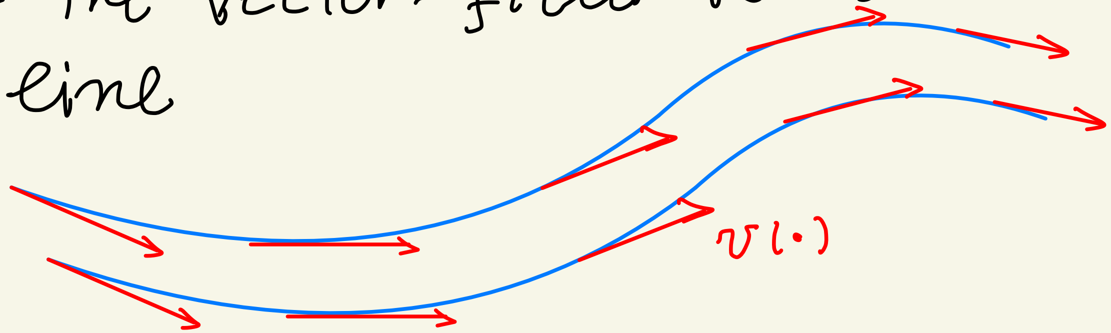

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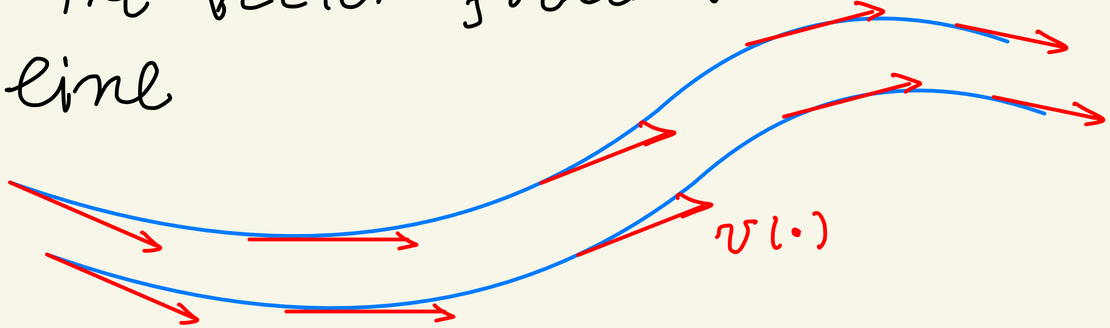


- THM (Lacey-Li, '06) Single-annulus estimates

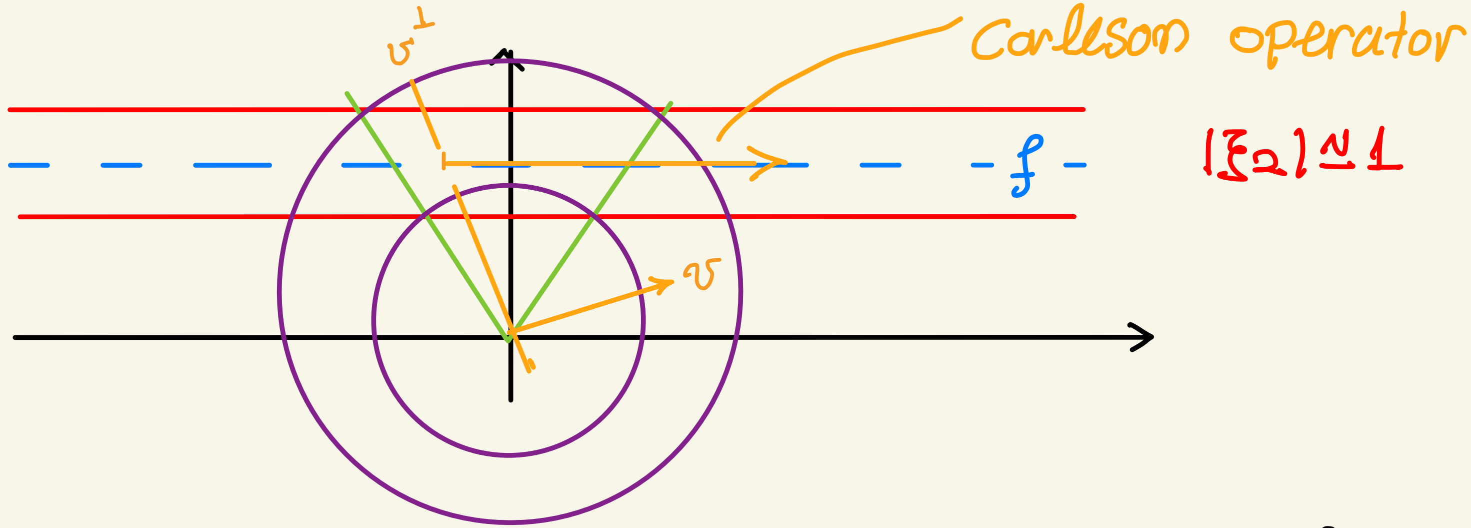
$$\| H_{v(\cdot)} \circ P_0 \|_{L^2(\mathbb{R}^2) \longrightarrow L^{2,\infty}(\mathbb{R}^2)} \lesssim 1$$

$$\| H_{v(\cdot)} \circ P_0 \|_{L^p(\mathbb{R}^2) \longrightarrow L^p(\mathbb{R}^2)} \lesssim 1, \quad 2<p<\infty$$

measurable v.f.

smooth projection on $|\xi| \simeq 1$.

LACEY-LI IMPLIES CARLESON



Single-annulus vector-field problem implies 1d Carleson.

open question: Vector-valued Lacey-Li: $f \mapsto \left(\sum_k |H_{u(\cdot)} \circ P_k f|^2 \right)^{\frac{1}{2}}$?

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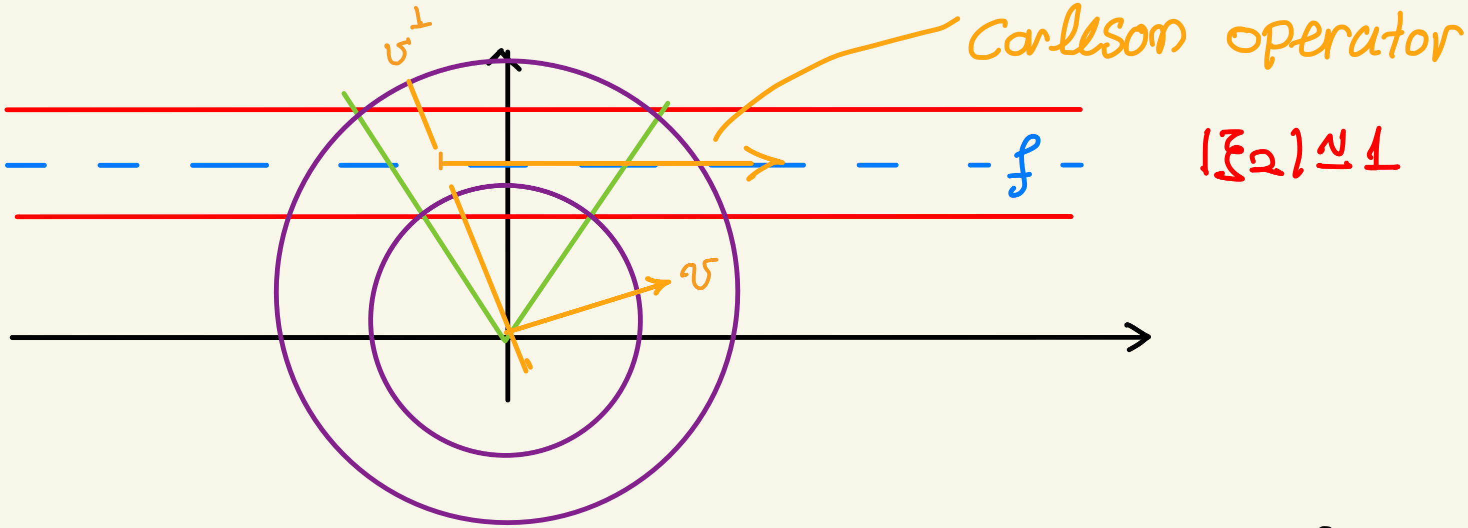

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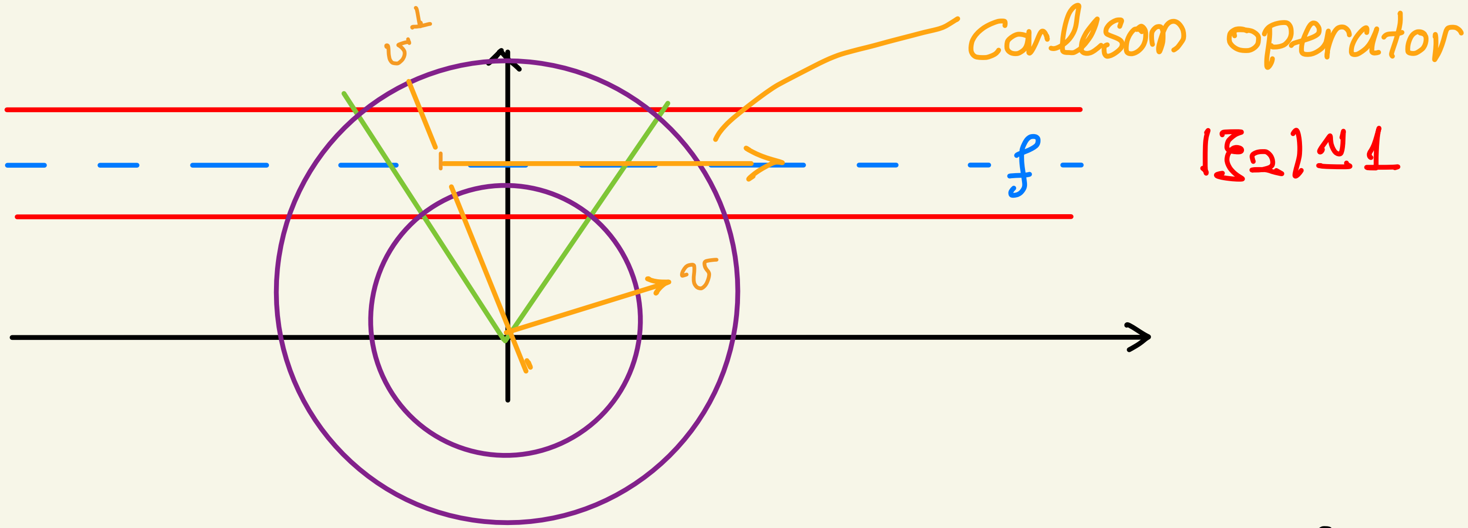

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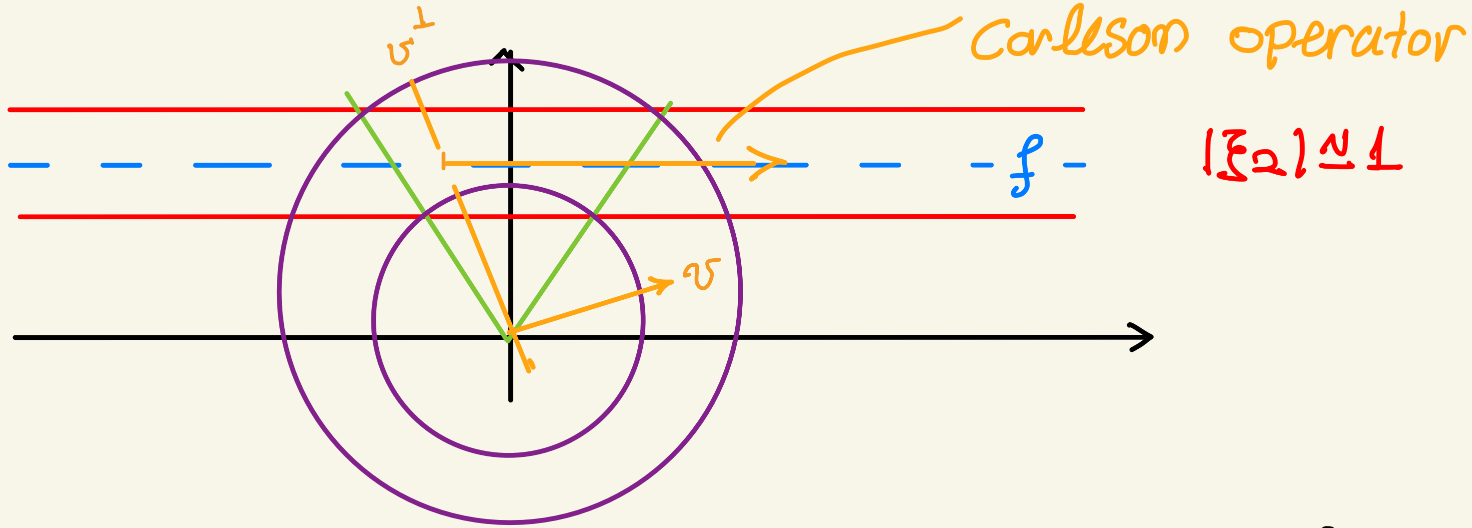

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- Di Plinio, Guo, Thiele, Zorin-Kranich ('18), More general proof.

III. CODIMENSION ONE DIRECTIONAL MULTIPLIERS

- Remember that in the case $n=2$

$$T_{m,V} f(x) := \sup_{v \in V} \left| \int_{\mathbb{R}^2} m(\xi \cdot v) \hat{f}(\xi) e^{2\pi i \xi \cdot x} \, d\xi \right|, \quad x \in \mathbb{R}^2$$

where $m \in HM(\mathbb{R}^{n-1})$: the kernel has dimension

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SINGLE ANNULUS ZYGMUND-STEIN IN CODIMENSION 1

THEOREM (O. BAKAS, F. DI PLINIO, I.P., L. RONCAL) If P_k is a smooth Littlewood-Paley projection onto $|\xi| \simeq 2^k$ and the map $\sigma \longmapsto m_\sigma$ is log-Hölder continuous, then

$$\sup_{k \in \mathbb{Z}} \| T^*_{m_{\sigma(\cdot)}} \circ P_k f \|_{L^{2,\infty}(\mathbb{R}^n)} \lesssim \|f\|_{L^2(\mathbb{R}^n)}$$

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Remark The main theorem implies the Carleson-Sjölin theorem $f \longmapsto \sup_N \left| \int_{\mathbb{R}^d} \hat{f}(\eta) m(\eta+N) e^{2\pi i \eta \cdot x} \, d\eta \right|$ bounded on $L^p(\mathbb{R}^d)$.

REMOVING THE ROTATIONAL INVARIANCE

- T_m^* is invariant under rotations $Q_\sigma \in SO(d)$ as they just rotate $(v_1^\sigma, \dots, v_d^\sigma)$ to a new ONB of σ (and the sup over $\sigma \in Gr(d,n)$ kills Q_σ)

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- This is done essentially as in the following model problem:

$$\boxed{\text{For } m \in HM(\mathbb{R}^d),\ f \in L^p(\mathbb{R}^d), \text{ is the map } f \longmapsto \sup_{Q \in SO(d)} \left| \int_{\mathbb{R}^d} m(Q\xi)\, \hat{f}(\xi)\, e^{2\pi i \xi \cdot x}\, d\xi \right| \text{ bounded ?}}$$

MAXIMALLY ROTATED MULTIPLIERS

- Let $d=2$ for simplicity and consider

$$Tf(x,\vartheta) := \int_{\mathbb{R}^2} \hat{f}(\eta)\, m(O(\theta)\eta)\, e^{2\pi i x\cdot\eta}\, d\eta, \qquad O(\theta) := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

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- Use the fundamental theorem of calculus to write

$$|Tf(x,\vartheta)| = \left| Tf(x,0) + \int_0^{\vartheta} \int_{\mathbb{R}^2} \hat{f}(\eta)\, \partial_\tau [m(O(\tau)\eta)]\, e^{2\pi i x\cdot\eta}\, d\eta\, d\tau \right|$$

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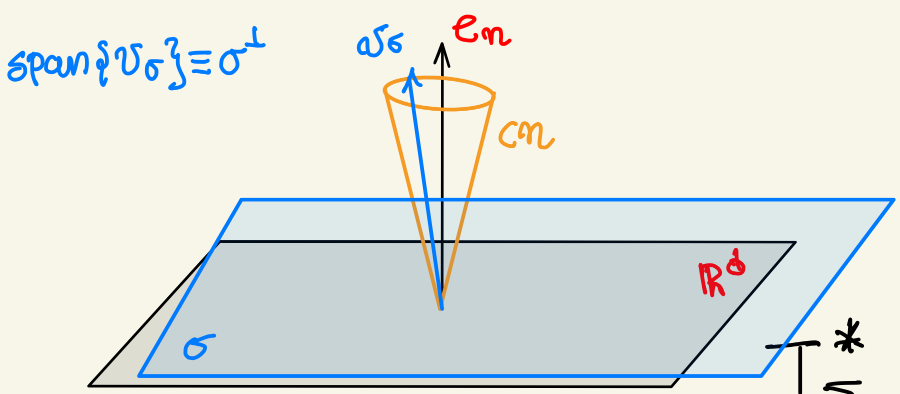
$$\leq |Tf(x,0)| + \int_0^{2\pi} |Sf(x,\tau)|\, d\tau$$

where $f \longmapsto Sf(\cdot,\tau)$ is the operator with Fourier multiplier

$$\eta \longmapsto \partial_\tau m(O(\tau)\eta) = \langle \nabla m(O(\theta)\eta), O'(\theta)\eta \rangle$$

and it is easy to check that these multipliers are uniformly (in τ) HM, with one derivative less than m.

INITIAL REDUCTIONS



▶ We have reduced matters to the study of the operator

$$T^*_{\Sigma,\mathbf{m}} f := \sup_{\sigma\in\Sigma} \Big| \int_{\mathbb{R}^n} \overbrace{m_\sigma(O_\sigma \Pi_\sigma \xi)\, \widehat{P_0 f}(\xi)\, e^{2\pi i \xi\cdot x}}^{T_\sigma P_0 f} \, d\xi \Big|$$

$\uparrow$ smooth LP on $|\xi|\simeq 1$

$\sigma \longmapsto O_\sigma$ C^1, $O_\sigma \sigma = \mathbb{R}^d$, $\Pi_\sigma : \mathbb{R}^n \longrightarrow \sigma$ orthogonal projection

$\Sigma := \{\sigma \in Gr(d,n) : \ \mathrm{dist}(\sigma, \mathbb{R}^d) \ll 1\}$ (by finite splitting)

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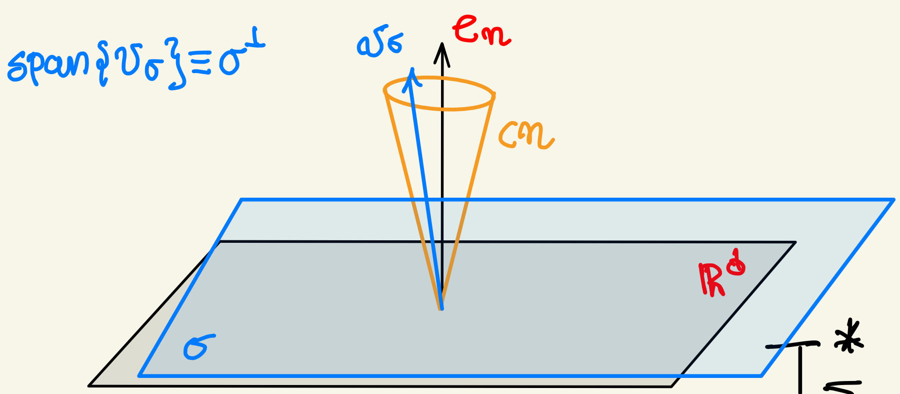

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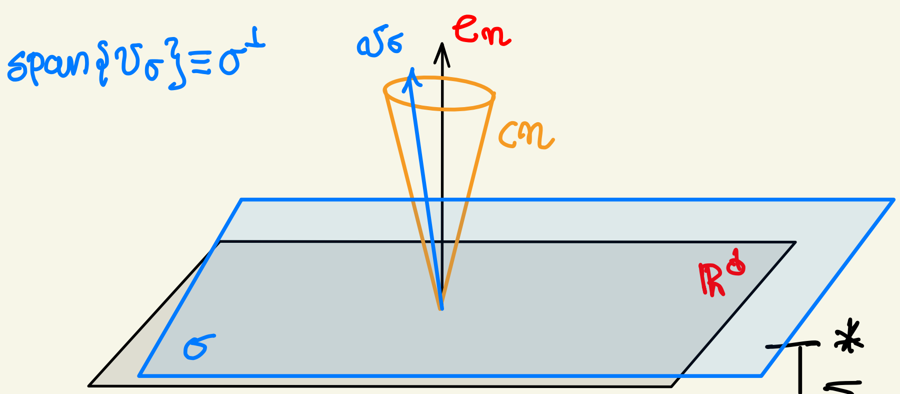

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P_0 ← smooth LP on $|\xi|\simeq 1$

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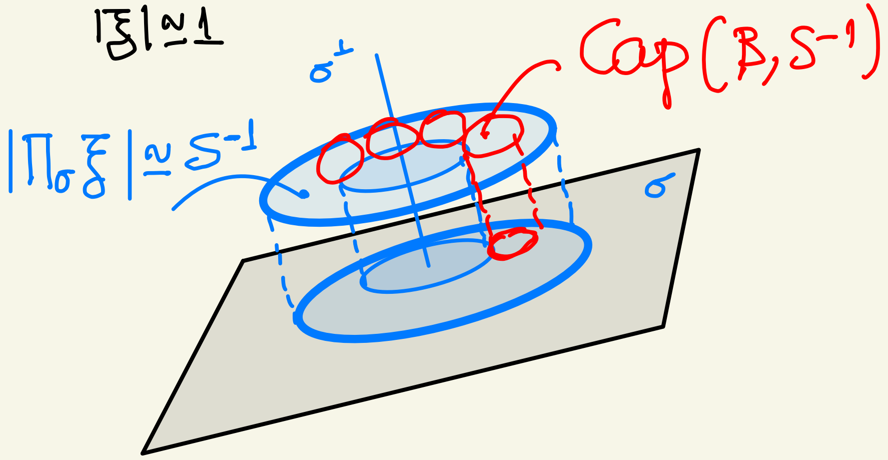
▶ From here on in we will assume that $f\in \mathcal{S}(\mathbb{R}^n)$ and $\mathrm{supp}(\widehat{f}) \subseteq \{|\xi|\simeq 1\} \cap C_n$.

TIME-FREQUENCY DISCRETIZATION

- Fix $\sigma \in \Sigma$; we do a LP-decomposition of the multiplier in the σ-plane

$$\int_{|\xi|\simeq 1} m_\sigma(O_\sigma \Pi_\sigma \xi)\, e^{2\pi i \xi\cdot x}\, d\xi = \sum_S \underbrace{\int_{|\xi|\simeq 1} m_\sigma(O_\sigma \Pi_\sigma \xi)\, \underbrace{\psi(S\Pi_\sigma \xi)}_{|\Pi_\sigma\xi|\simeq S^{-1}}\, e^{2\pi i \xi\cdot x}\, d\xi}_{\psi_S(x,\sigma)}$$



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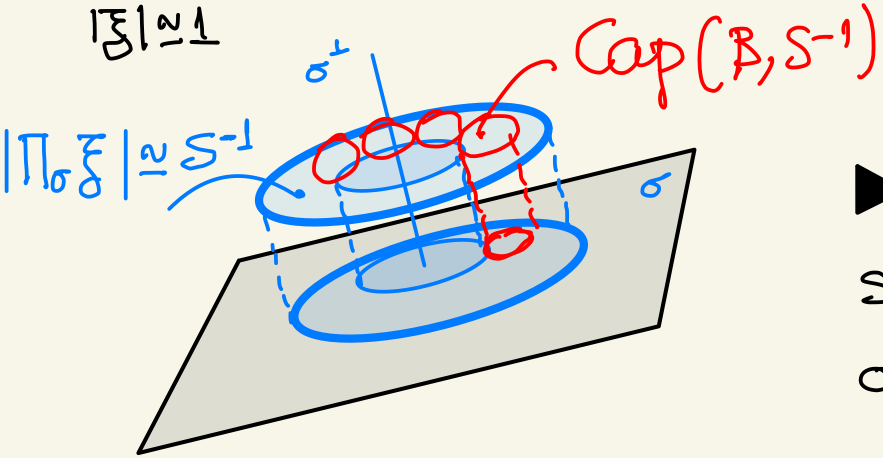

- For every $s \gg 1$ we perform a single-scale Gabor decomposition of F (at frequency scale s^{-1})

$$F = P_0 \circ P_{cn} f = \sum_{\beta \in \Delta_{s^{-1}}} F * \varphi_\beta * \varphi_\beta \;;$$

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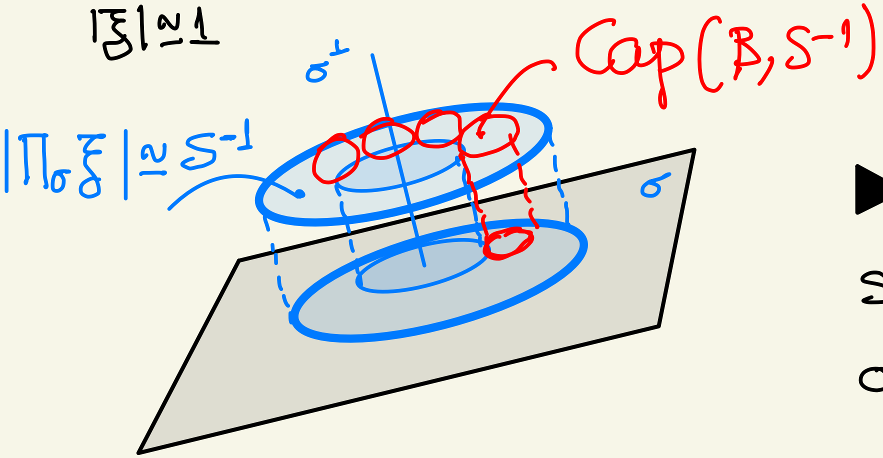

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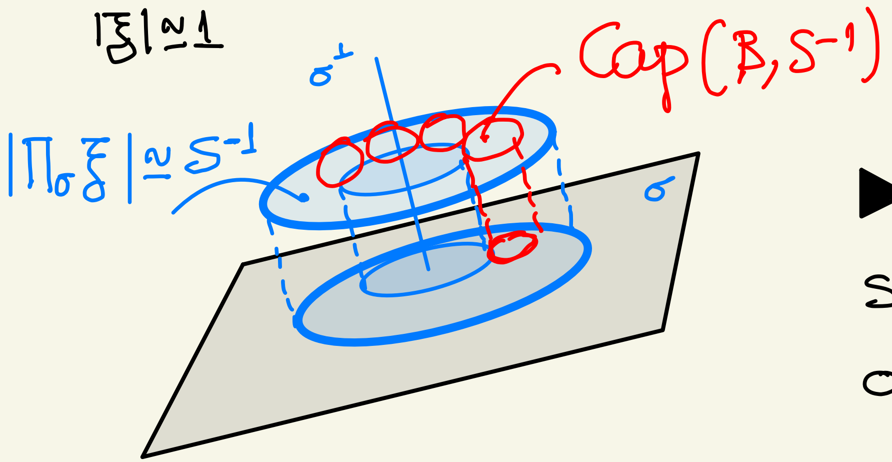

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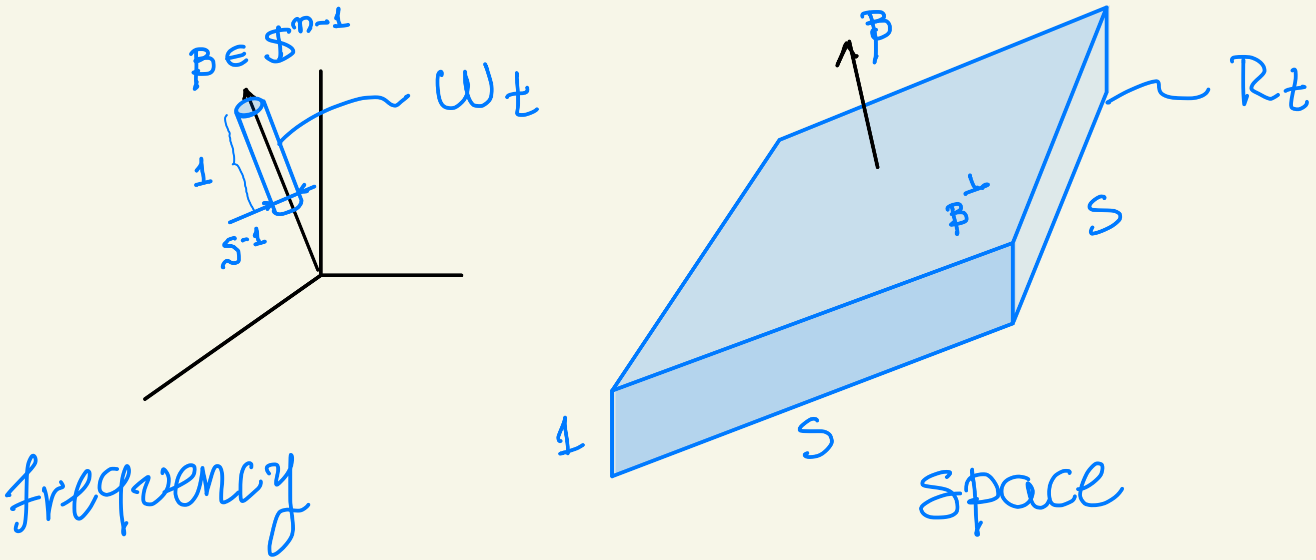
- $\Delta_{s^{-1}}$ is a s^{-1}-net on $\Sigma \subseteq S^{n-1}$;
- $\text{SUPP}\, \varphi_\beta \subseteq \left\{ \xi : |\xi| \simeq 1 : \xi/|\xi| \in Cap(\beta, s^{-1}) \right\}$.

TIME-FREQUENCY DISCRETIZATION

- Each piece in frequency is a tube pointing along $\beta \in S^{n-1}$, of length ~ 1 and cross section $s^{-1} \ll 1$



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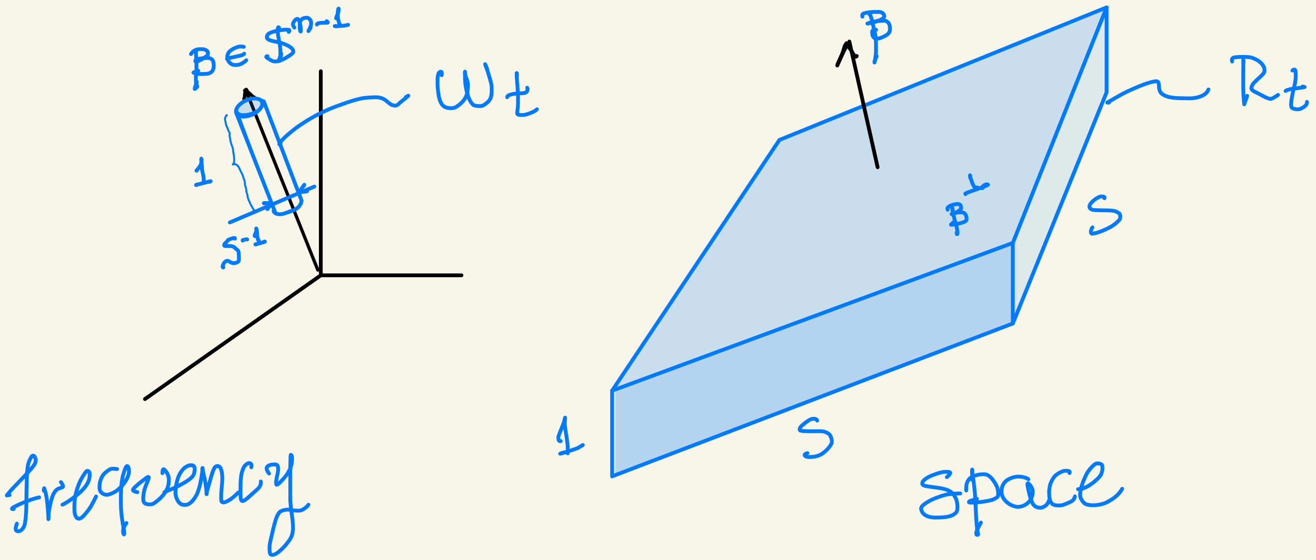

Tile

$$t := R_t \times \omega_t$$

$$|R_t|\,|\omega_t| \simeq 1$$

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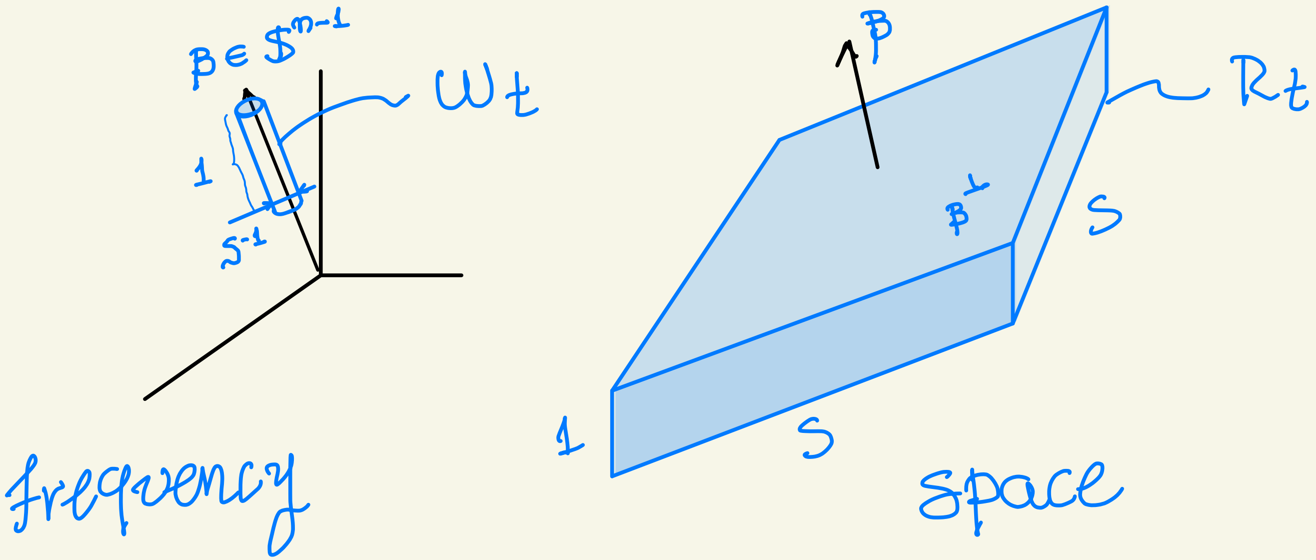

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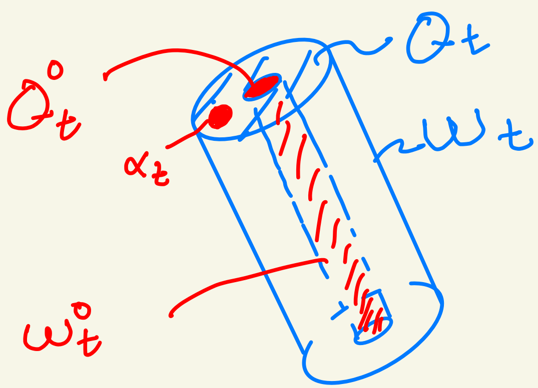
$$T_\sigma f := \sum_{t \in \mathbb{P}} \langle f, \varphi_t \rangle \, \psi_s(\cdot, \sigma) * \varphi_t$$

$$=: \sum_{t \in \mathbb{P}} \langle f, \varphi_t \rangle \, \vartheta_t(\cdot, \sigma)$$

TIME-FREQUENCY DISCRETIZATION

- Each φ_t is time-frequency localized on the tile $t = R_t \times \omega_t$



tube, cap

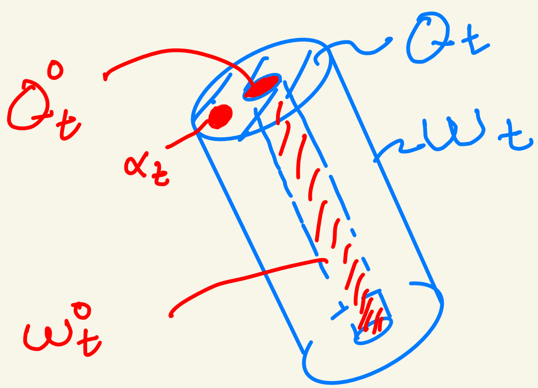
- $\omega_t = \{|\xi| \simeq 1 \,.\, \xi/|\xi| \in Q_t\}$

central cap

$$\operatorname{supp}(\widehat{\varphi_t}) \subseteq \left\{|\xi| \simeq 1 \,,\, \frac{\xi}{|\xi|} \in Q_t^0 \right\} =: \omega_t^0$$

central tube

- Eeach g_t is time-frequency localized on the tile $t = R_t \times \omega_t$

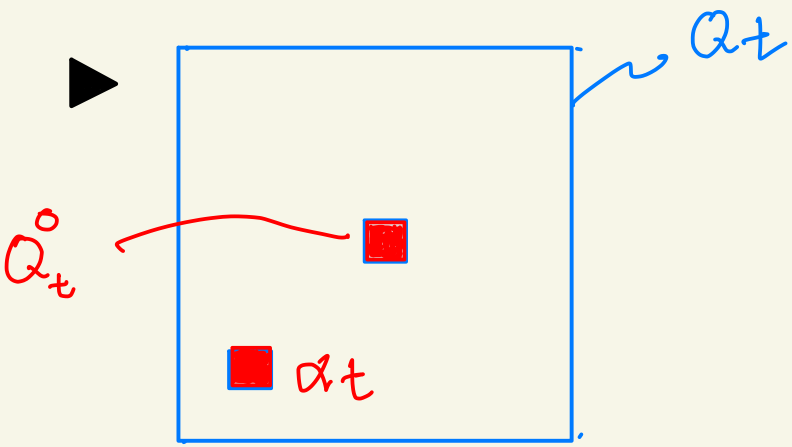


tube → $\omega_t = \{|\xi| \simeq 1 \,.\, \xi/|\xi| \in Q_t\}$ ← cap

central cap → Q_t°

$$\operatorname{supp}(\widehat{g_t}) \subseteq \left\{|\xi| \simeq 1,\ \frac{\xi}{|\xi|} \in Q_t^\circ\right\} =: \omega_t^\circ$$

↑ central tube

- 

Directional support

$$\vartheta_t(\cdot, \sigma) \neq 0 \Rightarrow v_\sigma \in \alpha_t$$

where v_σ is the unit normal to $\sigma \in \Sigma$.

The maximal multiplier is controlled in duality form by

$$\left| \left\langle \sup_{\sigma \in \Sigma} \left| T_\sigma \circ (P_0 \circ P_{cn}) f \right|, g \right\rangle \right|$$

$|\tilde{g}| = |g|$

$$\leq \left| \left\langle \sum_{t \in \Pi} \langle f, \varphi_t \rangle \, \vartheta_t(\cdot, \sigma(\cdot)), \tilde{g} \right\rangle \right|$$

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$$\leq \sum_{t\in\Pi} |\langle f, \varphi_t\rangle| \left|\left\langle \vartheta_t(\cdot, \sigma(\cdot))\, \mathbb{1}_{\alpha_t}(v_{\sigma(x)}), \tilde{g}\right\rangle\right|$$

$\sigma : \mathbb{R}^n \longrightarrow \Sigma \subseteq Gr(d,n)$ measurable.

▶ A tree is a collection of tiles $\mathbf{T}$ such that there exists $(\xi_{\mathbf{T}}, R_{\mathbf{T}})$, $\xi_T \in S^{n-1}$, R_T an admissible parallelepiped such that

$$\xi_{\mathbf{T}} \in Q_t, \quad R_t \cap R_{\mathbf{T}} \neq \emptyset, \quad scl(R_t) \leq scl(R_{\mathbf{T}}) \quad \forall \; t \in \mathbf{T}.$$

TOP OF THE TREE

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- $\xi_{\mathbf{T}} \in \alpha_t \quad \forall\ t \in \mathbf{T}$: lacunary tree

  $\xi_{\mathbf{T}} \in Q_t \setminus \alpha_t \quad \forall\ t \in \mathbf{T}$: overlapping tree

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- $\xi_{\mathbf{T}} \in \alpha_t \quad \forall\ t \in \mathbf{T}$: lacunary tree

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- For a single (lacunary) tree $\mathbf{T}$, assume WLOG $\xi_{\mathbf{T}} = e_n$. We want to be able to pretend that $\sigma = \mathbb{R}^d$ (everything behaves like the top):

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  For $\mathrm{scl}(t)\, \mathrm{dist}(\sigma, \rho) \lesssim 1$, $\sigma, \rho \in Gr(d,n)$ we have

$$|\vartheta_t(\cdot, \sigma) - \vartheta_t(\cdot, \rho)| \leq \max\left(\mathrm{scl}(t)\, \mathrm{dist}(\sigma, \rho), \frac{1}{\log(e + [\mathrm{dist}(\sigma, \rho)]^{-1})}\right) \chi^{(2)}_{R_t}$$

APPENDIX: THE COROLLARY FOR $\#\Sigma < +\infty$

▶ Let $m_1, \dots, m_N$ be Fourier multipliers, $\|m_j\|_\infty \leq 1$.

It is a result of Grafakos, Honzík, Seeger ('05) with the additions of Demeter ('10) that

$$\Big\| \sup_{1 \leq j \leq N} |T_{m_j} f| \Big\|_{L^p(\mathbb{R}^n)} \lesssim \sqrt{\log(e+N)} \, \Big\| \Big(\sum_k \sup_{1 \leq j \leq N} |T_{m_j} \circ P_k f|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\mathbb{R}^n)}$$

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▶ The main theorem and interpolation arguments imply

$$\forall\, k \in \mathbb{Z} \qquad \Big\| \sup_{\sigma\in\Sigma} |(T_\sigma \circ P_k) f| \Big\|_{L^2(\mathbb{R}^n)} \lesssim \sqrt{\log(\#\Sigma)}\, \| P_k \circ f \|_{L^2(\mathbb{R}^n)}$$

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CWN

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$$\lesssim [\log(\#\Sigma)]^{1/2} \Big(\sum_k \Big\| \sup_{\sigma\in\Sigma} |(T_\sigma\circ P_k)f| \Big\|^2_{L^2(\mathbb{R}^n)} \Big)^{1/2}$$

$$\underset{\text{corollary of main theorem}}{\lesssim} \log(\#\Sigma) \Big(\sum_k \|P_k f\|^2_{L^2(\mathbb{R}^n)} \Big)^{1/2}$$

$$\lesssim \log(\#\Sigma)\, \|f\|_{L^2(\mathbb{R}^n)}.$$