Bilinear Bochner-Riesz operator for convex domains

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Introduction

Bochner-Riesz Operator

- Let $\alpha \geq 0$ and $n \geq 1$,

$$B_R^\alpha(f)(x) = \int_{\mathbb{R}^n} \left(1 - \frac{|\xi|^2}{R^2}\right)_+^\alpha \widehat{f}(\xi)\, e^{2\pi i x \cdot \xi}\, d\xi.$$

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- $\left(1 - \frac{|\xi|^2}{R^2}\right)_+ = \begin{cases} 1 - \frac{|\xi|^2}{R^2} & |\xi| \leq R, \\ 0 & |\xi| > R. \end{cases}$

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- We can also write

$$B_R^\alpha(f)(x) = K_R^\alpha * f(x).$$

- $K_R^\alpha(x) = c_{n,\alpha} R^n \frac{\mathcal{J}_{\frac{n}{2}+\alpha}(2\pi R|x|)}{(R|x|)^{\frac{n}{2}+\alpha}}.$

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- When $R = 1$, we write $B_R^\alpha = B^\alpha$.

Bochner-Riesz Operator

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Theorem (Bochner-Riesz conjecture)

B^α is bounded from $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ if and only if $\alpha > \alpha(p)$.

Bilinear Bochner-Riesz Operator

- Let $\alpha \geq 0$ and $n \geq 1$, the bilinear Bochner-Riesz operator is defined by

$$\mathcal{B}_R^\alpha(f,g)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left(1 - \frac{|\xi|^2 + |\eta|^2}{R^2}\right)_+^\alpha \widehat{f}(\xi)\widehat{g}(\eta)\, e^{2\pi i x\cdot(\xi+\eta)}\, d\xi\, d\eta.$$

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where $(f \otimes g)(x,y) = f(x)g(y)$.

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- When $R = 1$, we denote $\mathcal{B}_R^\alpha$ by $\mathcal{B}^\alpha$.

Bilinear Bochner-Riesz Operator

- We have the kernel estimate

$$|K^\alpha(y,z)| \le \frac{1}{(1+|y|)^{\frac{n+\alpha+\frac{1}{2}}{2}}} \frac{1}{(1+|z|)^{\frac{n+\alpha+\frac{1}{2}}{2}}}.$$

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- Thus, for $\alpha > n - \frac{1}{2}$, we have

$$\mathcal{B}^\alpha : L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \to L^p(\mathbb{R}^n),$$

whenever $1 \le p_1, p_2 \le \infty$ satisfying Hölder relation $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$.

Bilinear Bochner-Riesz Operator

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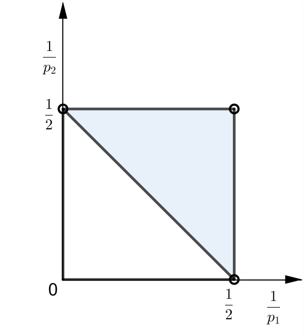

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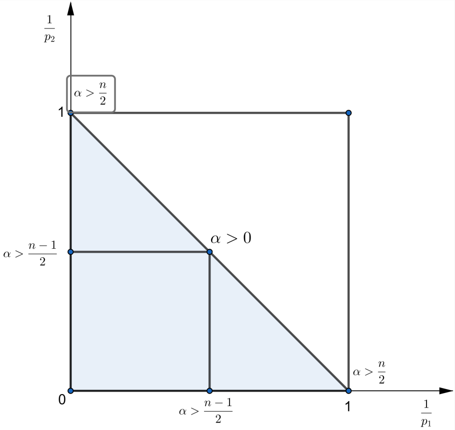
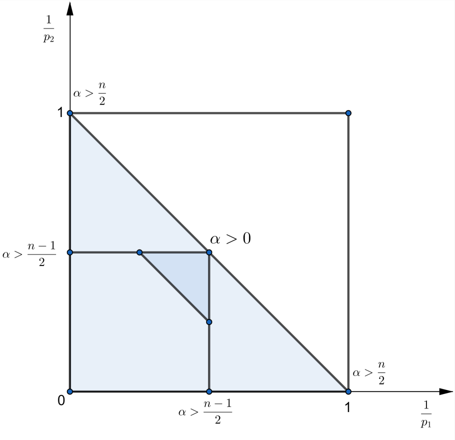
- **Diestel and Grafakos (Nagoya Math. J., 2007)**
  When $n \geq 2$, $\mathcal{B}^0$ is not bounded if exactly one of p_1, p_2, or p' less than 2.

Some results on $L^{p_1} \times L^{p_2} \to L^p$-boundedness of bilinear Bochner-Riesz operator for $p \geq 1$ were first given by F. Bernicot et al. (J. Anal. Math.,2015)



Jeong, Lee and Vargas (Math. Ann., 2018) improved the range of exponent α when $p_1, p_2 \geq 2$.



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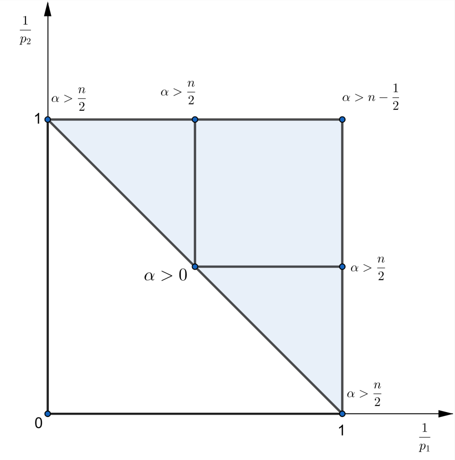
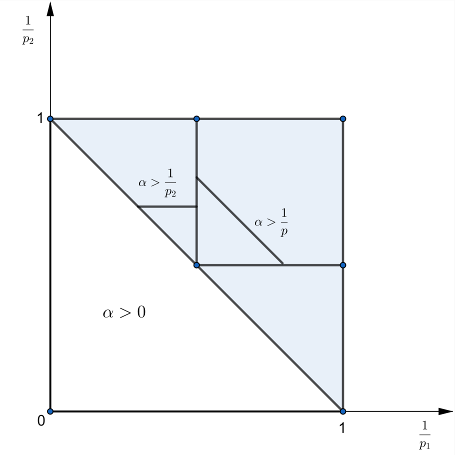
Liu and Wang (Proc. Amer. Math. Soc., 2020) extended the boundedness results in the non-Banach triangle (i.e. $p < 1$).



Kaur and Shrivastava (Adv. Math., 2022) obtained boundedness in non-Banach range in dimension $n = 1$.



Bilinear Bochner-Riesz means for convex domain in the plane

Bilinear Bochner-Riesz means

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$$B^{\alpha}_{\Omega} f(x) = \int_{\mathbb{R}^2} (1 - \rho(\xi))^{\alpha}_{+} \widehat{f}(\xi) e^{2\pi i x\cdot\xi} \, d\xi.$$

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- Seeger and Ziesler extended the study to open and bounded convex domains in the plane.

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Demeter and Gautam proved $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \to L^{p}(\mathbb{R})$-boundedness for $1 < p_1, p < 2$ and $2 < p_2 < \infty$.

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- $\Omega = \{(\xi,\eta) \in \mathbb{R}^2 : \ \xi \leq 0, 2^{\xi} \leq \eta < 1\}$,
  Saari and Thiele proved $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \to L^{p}(\mathbb{R})$-boundedness in Local L^2-range.

Bilinear Bochner-Riesz means

- If Ω has a smooth boundary in the plane, Bernicot and Germain proved $L^{p_1} \times L^{p_2} \to L^p-$boundedness of $\mathcal{B}_\Omega^\alpha$ for $\alpha > 0$, when $p_1, p_2 \geq 2$.

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- Our result concerns open and bounded convex domains in the plane.

Theorem (A. Bhojak,__, S. Shrivastava; to appear Math. Ann.)

Let $\alpha > 0$ and $p_1, p_2 \geq 2$ with $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. Then $\mathcal{B}_\Omega^\alpha$ maps $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ into $L^p(\mathbb{R})$, i.e., there exists a constant $C = C(\Omega, \alpha, p_1, p_2) > 0$ such that

$$\|\mathcal{B}_\Omega^\alpha(f, g)\|_p \leq C\|f\|_{p_1}\|g\|_{p_2}, \quad f, g \in \mathcal{S}(\mathbb{R}).$$

Brief idea of the proof

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- We also assume that no portion of the boundary $\partial\Omega$ is parallel to the coordinates axes.

Cont.

- Let $\phi \in C_c^\infty([-\frac{3}{4}, \frac{3}{4}])$ and $\psi \in C_c^\infty([\frac{1}{2}, 2])$, be such that

$$\phi(t) + \sum_{l=1}^{\infty} \psi(2^l(1-t)) = 1, \quad \text{for all } t \in [0,1).$$

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- We have the following decomposition of the multiplier.

$$\begin{aligned}
&(1-\rho(\xi,\eta))_+^\alpha \\
&= \phi(\rho(\xi,\eta))(1-\rho(\xi,\eta))_+^\alpha + \sum_{l=1}^{\infty} 2^{-\alpha l}\psi(2^l(1-\rho(\xi,\eta)))(2^l(1-\rho(\xi,\eta)))_+^\alpha \\
&= m_0(\xi,\eta) + \sum_{l=1}^{\infty} 2^{-\alpha l} m_l(\xi,\eta).
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- In each sector, we further refine boundary decomposition depending on the curvature of the parametrized curve in each sector and obtain

$$m_l(\xi,\eta) = \sum_{p=1}^{2^{2M}} \sum_{j,\nu} m_{l,p,j,\nu}(\xi,\eta),$$

where $\nu = -2M - l, \ldots, 2M + l, \quad j = 1, \ldots, Q$, and $Q \lesssim C_M 2^{\frac{l}{2}}$.

Cont.

- Let $K_{l,p,j,\nu} = \mathcal{F}^{-1}(m_{l,p,j,\nu})$ and $A_k = B(0, 2^k) \setminus B(0, 2^{k-1})$. We write

$$\begin{aligned} K_{l,p,j,\nu} &= \sum_{k=0}^{10l} K_{l,p,j,\nu} \chi_{A_k} + \sum_{k=10l+1}^{\infty} K_{l,p,j,\nu} \chi_{A_k} \\ &= K^1_{l,p,j,\nu} + K^2_{l,p,j,\nu}. \end{aligned}$$

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$$\begin{aligned} \left\| \sum_{l=1}^{\infty} 2^{-\lambda l} \sum_{p=1}^{2^{2M}} \sum_{j,\nu} K^2_{l,p,j,\nu} \right\|_1 &\lesssim \sum_{l=1}^{\infty} 2^{-(\lambda+3)l} lQ \\ &\lesssim \sum_{l=1}^{\infty} 2^{-(\lambda+3-\frac{1}{2})l} l \lesssim 1. \end{aligned}$$

Cont.

- Let $P^1_{l,p,j,\nu}$ and $P^2_{l,p,j,\nu}$ be the projection of the support of the multiplier $m_{l,p,j,\nu}$ onto ξ-axis and η-axis respectively.

$$K^1_{l,p,j,\nu} * (f, g)(x, x) = K^1_{l,p,j,\nu} * (f_{l,p,j,\nu}, g_{l,p,j,\nu})(x, x),$$

where $\widehat{f}_{l,p,j,\nu} = \chi_{P^1_{l,p,j,\nu}} \widehat{f}$ and $\widehat{g}_{l,p,j,\nu} = \chi_{P^2_{l,p,j,\nu}} \widehat{g}$.

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where $\widehat{f}_{l,p,j,\nu} = \chi_{P^1_{l,p,j,\nu}} \widehat{f}$ and $\widehat{g}_{l,p,j,\nu} = \chi_{P^2_{l,p,j,\nu}} \widehat{g}$.

- We have the following estimate

$$K^1_{l,p,j,\nu} * (f, g)(x, x) \lesssim l\mathcal{M}_{2^{30l}}(Mf_{l,p,j,\nu}, Mg_{l,p,j,\nu}),$$

where M is the Hardy-Littlewood maximal function and $\mathcal{M}_{2^{30l}}$ is the bilinear Kakeya maximal function.

Cont.

$$
\begin{aligned}
&\left\| \sum_{l=1}^{\infty} 2^{-\lambda l} \sum_{p=1}^{2^{2M}} \sum_{j,\nu} K^1_{l,p,j,\nu} * (f_{l,p,j,\nu}, g_{l,p,j,\nu}) \right\|_p \\
\lesssim &\sum_{l=1}^{\infty} 2^{-\lambda l} \sum_{p=1}^{2^{2M}} \left\| \sum_{j,\nu} l \mathcal{M}_{2^{30l}} (M f_{l,p,j,\nu}, M g_{l,p,j,\nu}) \right\|_p .
\end{aligned}
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$$\lesssim \sum_{l=1}^{\infty} 2^{-\lambda l} \sum_{p=1}^{2^{2M}} \left\| \sum_{j,\nu} l \mathcal{M}_{2^{30l}} (M f_{l,p,j,\nu}, M g_{l,p,j,\nu}) \right\|_p .$$

Using the vector valued boundedness of bilinear Kakeya maximal function and that of Hardy-Littlewood maximal function, the above term can be dominated by

$$\sum_{l=1}^{\infty} 2^{-\frac{\lambda l}{2}} l^2 \sum_{p=1}^{2^{2M}} \left\| \Big(\sum_{j,\nu} |M f_{l,p,j,\nu}|^2 \Big)^{\frac{1}{2}} \right\|_{p_1} \left\| \Big(\sum_{j,\nu} |M g_{l,p,j,\nu}|^2 \Big)^{\frac{1}{2}} \right\|_{p_2}$$

$$\lesssim \sum_{l=1}^{\infty} 2^{-\frac{\lambda l}{2}} l^2 \sum_{p=1}^{2^{2M}} \left\| \Big(\sum_{j,\nu} |f_{l,p,j,\nu}|^2 \Big)^{\frac{1}{2}} \right\|_{p_1} \left\| \Big(\sum_{j,\nu} |g_{l,p,j,\nu}|^2 \Big)^{\frac{1}{2}} \right\|_{p_2} \lesssim \|f\|_{p_1} \|g\|_{p_2} .$$

Bilinear Kakeya maximal function

Maximal function

Let $\mathfrak{F}$ be a collection of finite measure sets in $\mathbb{R}^n$. Consider the maximal averaging operator associated with the collection $\mathfrak{F}$ defined by

$$M_{\mathfrak{F}} f(x) = \sup_{F \in \mathfrak{F}:\ x \in F} \frac{1}{|F|} \int_F |f(y)|\ dy.$$

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- If $\mathfrak{F}$ is the collection of cubes (or balls) in $\mathbb{R}^n$, then $M_{\mathfrak{F}}$ (Hardy-Littlewood maximal operator), maps $L^p(\mathbb{R}^n)$ into itself for all $1 < p \leq \infty$ with a weak-type boundedness at $p = 1$.

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- If $\mathfrak{F}$ is the collection of all rectangles in $\mathbb{R}^n$, then by a well-known Besicovitch set construction, it is known that the corresponding operator $M_{\mathfrak{F}}$ fails to be L^p-bounded for all $1 \leq p < \infty$.

Kakeya maximal function

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For an integer $N > 1$ and $\delta > 0$, let $\mathcal{R}_{\delta,N}$ be the class of all rectangles in $\mathbb{R}^2$ with dimensions $\delta \times \delta N$ and $\mathcal{R}_N = \cup_{\delta>0}\mathcal{R}_{\delta,N}$.

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- Córdoba proved that

$$\|M_{\mathcal{R}_{1,N}}\|_{L^2\to L^2} \lesssim (\log N)^{\frac{1}{2}},$$

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- Strömberg proved the following sharp bounds for the maximal operator $M_{\mathcal{R}_N}$.

$$\|M_{\mathcal{R}_N}\|_{L^2\to L^2} \lesssim \log N.$$

Bilinear Kakeya maximal function

- The bilinear Kakeya maximal function associated with the collection $\mathcal{R}_N$ is defined by

$$\mathcal{M}_{\mathcal{R}_N}(f,g)(x) = \sup_{k \le N} \sup_{\substack{R \in \mathcal{R}_k \\ (x,x) \in R}} \frac{1}{|R|} \int_R |f(y_1)||g(y_2)| \, dy_1 \, dy_2.$$

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- We can see that the bilinear Kakeya maximal function $\mathcal{M}_{\mathcal{R}_N}(f,g)$ can be obtained by restricting the (linear) two-dimensional Kakeya maximal function $M_{\mathcal{R}_N}(f \otimes g)$ to the diagonal $\{(x,x) : x \in \mathbb{R}\}$.

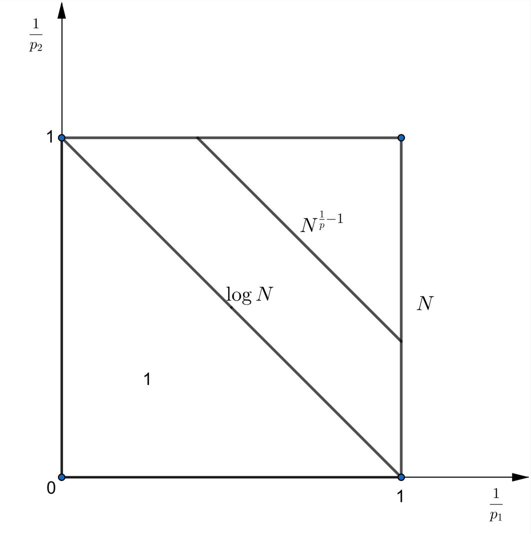
Bilinear Kakeya maximal function

Theorem

Let $1 \leq p_1, p_2 \leq \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. The following bounds hold.

1. **Banach case:**
   a) *If $p > 1$, $\|\mathcal{M}_{\mathcal{R}_N}\|_{L^{p_1} \times L^{p_2} \to L^p} \lesssim 1$.*
   b) *If $p_3 = 1$, $\|\mathcal{M}_{\mathcal{R}_N}\|_{L^{p_1} \times L^{p_2} \to L^1} \lesssim \log N$.*
   Moreover, the bound $\log N$ is sharp.
2. **Non-Banach case:**
   a) *For $1 < p_1, p_2 \leq \infty$ and $\frac{1}{2} < p < 1$, we have*
   $\|\mathcal{M}_{\mathcal{R}_N}\|_{L^{p_1} \times L^{p_2} \to L^p} \lesssim N^{\frac{1}{p}-1}$.
   b) **End-point case:** *If atleast one of p_1 or p_2 is 1 then*
   $\|\mathcal{M}_{\mathcal{R}_N}\|_{L^{p_1} \times L^{p_2} \to L^{p,\infty}} \lesssim N$.

Bilinear Kakeya maximal function



Bilinear Kakeya maximal function

Theorem

Let $1 < p_1, p_2 < \infty$*,* $1 \leq p < \infty$ *and* $1 < r_1, r_2 \leq \infty$*,* $1 \leq r \leq \infty$ *satisfy* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ *and* $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$*. Then for any* $\epsilon > 0$*, we have*

$$\left\| \left(\sum_j \left| \mathcal{M}_{\mathcal{R}_N} (f_j, g_j) \right|^r \right)^{\frac{1}{r}} \right\|_p \lesssim N^{\epsilon} \left\| \left(\sum_j |f_j|^{r_1} \right)^{\frac{1}{r_1}} \right\|_{p_1} \left\| \left(\sum_j |g_j|^{r_2} \right)^{\frac{1}{r_2}} \right\|_{p_2}.$$

Brief idea of the proof for non-Banach case

- We observe that any rectangle $R \in \mathcal{R}_{\delta,N}$, we can dominate the bilinear average over R by a bilinear average over square with its side-length comparable to δN and containing R. This gives us

$$\mathcal{M}_{\mathcal{R}_{\delta,N}}(f,g)(x) \leq N\, Mf(x)Mg(x).$$

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- Further, we have

$$\frac{1}{|R|}\int_R |f(x-y_1)||g(x-y_2)|\; dy_1\, dy_2 \lesssim M_s f(x) M_{s'}\, g(x), \quad 1 \leq s \leq \infty,$$

where $M_s f(x) = (M(f^s)(x))^{\frac{1}{s}},\; 1 \leq s < \infty$ and $M_\infty f(x) = \|f\|_\infty$.

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where $M_s f(x) = (M(f^s)(x))^{\frac{1}{s}},\; 1 \leq s < \infty$ and $M_\infty f(x) = \|f\|_\infty$.

- We get the boundedness in non-Banach range $(\frac{1}{2} < p_3 < 1)$ with constant $N^{\frac{1}{p_3}-1}$ by interpolating weak-type estimates at points $(1,\infty,1), (1,1,\frac{1}{2})$ and $(\infty,1,1)$.

Brief idea of the proof for Banach case

Lemma

Let $1 < s < \infty$. Suppose T is a bi-sublinear operator satisfying

$$\|T\|_{L^{p_1} \times L^{p_2} \to L^{p_3,\infty}} \lesssim A,$$

for the following Hölder indices (p_1, p_2, p_3):

1. (∞, ∞, ∞), (∞, s', s'), (s, ∞, s), $(s, s', 1)$, $(\infty, \frac{3s'}{s'+2}, \frac{3s'}{s'+2})$, *and* $(\frac{3s}{s+2}, \infty, \frac{3s}{s+2})$ *with* $A = 1$.
2. $(\frac{3s}{s+2}, \frac{3s'}{s'+2}, \frac{3}{4})$ *with* $A = N^{\frac{1}{3}}$.
3. $(s, \frac{3s'}{s'+2}, \frac{3s}{3s+1})$ *with* $A = N^{\frac{1}{3s}}$.
4. $(\frac{3s}{s+2}, s', \frac{3s'}{3s'+1})$ *with* $A = N^{\frac{1}{3s'}}$.

Then, we have the following strong type estimate,

$$\|T\|_{L^s \times L^{s'} \to L^1} \lesssim \log N.$$

References

Bhojak, A.; Choudhary, S.S. and Shrivastava, S., *Bilinear Bochner-Riesz means for convex domains and Kakeya Maximal function. Math. Ann. (2024).*

Bernicot F.; Grafakos L.; Song L. and Yan L., *The bilinear Bochner-Riesz problem, J. Anal. Math. 127 (2015), 179–217.*

Jeong E.; Lee S. and Vergas A., *Improved bound for the bilinear Bochner-Riesz operator, Math. Ann. 372 (2018), no. 1-2, 581–609.*

Jotsaroop K. and Shrivastava S., *Maximal estimates for bilinear Bochner-Riesz means*, Adv. Math. 395(2022), Paper No. 108100.

THANK YOU!