L^p bounds of maximal operators given by Fourier multipliers with some dilation sets

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Outline

Maximal Operators and
Fourier Multipliers

Result and applications; $E = (0, \infty)$

Result and applications; $\dim(E) \in [0, 1]$

Sketch of Proofs

- A general strategy; the case of $E = (0, \infty)$
- Dimensions and square functions; the case of $\kappa(E) \in [0, 1]$

Maximal Operators and Fourier Multipliers

Maximal Operators in Harmonic analysis

- Hardy-Littlewood maximal function:

$$\mathcal{M}_{HL}f(x) = \sup_{t>0} \frac{1}{|B(x,t)|} \int_{B(x,t)} |f(y)| \mathrm{d}y.$$

- Spherical maximal function:

$$\mathcal{M}_{sph}f(x) = \sup_{t>0} \Big| \int_{\mathbb{S}^{d-1}} f(x - ty) \mathrm{d}\sigma(y) \Big|.$$

- Maximal Bochner-Riesz operator:

$$S_*^{\delta}(f)(x) = \sup_{t>0} \Big| (m^{\delta}(\cdot/t)\widehat{f})\check{}(x) \Big|, \quad m^{\delta}(\xi) = \Big(1 - |\xi|^2\Big)_+^{\delta}.$$

- Maximal operator associated with Schrödinger equation:

$$\mathfrak{M}(f)(x) = \sup_{t>0} \Big| e^{it\Delta} f(x) \Big|$$

- etc...

L^p boundedness and Pointwise Convergence

- $\|\mathcal{M}_{HL}f\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}, \quad 1 < p \leq \infty$.
  => $\int_{B(0,1)} f(x+ty)\mathrm{d}y \to f(x)$ as $t \to 0$ for almost every $x \in \mathbb{R}^d$ whenever $f \in L^1_{loc}$.
- $\|\mathcal{M}_{sph}f\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}, \quad \frac{d}{d-1} < p \leq \infty$, $d \geq 2$.
  => $\int_{\mathbb{S}^{d-1}} f(x+ty)\mathrm{d}\sigma(y) \to f(x)$ as $t \to 0$ for almost every $x \in \mathbb{R}^d$ whenever $f \in L^p$.
- $\|S^\delta_*(f)\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}, \quad \delta > (d-1)\left|\frac{1}{p} - \frac{1}{2}\right|$.
  Results :
  - $p \geq 2$; Carbery($d = 2$, Duke Math. J. 1983), Lee($d \geq 3$, Duke Math. J. '04), Gan-Jing-Wu($d = 3$, arXiv '21), Gan-Oh-Wu($d \geq 4$, arXiv '21),
  - $p \leq 2$; Tao($d \geq 3$, Indiana Univ. Math. J. 1998), and Li-Wu($d = 2$, Math. Ann. '20, arXiv '24).

  => $S^\delta_t(f)(x) \to f(x)$ as $t \to \infty$ for almost every $x \in \mathbb{R}^d$.
  * The convergence itself($p \geq 2$) is fully resolved by Carbery-Rubio de Francia-Vega, 1988.

L^p boundedness and Pointwise Convergence

$$e^{it\Delta}f(x) = \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} e^{-it(2\pi|\xi|)^2}\widehat{f}(\xi)\mathrm{d}\xi.$$

$$\frac{1}{i}\partial_t u = \Delta u; \quad u(x,0) = f(x); \quad x\in\mathbb{R}^d, t\in(0,\infty).$$

- Sharp L^2 maximal estimates

$$\left\|\mathfrak{M}(f)\right\|_{L^2(B(0,1))} = \left\|\sup_{t\in(0,1]}|e^{it\Delta}f|\right\|_{L^2(B(0,1))} \leq C(s)\|f\|_{H^s(\mathbb{R}^d)}, \quad s > \frac{d}{2(d+1)}.$$

- (PWC)

$$\lim_{t\to 0} e^{it\Delta}f(x) = f(x), \quad a.e.\ x\in\mathbb{R}^d,$$

whenever $f\in H^s(\mathbb{R}^d)$ and $s > \frac{d}{2(d+1)}$.

* $s \geq \frac{d}{2(d+1)}$ is necessary for (PWC) due to J. Bourgain.

Maximal operators and Fourier multipliers

All $\mathcal{M}_{HL}, \mathcal{M}_{sph}, S_*^\delta, \mathfrak{M}$ are given in terms of Fourier multipliers.

$$T_m(f)(x) = \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} m(\xi) \widehat{f}(\xi) \mathrm{d}\xi,$$
$$\mathcal{M}_m(f)(x) = \sup_{t>0} \left| T_{m(t\cdot)}(f)(x) \right|.$$

For example,

$$\mathcal{M}_{HL} : m(\xi) = \widehat{\chi_{B(0,1)}}(\xi) = \frac{J_{n/2}(|\xi|)}{|\xi|^{n/2}},$$
$$\mathcal{M}_{sph} : m(\xi) = \widehat{\mathrm{d}\sigma}(\xi) = \frac{J_{\frac{n-2}{2}}(|\xi|)}{|\xi|^{\frac{n-2}{2}}},$$
$$S_*^\delta : m(\xi) = (1-|\xi|^2)_+^\delta, \quad t \to t^{-1},$$
$$\mathfrak{M} : m(\xi) = e^{2\pi i |\xi|^2}, \quad t \to t^{1/2}.$$

Question : For which condition, is $\mathcal{M}_m$ bounded on $L^p(\mathbb{R}^d)$?

Abstract Theory for m; Hörmander-Mikhlin multipliers

Recall that

$$T_m(f)(x) = (m\widehat{f}\,)^{\vee}(x),$$
$$T_m(f)(x,t) = \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} m(t\xi)\widehat{f}(\xi)\mathrm{d}\xi,$$
$$\mathcal{M}_m(f)(x) := \sup_{t>0} \Big| T_m(f)(x,t) \Big|.$$

T_m satisfies L^p-boundedness for all $p \in (1,\infty)$ whenever

$$\sup_{j\in\mathbb{Z}} \Big\| (I-\Delta)^{s/2} m(2^j\cdot)\widehat{\psi} \Big\|_{L^2(\mathbb{R}^d)} = \sup_{j\in\mathbb{Z}} \Big\| m(2^j\cdot)\widehat{\psi} \Big\|_{L^2_s(\mathbb{R}^d)} < \infty, \quad s > \frac{d}{2}. \quad \text{(HM)}$$

(HM) is not sufficient for L^p-boundedness of $\mathcal{M}_m f$.

It requires for $p, q \in (0,\infty)$, $r = \min(p,2)$

$$\Big(\sum_j \| m(2^j\cdot)\widehat{\psi} \|^q_{L^r_s(\mathbb{R}^d)} \Big)^{1/q} < \infty, \quad s > \frac{d}{r}. \qquad \text{(CGHS)}$$

* (CGHS) is given by Christ-Grafakos-Honzík-Seeger('05, Math. Z., '06, Adv. Math.).

Remark on (CGHS)

The condition (CGHS),

$$\left(\sum_j \|m(2^j\cdot)\widehat{\psi}\|^q_{L^r_s(\mathbb{R}^d)}\right)^{1/q} < \infty, \quad s > \frac{d}{r}$$

cannot cover both $\mathcal{M}_{sph}$ and $\sup_t \left|e^{it\Delta}f\right|$.

Intuition for (CGHS):

$$|\partial^\alpha m(\xi)| \lesssim (1+|\xi|)^{-|\alpha|-\varepsilon}, \quad \varepsilon > 0, |\alpha| > \frac{d}{2}.$$

Intuition for $\mathcal{M}_{sph}$ and $e^{it\Delta}$:

$$|\partial^\alpha m(\xi)| \lesssim |\xi|^{-\gamma}, \quad \gamma > 0, \quad \text{or even worse.}$$

Motivation in Potential Theory

Let f be a twice differentiable function.

Then we have

$$\begin{aligned}\int_{\mathbb{S}^{d-1}} f(x+ty)\mathrm{d}\sigma(y) - f(x) &= A_S(f)(x,t) - f(x) \\ &= C\int_0^t \tau\Big(\int_{B(0,1)} (\Delta f)(x+ty)\mathrm{d}y\Big)d\tau,\end{aligned}$$

which gives

$$\sup_{t>0}\left|\frac{1}{t^2}\Big(A_S(f)(x,t) - f(x)\Big)\right| \lesssim \mathcal{M}(\Delta f)(x).$$

Therefore, for f and Δf in L^p we have

$$A_S(f)(x,t) - f(x) = O(t^2).$$

Recall that $A_S(f)(x,t) - f(x) = O(1)$ whenever $f \in L^p$.

Q : $A_S(f)(x,t) - f(x) = O(t^\alpha)$ whenever $f, \Delta^{\alpha/2} f \in L^p$ for $\alpha \in (0,2]$?

Motivation in Potential Theory

Let

$$A_S(f)(x,t) = \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} m(t\xi) \widehat{f}(\xi) \mathrm{d}\xi, \quad m(\xi) = \widehat{\mathrm{d}\sigma}(\xi).$$

Then we have

$$\frac{1}{t^\alpha}\Big(A_S(f)(x,t) - f(x)\Big) = \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} \frac{m(t\xi)-1}{|t\xi|^\alpha} \widehat{\Delta^{\alpha/2} f}(\xi) \mathrm{d}\xi.$$

Therefore, our goal is to show

$$\|\sup_{t>0} |T_{m_\alpha(t\cdot)} f|\|_{L^p} \lesssim \|f\|_{L^p}, \quad m_\alpha(\xi) = \frac{m(\xi)-1}{|\xi|^\alpha}$$

We also want to suggest a condition for m_α which guarantees such L^p boundedness.

Maximal operators with dilation sets $E \subset (0, \infty)$

Let E be a subset of $(0, \infty)$. Then we define $\mathcal{M}_m^E$ by

$$\mathcal{M}_m^E(f)(x) := \sup_{t \in E} |T_{m(t\cdot)}(f)(x)|. \tag{1}$$

- E could be $\{2^j\}_{j \in \mathbb{Z}}$, Cator sets, etc.
- Differentiation theorem will hold for associated dilation sets.
- We will use $\kappa = \kappa(E)$ to denote a dimension quantity.

Until explained thoroughly, please regard $\kappa(E)$ as just a dimension of E.

Spherical averages with dilation in E

Let $\dim(E) \in [0,1)$ and define

$$M^E_{\mathbb{S}^{d-1}} f(x) := \sup_{t \in E} \left| \int_{\mathbb{S}^{d-1}} f(x - ty) \, \mathrm{d}\sigma(y) \right|. \tag{2}$$

Then we have

Theorem (Seeger-Wainger-Wright, 1995)

For $d \geq 2$, $\|M^E_{\mathbb{S}^{d-1}}\|_{p \to p} < \infty$, if $p > 1 + \frac{\kappa(E)}{d-1}$.

- If $\kappa(E) = 1$, then $1 + \frac{\kappa(E)}{d-1} = 1 + \frac{1}{d-1} = \frac{d}{d-1}$.
- If $d = 2$ and $\kappa(E) < 1$, then $1 + \frac{\kappa(E)}{d-1} = 1 + \kappa(E) < 2$.
- For $E \subset [1,2]$, $L^p \to L^q$ bounds of $M^E_{\mathbb{S}^{d-1}}$ are studied by Anderson-Hughes-Roos-Seeger($d \geq 3$, '20), Roos-Seeger($d = 2$, '23).

Averages over measures and Fourier multipliers

Theorem (Duoandikoetxea-Vargas, 1998)

1. *Let* m *be the Fourier transform of a compactly supported finite Borel measure, and* $E \subset (0, \infty)$. *Assume that* $|m(\xi)| \leq C|\xi|^{-a}$ *for some* $a > \kappa(E)/2$. *Then* $\mathcal{M}_m^E$ *is bounded on* $L^p(\mathbb{R}^d)$ *for* $p > 1 + \frac{\kappa(E)}{2a}$.
2. *Let* $s = [d/2] + 1$, $m \in C^{s+1}$, *and* $E \subset (0, \infty)$. *Assume that* $|\partial^\gamma m(\xi)| \leq C|\xi|^{-a}$ *for all* $|\gamma| \leq s + 1$ *and some* $a > \kappa(E)/2$. *Then* $\mathcal{M}_m^E$ *is bounded on* $L^p(\mathbb{R}^d)$ *for*

$$\frac{2d}{d + 2a - \kappa(E)} < p < 2\frac{d - \kappa(E)}{d - 2a}.$$

- By taking $a = \frac{d-1}{2}$ and $\kappa(E) \in [0, 1)$, this theorem recovers the result fo Seeger-Wainger-Wright.
- The range of p in the second statement seems sharp, we will introduce a result with improved regularity condition on m.

Result and applications;

$E = (0, \infty)$

Main Theorem

Let X be a Banach space of functions on $\mathbb{R}^d$. Then we define $\Sigma^2_\theta(X)$ with norm given by

$$\|f\|_{\Sigma^2_\theta(X)} = \Big(\sum_j (2^{j\theta}\|f(2^j\cdot)\widehat{\psi}\|_X)^2\Big)^{1/2}.$$

If $\theta = 0$, we simply write $\Sigma^2(X)$.

Theorem (L.-Seo, JFA, '23)

Let $p \in (1,\infty)$, $\frac{1}{p_0} = \left|\frac{1}{p} - \frac{1}{2}\right|$ *and* $s > \frac{d}{p_0} + \min\{\frac{1}{2}, \frac{1}{p}\}$. *Then we have*

$$\Big\|\mathcal{M}_m f\Big\|_{L^p(\mathbb{R}^d)} \lesssim \|m\|_{\Sigma^2(B^s_{p_0})}\|f\|_{L^p(\mathbb{R}^d)}. \qquad \text{(L-S)}$$

* $B^s_p = B^s_{p,p}$ denotes the Besov space.

$$\|f\|_{B^s_p} \sim \|S_0 f\|_{L^p} + \Big(\sum_{j\geq 0}(2^{js}\|\psi_j * f\|_{L^p})^p\Big)^{1/p}.$$

$\psi \in \mathscr{S}$ is chosen so that supp$(\widehat{\psi}) \subset \{\frac{1}{2} < |\xi| < 2\}$ and $\sum_{j\in\mathbb{Z}}\widehat{\psi}(2^{-j}\xi) = 1$.

Example of Σ^2 space; Weighted Sobolev spaces

We introduce a weighted Sobolev space, $H_\theta^{p,\gamma}(\mathbb{R}^d)$.

$$\|f\|^p_{H_\theta^{p,\gamma}(\mathbb{R}^d\setminus\{0\})} = \sum_{l=0}^{\gamma}\int_{\mathbb{R}^d} |D^l f(x)|^p |x|^{pl+\theta-d} \mathrm{d}x.$$

By dyadic decomposition around the origin, it follows that

$$\sum_{l=0}^{\gamma}\int_{\mathbb{R}^d} |D^l f(x)|^p |x|^{pl+\theta-d} \mathrm{d}x \sim \sum_{l=0}^{\gamma}\sum_{j\in\mathbb{Z}} 2^{j\theta} \int_{\mathbb{R}^d} |D^l(f(2^j x)\widehat{\psi}(x))|^p \mathrm{d}x,$$

which yields

$$\|f\|_{H_\theta^{p,\gamma}(\mathbb{R}^d\setminus\{0\})} \sim \|f\|_{\Sigma^p_{\theta/p}(L^p_\gamma)}.$$

The weighted Sobolev spaces is used to study boundary behavior of a function mostly appeared in theory of partial differential equations.

Regularity near the origin

Corollary

Let $p \in (1, \infty)$ *and* m *be of class* $B^s_{p_0}(\mathbb{R}^d)$ *where* $s > \frac{d}{p_0} + \min\{\frac{1}{2}, \frac{1}{p}\}$. *Then for any fixed* $\phi_0 \in C_c^\infty(\mathbb{R}^d)$ *we have*

$$\|\mathcal{M}_{m\phi_0}(f)\|_{L^p(\mathbb{R}^d)} \lesssim \|m\|_{B^s_{p_0}(\mathbb{R}^d)} \|f\|_{L^p(\mathbb{R}^d)}.$$

If m is of class $L^2_s(\mathbb{R}^d)$ with $s > \frac{d+1}{2}$, then $\|\mathcal{M}_{m\phi_0}\|_{L^p \to L^p}$ is bounded for any $p \in (1, \infty)$.

By the range $s > \frac{d+1}{2}$, we improve the result of Rubio de Francia(1986, Adv. Math., Duke Math. J.) which requires $m \in C_c^{[\frac{d}{2}]+2}(\mathbb{R}^d)$ for $\mathcal{M}_m$ to be bounded on L^p.

(CGHS) and (L-S)

Two conditions cannot be directly compared!

However,

when $p \leq 2$

(CGHS) :

$$\sum_{j\in\mathbb{Z}} \|m(2^j\cdot)\widehat{\psi}\|_{L^r_s(\mathbb{R}^d)}^q < \infty, \quad r = p, \ s > \frac{d}{p}.$$

(L-S) :

$$\sum_{j\in\mathbb{Z}} \|m(2^j\cdot)\widehat{\psi}\|_{B^s_{p_0}(\mathbb{R}^d)}^2 < \infty, \quad \frac{1}{p_0} = \frac{1}{p} - \frac{1}{2}, \ s > \frac{d}{p} - \frac{d}{2} + \frac{1}{2}.$$

There is $\frac{d-1}{2}$ gain in the sense of regularity.

Applications

- Let $\alpha \in (0,1)$ and $\beta > \frac{1}{2}$. Define

$$\mathfrak{M}_{\alpha,\beta} f = \sup_{t>0} \left| (m_{\alpha,\beta}(t\cdot)\widehat{f})^{\vee} \right|, \quad m_{\alpha,\beta}(\xi) = e^{i|\xi|^{\alpha}} m_{\beta}(\xi),$$

  where m_β vanishes near the origin and satisfies $|\partial^\gamma m_\beta(\xi)| \lesssim |\xi|^{-\beta-|\gamma|}$. Then $\mathfrak{M}_{\alpha,\beta}$ satisfies L^p-boundedness for $\frac{d-2\beta/\alpha}{2(d-1)} < \frac{1}{p} < \frac{d-1+2\beta/\alpha}{2d}$.
- Let

$$U_{\alpha,\beta}(f)(x,t) = \frac{e^{it(-\Delta)^{\alpha/2}} f(x) - f(x)}{t^{\beta}}.$$

  Then for almost all $x \in \mathbb{R}^d$,

$$e^{it(-\Delta)^{\alpha/2}} f(x) - f(x) = O(t^{\beta}).$$

Applications

- Let $m \in C_{loc}^{[\frac{d+1}{2}]+1}(\mathbb{R}^d)$, $m(0) = 1$, and $|\partial^\gamma m(\xi)| \lesssim (1 + |\xi|)^{-\beta}$. Then, for $\alpha \in (0, 1)$ and $f \in \dot{L}^p_\alpha(\mathbb{R}^d)$ with $\frac{d-2(\alpha+\beta)}{2(d-1)} < \frac{1}{p} < \frac{d-1+2(\alpha+\beta)}{2d}$, we have

$$f(x) - T_{m(t\cdot)}f(x) = O(t^\alpha), \text{for almost } x \in \mathbb{R}^d.$$

- For $\beta = \frac{d-1}{2}$ we have $\frac{1-2\alpha}{2(d-1)} < \frac{1}{p} < \frac{2d-2+\alpha}{2d}$
  and $\widehat{\mathrm{d}\sigma}$ satisfies $|\partial^\gamma \widehat{\mathrm{d}\sigma}(\xi)| \lesssim (1 + |\xi|)^{-\frac{d-1}{2}}$.
  That is, it follows that for almost every x

$$f(x) - A_S(f)(x, t) = O(t^\alpha),$$

  whenever $f \in L^p_\alpha(\mathbb{R}^d)$.

  * For direct computation on $\mathrm{d}\sigma$ gives us $\alpha \in (0, 2)$ case.

Result and applications;

$\dim(E) \in [0, 1]$

First of all, we define the dimension quantity, $\kappa(E)$.

$$\kappa(E) := \lim_{\delta \to 0} \sup_{j \in \mathbb{Z}} \frac{\log N(E_j, \delta)}{-\log \delta}, \tag{3}$$

where $E_j := (2^{-j}E) \cap [1,2]$, and $N(E,\delta)$ is the entropy number of E,

$$N(E,\delta) := \#\{k \in \mathbb{N} \mid E \cap [k\delta, (k+1)\delta] \neq \emptyset\} \sim \delta^{-D}.$$

- Let $E = \{1 + n^{-a} \mid n = 1, 2, \dots\}$, $a > 0$. Then $\kappa(E) = (1+a)^{-1}$.
- For a compact E, $\kappa(E)$ equals the Minkowski dimension of E,

$$\dim_{\mathcal{M}}(E) := \inf\{a > 0 \mid \exists C > 0, \text{ such that } \forall \delta \in (0,1),\ N(E,\delta) \leq C\delta^{-a}\}$$

Main result

Theorem (L.-Seo, 2024+)

Let $p \in (1, \infty)$ *and* $E \subset (0, \infty)$ *satisfy* $\kappa(E) < 1$*. Suppose that* m *is of class* $\Sigma^2(B^s_{p_0})$ *with* $1/p_0 = |1/2 - 1/p|$ *and*

$$s > \frac{d}{p_0} + \kappa(E) \min\left\{\frac{1}{2}, \frac{1}{p}\right\}.$$

Then $\mathcal{M}^E_m$ *is bounded on* $L^p(\mathbb{R}^d)$*.*

- For $E = (0, \infty)$, we have $s > \frac{d}{p_0} + \min\left\{\frac{1}{2}, \frac{1}{p}\right\}$.
- We recover the result (2) of Duoandikoetxea-Vargas.

A pointwise convergence result

Let $m_{\alpha,\beta}(\xi) := \mathrm{e}^{2\pi i|\xi|^\alpha} m_\beta(\xi)$, where

- $0 < \alpha < 1$ and $\beta > 0$,
- m_β vanishes near the origin, and

$$|\partial^\gamma m_\beta(\xi)| \lesssim |\xi|^{-\beta-|\gamma|} \quad \text{for any multi-index } \gamma\,.$$

Then we have

$$\|\mathcal{M}_{\mathfrak{m}_{\alpha,\beta}}\|_{L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^d)} < \infty, \tag{4}$$

whenever $\frac{d-2\beta/\alpha}{2(d-\kappa(E))} < \frac{1}{p} < \frac{d+2\beta/\alpha-\kappa(E)}{2d}$.

When $\kappa(E) = 1$, it follows that $\frac{d-2\beta/\alpha}{2(d-1)} < \frac{1}{p} < \frac{d+2\beta/\alpha-1}{2d}$, which is the range of p in the Applications 1

Using the estimates of the previous slide, one can obtain a convergence result of solutions to fractional Schrödinger equations:

Let $\alpha \in (0,1)$, $\beta \in \big(\kappa(E)/2, 1\big)$. Suppose $f \in \dot{L}^p_{\alpha\beta}$ for $\frac{d-2\beta}{2(d-\kappa(E))} < \frac{1}{p} < \frac{d+2\beta-\kappa(E)}{2d}$. Then we have

$$|\mathrm{e}^{-it(-\Delta)^{\alpha/2}} f(x) - f(x)| = O(t^\beta), \quad t \to 0 \text{ on } E\,,$$

where $\mathrm{e}^{-it(-\Delta)^{\alpha/2}} f$ denotes a solution to fractional Schrödinger(half-wave) equations with initial data f. Note that one needs $\beta > \kappa(E)/2$ for $f \in L^2_{\alpha\beta}$, which yields

$$s = \alpha\beta > \frac{\alpha\kappa(E)}{2}.$$

By this observation, we recover the convergence result of Cho-Ko-Koh-Lee('23) for $\alpha \in (0,1)$ and $d \geq 2$:

$$\lim_{n\to\infty} \mathrm{e}^{it_n(-\Delta)^{\alpha/2}} f(x) = f(x) \quad \text{a.e. } x, \quad \forall f \in L^2_s,$$

whenever $s > \frac{\alpha\kappa(E)}{2}$ and $\dim_{\mathcal{M}}(\{t_n\}_n) = \kappa(E)$.

Sketch of Proofs

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Sketch of Proofs

A general strategy; the case of $E = (0, \infty)$

Dimensions and square functions; the case of $\kappa(E) \in [0, 1]$

Let us define for $\alpha \in (0,1)$

$$I^{\alpha}_{0+} f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \mathrm{d}s,$$
$$D^{\alpha}_{0+} F(t) := \frac{\mathrm{d}}{\mathrm{d}t} \Big(I^{1-\alpha}_{0+} F \Big)(t).$$

Then we have

Lemma

$$F(t) = I^{\alpha}_{0+} D^{\alpha}_{0+} F(t) + \frac{t^{\alpha}}{\Gamma(\alpha)} F(0).$$

By this lemma, we have $m(t\xi) = \frac{1}{\Gamma(\frac{1}{2}-\varepsilon)} I^{\frac{1}{2}+\varepsilon}_{0+} \Big((\cdot)^{-\frac{1}{2}-\varepsilon} \widetilde{m}(\cdot\xi) \Big)(t)$,

where

$$\widetilde{m}(\xi) = m(\xi) + \left(\frac{1}{2} + \varepsilon\right) \int_0^1 \frac{m(\xi) - m(s\xi)}{(1-s)^{\frac{3}{2}+\varepsilon}} \mathrm{d}s.$$

We simply put $m(t\xi) = \frac{1}{\Gamma(\frac{1}{2}-\varepsilon)} I_{0+}^{\frac{1}{2}+\varepsilon}\Big((\cdot)^{-\frac{1}{2}-\varepsilon}\widetilde{m}(\cdot\xi)\Big)(t)$ in $T_{m(t\cdot)}f$.

$$\begin{aligned}
\big|T_{m(t\cdot)}f\big|^2 &= \Big|\frac{1}{\Gamma(\frac{1}{2}-\varepsilon)} I^{\frac{1}{2}+\varepsilon}\big(t^{-\frac{1}{2}-\varepsilon}T_{\widetilde{m}(t\cdot)}f\big)\Big|^2 \\
&= \left|\frac{1}{\Gamma(\frac{1}{2}-\varepsilon)}\int_0^t (t-s)^{-\frac{1}{2}+\varepsilon}s^{-\frac{1}{2}-\varepsilon}T_{\widetilde{m}(s\cdot)}f\,ds\right|^2 \\
&\lesssim \int_0^t (t-s)^{-1+2\varepsilon}s^{-2\varepsilon}ds \times \int_0^t \big|T_{\widetilde{m}(s\cdot)}f\big|^2\frac{ds}{s}.
\end{aligned}$$

Thus we have

$$\big|\mathcal{M}_m f\big|^2 \lesssim \int_0^\infty \big|T_{\widetilde{m}(t\cdot)}f\big|^2\frac{dt}{t} =: G_{\widetilde{m}}(f)^2.$$

Vector-valued operators, Embeddings, and Bilinear interpolation

- Vector-valued harmonic analysis for singular integral operators.

$$\|G_{\widetilde{m}}(f)\|_{L^2} \leq \|\widetilde{m}\|_X \|f\|_{L^2}$$

$$\|G_{\widetilde{m}}(f)\|_{L^1} \leq \|\widetilde{m}\|_Y \|f\|_{H^1}, \quad \|G_{\widetilde{m}}(f)\|_{BMO} \leq \|\widetilde{m}\|_Y \|f\|_{L^\infty}$$

- Embeddings.

$$\|\widetilde{m}\|_X \lesssim \|m\|_{X_{1/2+\varepsilon}}, \quad \|\widetilde{m}\|_Y \lesssim \|m\|_{Y_{1/2+\varepsilon}}$$

- Bilinear interpolation of A. P. Calderón.

$$\|B(f,g)\|_{C_0} \leq M_0 \|f\|_{A_0} \|g\|_{B_0},$$
$$\|B(f,g)\|_{C_1} \leq M_1 \|f\|_{A_1} \|g\|_{B_1}$$

$$\downarrow$$

$$\|B(f,g)\|_{[C_0,C_1]_\theta} \leq M_0^{1-\theta} M_1^{\theta} \|f\|_{[A_0,A_1]_\theta} \|g\|_{[B_0,B_1]_\theta}.$$

Vector-valued harmonic analysis

Let $T_m(f)(x,t)$ be an $L^2(\mathbb{R}_+, \frac{\mathrm{d}t}{t})$-valued operator and $\mathcal{H} = L^2(\mathbb{R}_+, \frac{\mathrm{d}t}{t})$.

- Then $L^2(\mathcal{H}) \to L^2(\mathcal{H})$ operator norm of T_m is

$$\sup_{\xi\neq 0}\Big(\int_0^\infty |m(t\xi)|^2 \frac{\mathrm{d}t}{t}\Big)^{1/2} = \|m\|_{L^\infty(\mathcal{H})}.$$

- For $H^1 \to L^1$ boundedness, we have for $\beta > \frac{d}{2}$

$$\int_{|x|>2|y|}\Big(\int_0^\infty |K_t(x-y) - K_t(x)|^2 \frac{\mathrm{d}t}{t}\Big)^{1/2} \mathrm{d}x \lesssim \sup_{j\in\mathbb{Z}} \|m(2^j t\xi)\widehat{\psi}(\xi)\|_{L^2_\beta(\mathcal{H})}.$$

For each operator norms we have

$$\|m\|_{L^\infty(\mathcal{H})} \lesssim \Big(\sum_{j\in\mathbb{Z}} \|m(2^j\cdot)\widehat{\psi}(\cdot)\|_{L^\infty}^2\Big)^{1/2} = \|m\|_{\Sigma^2(L^\infty)},$$

$$\sup_{j\in\mathbb{Z}} \|m(2^j t\xi)\widehat{\psi}(\xi)\|_{L^2_\beta(\mathcal{H})} \lesssim \|m\|_{\Sigma^2(L^2_\beta)}.$$

Embeddings

From the previous slide, we have

$$\|G_{\widetilde{m}}\|_{L^2\to L^2} \leq \|\widetilde{m}\|_{\Sigma^2(C^{0,\varepsilon})},$$

and $\quad \|G_{\widetilde{m}}\|_{H^1\to L^1}, \|G_{\widetilde{m}}\|_{L^\infty\to BMO} \leq \|\widetilde{m}\|_{\Sigma^2(L^2_\beta)}, \quad \beta > \frac{d}{2}.$

For these spaces, we have shown that

$$\|\widetilde{m}\|_{\Sigma^2(C^{0,\varepsilon})} \lesssim \|m\|_{\Sigma^2(C^{0,\frac{1}{2}+\varepsilon})},$$

$$\|\widetilde{m}\|_{\Sigma^2(L^2_{\frac{d}{2}+\varepsilon})} \lesssim \|m\|_{\Sigma^2(L^2_{\frac{d+1}{2}+\varepsilon})}$$

That is, the mapping $m \to \widetilde{m}$ yields $\frac{1}{2}$-derivative in terms of $\Sigma^2(X)$ spaces.

Putting all together

1. Control $\mathcal{M}_m(f)$ by $G_{\widetilde{m}} f$.
2. Consider $G_{\widetilde{m}} f$ as a bilinear map $B(m, f)$.
3. Put $X = \Sigma^2(C^{0,\frac{1}{2}+\varepsilon})$, $Y = \Sigma^2(L^2_{\frac{d}{2}+\frac{1}{2}+\varepsilon})$, which gives

$$\|\mathcal{M}_m f\|_{L^2} \lesssim \|m\|_X \|f\|_{L^2},$$
$$\|\mathcal{M}_m f\|_{L^1} \lesssim \|m\|_Y \|f\|_{H^1}, \quad \text{and} \quad \|\mathcal{M}_m f\|_{BMO} \lesssim \|m\|_Y \|f\|_{L^\infty}.$$

4. Apply Calderón's bilinear interpolation.

$$[\Sigma^2(C^{0,\frac{1}{2}+\varepsilon}), \Sigma^2(L^2_{\frac{d+1}{2}+\varepsilon})]_\theta = \Sigma^2\Big([C^{0,\frac{1}{2}+\varepsilon}, L^2_{\frac{d+1}{2}+\varepsilon}]_\theta\Big)$$

Taking $\theta = 2\left|\frac{1}{p} - \frac{1}{2}\right|$ yields $[C^{0,\frac{1}{2}+\varepsilon}, L^2_{\frac{d+1}{2}+\varepsilon}]_\theta = B^s_{p_0}$
with $s > \frac{d}{p_0} + \frac{1}{2}$ and $\frac{1}{p_0} = |\frac{1}{p} - \frac{1}{2}|$.

5. $\|\mathcal{M}_m f\|_{L^p(\mathbb{R}^d)} \lesssim \|m\|_{\Sigma^2(B^s_{p_0})} \|f\|_{L^p(\mathbb{R}^d)}$.

Outline

Maximal Operators and
Fourier Multipliers

Result and applications; $E = (0, \infty)$

Result and applications; $\dim(E) \in [0, 1]$

Sketch of Proofs

- A general strategy; the case of $E = (0, \infty)$
- Dimensions and square functions; the case of $\kappa(E) \in [0, 1]$

The Assouad & Aikawa dimensions

We call $\kappa(E)$ the Assouad dimension of E, $\dim_{AS}(E)$.

Definition (Aikawa dimension)

Let $X = (X, \mu, \mathrm{d})$ be a general metric space and its doubling dimension is n. For $E \subset X$, let $G(E)$ be the set of $t > 0$ for which there exists a constant c_t such that

$$\int_{B(x,r)} \operatorname{dist}(y, E)^{t-n} \, \mathrm{d}\mu \leq c_t r^{t-n} \mu(B(x, r))$$

for every $x \in E$ and all $r \in (0, \operatorname{diam}(E))$. Then the Aikawa dimension of E is defined to be $\dim_{AI}(E) = \inf G(E)$.

Theorem

Let X be a Q-regular metric measure space, and $E \subset X$. Then $\dim_{AS}(E) = \dim_{AI}(E)$.

A measure μ is Q-regular for $Q > 1$, if there is a constant $c_Q \geq 1$ such that

$$c_Q^{-1} r^Q \leq \mu(B(x, r)) \leq c_Q r^Q, \quad \forall x \in X, r \in (0, \mathrm{diam}(X)).$$

Then we have

- $\dim_{AS}(E) \leq \dim_{AI}(E)$ holds for a doubling metric measure space.
- $\dim_{AI}(E) \leq \dim_{AS}(E)$ holds for a Q-regular metric measure space.

Bounds of $\mathcal{M}_m^E$ using square functions

Lemma (Lemma 3.1, L.-Seo, '24+)

For $E \subset (0,\infty)$, *let* $E_j = 2^{-j}E \cap [1,2]$. *Suppose* $\dim_{AS}(E) \le D_0$ *for some* $D_0 \in (0,1]$. *Then for* $\alpha, \beta \in \mathbb{R}$ *with* $\frac{D_0}{2} < \alpha < \beta \le 1$ *and* $F \in C_{loc}([0,\infty)) \cap C_{loc}^{0,\beta}((0,\infty))$, *we have*

$$\sup_{t\in E} |F(t)|^2 \lesssim \sum_{j\in\mathbb{Z}} \int_1^2 \mathit{dist}(s, E_j)^{-1+2\beta} |D_{0+}^{\alpha} F_j(s)|^2 \,\mathrm{d}s, \quad F_j(s) = F(2^j s).$$

By taking $F(s) = T_{m(s)}f$, we obtain a square function $\mathcal{G}^E(m,f)$.

Lemma (Lemma 3.2, L.-Seo, '24+)

Let E *be a non-empty set in* $[1,2]$. *For any* $a \in (0,1)$, *we have*

$$\sup_{0<\delta\le 1} \delta^a N(E,\delta) \lesssim_a \int_1^2 \mathit{dist}(t,E)^{-1+a} \,\mathrm{d}t \lesssim_a 1 + \int_0^1 \lambda^a N(E,\lambda) \,\frac{\mathrm{d}\lambda}{\lambda}.$$

Following general strategy, one can obtain

1. $\|\mathcal{G}^E(m,f)\|_{L^1} \lesssim \|m\|_{\Sigma^2(L^2_{d/2+\alpha})}\|f\|_{H^1}$,
2. $\|\mathcal{G}^E(m,f)\|_{BMO} \lesssim \|m\|_{\Sigma^2(L^2_{d/2+\alpha})}\|f\|_{L^\infty}$,
3. $\|\mathcal{G}^E(m,f)\|_{L^2} \lesssim \|m\|_{\Sigma^2(C^{0,\alpha})}\|f\|_{L^2}$.

We apply bilinear interpolation on $(1,3)$ and $(2,3)$ to prove the theorem.

Thank you so much!