Quantitative characterizations of weights and parabolic boundary value problems

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Elliptic equations

Background: Harmonic measure

Consider a bounded and simply connected domain $\Omega \subset \mathbb{C}$. By Perron's method, there exists for each $f \in C(\partial\Omega)$ a function $u_f \in C(\overline{\Omega})$ such that

$$\begin{aligned} -\Delta u(z) &= 0, \quad z \in \Omega \\ \lim_{z \to \zeta} u(z) &= f(\zeta), \quad \zeta \in \partial\Omega. \end{aligned}$$

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$$\lim_{z \to \zeta} u(z) = f(\zeta), \quad \zeta \in \partial\Omega.$$

By maximum principle, for any $z \in \Omega$, the mapping $f \mapsto u_f(z)$ is a positive bounded linear functional. The Radon measure

$$\omega_\Omega^z(E) = u_{1_E}(z)$$

is called the harmonic measure with pole at z.

Probabilistic interpretation: Let (X, μ) be a probability space and $B : X \times [0, \infty) \to \Omega$ a Brownian motion starting at $z \in \Omega$. Then

$$\omega_\Omega^z(E) = \mu(\{x \in X : B(x, \tau) \in E\}),$$

where $\tau = \inf\{t \in [0, \infty) : B(x, t) \notin \Omega\}$.
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Change of pole: If $z, w \in \Omega$, then there is a constant $C(z, w)$ such that for all Borel sets $E \subset \partial\Omega$

$$\frac{1}{C(z, w)} \omega_\Omega^z(E) \leq \omega_\Omega^w(E) \leq C(z, w) \omega_\Omega^z(E).$$

Measure theoretic properties do not depend (much) on z.

Classical results on plane

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- Assume $\partial\Omega$ is a quasicircle. The harmonic measures $\{\omega_{\varphi(B(0,r))} : 0 < r < 1\}$ are A_∞ w.r.t. arclength measure with uniform constants **if and only if** Ω is a chord-arc domain (corollary of work of several authors, book of Garnett–Marshall).

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Further, Dirichlet problem (non-tangential a.e. limits)

is solvable in $L^p(\partial\Omega)$ iff every ω_Ω^z is in $\mathrm{RH}_{p'}(\partial\Omega)$.
In particular, it is solvable for some $p \in (1, \infty)$ if $\omega_\Omega^z \in A_\infty$.

Elliptic equations

Divergence form elliptic equations

$$\begin{aligned} \operatorname{div} A\nabla u &= 0, \quad \text{in } \Omega \\ u &= g, \quad \text{on } \partial\Omega \end{aligned}$$

with $A : \Omega \to \mathbb{R}^{n\times n}$ satisfying

- condition on boundedness and measurability;
- ellipticity condition: there is $M_0 > 0$ such that for all $\xi \in \mathbb{R}^n$

$$\inf_{x\in\Omega} A(x)\xi \cdot \xi \geq M_0|\xi|^2;$$

- something more.

$A = I$ in rough domains $\sim$ rough A in smooth domains

Dahlberg–Kenig–Stein pullback

Consider an L-Lipschitz function $F : \mathbb{R}^n \to \mathbb{R}$ and a domain $\Omega = \{x \in \mathbb{R}^{n+1} : x_{n+1} > F(x_1, \ldots, x_n)\}$.

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$$\rho(x, t) = (x_1, \ldots, x_n, ct + \theta_t * F(x)).$$

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Then for c large enough

- ρ is an injection $\mathbb{R}^{n+1}_+ \to \Omega$;
- the pull back of Laplacian under ρ is $\operatorname{div} A\nabla\cdot$ (for some A);
- it holds $|\nabla A(x, t)| \leq C/t$;
- $t|\nabla A(x, t)|^2 \, dxdt$ is Carleson measure.

Definition (Carleson measure)

Denote $R(x,r) = B(x,r) \times (0,r)$. A measure on $\mathbb{R}^n \times (0,\infty)$ is Carleson if

$$\|\mu\|_{\mathcal{C}} := \sup_{x,r} \frac{\mu(R(x,r))}{r^n} < \infty.$$

Dahlberg's questions on A_∞

- Perturbations: small deviation of A from a good coefficient matrix A_0 as

$$\mu(x', x_{n+1}) = \frac{1}{x_{n+1}} \sup_{y \in B(x', x_{n+1}/2)} |A(y) - A_0(y)|^2, \quad \|\mu\|_{\mathcal{C}} < \infty.$$

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- Smoothness:

$$\mu(x', x_{n+1}) = \frac{1}{x_{n+1}} \inf_{A_0} \sup_{y \in B(x', x_{n+1}/2)} |A(y) - A_0(y)|^2, \quad \|\mu\|_{\mathcal{C}} < \infty.$$

  These have elliptic measure in A_∞ (Kenig–Pipher 2001).

Weak Dahlberg–Kenig–Pipher condition

Set

$$\alpha_{A,2}(x,t) = \inf_{A_0} \left(\frac{1}{t^{n+1}} \iint_{(t/2,t)\times B(x,t)} |A(x,s) - A_0|^2 \, dxds \right)^{1/2}.$$

If

$$d\mu_{A,2} = \alpha_{A,2}(x,t)^2 \frac{dtdx}{t}$$

is density of a Carleson measure, then A is a weak **D**-matrix (or DKP).

Small A_∞ constants

- For every $p > 1$ there is $\delta > 0$, so that if $\|\mu_{A,\infty}\|_{\mathcal{C}} < \delta$, then the Dirichlet problem is L^p solvable (Dindos–Petermichl–Pipher 2007).

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- If
$$\lim_{s\to 0} \sup_y \|1_{B(y,s)\times(0,s)}\mu_{A,2}\|_{\mathcal{C}} = 0,$$
then the logarithm of the Poisson kernel is locally in VMO (Bortz–Toro–Zhao 2021).

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- If $\|\mu_{A,2}\|_{\mathcal{C}} < \delta$ for $\delta > 0$ small, then the elliptic measure with pole at infinity w has $\log[w]_{A_\infty} \lesssim \|\mu_{A,2}\|_{\mathcal{C}}^{1/2}$ (Bortz-Egert-S 2021/2022).

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- Chord-arc domains with small constants have harmonic measures with small A_∞ constants (David–Li–Mayboroda 2022).

Proof scheme I: Green's function (D-L-M)

- If A is weak DKP, then A-Green's function (with pole at infinity) is approximable by affine functions (DLM).

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For $(x_0, \lambda_0) \in \mathbb{R}^{n+1}_+$

$$\frac{1}{\lambda_0^{n+1}} \iint_{R(x_0,\lambda_0)} \beta(x,\lambda)^2 \frac{dxd\lambda}{\lambda} \leq C\|\mu_A\|_{\mathcal{C}}$$

where

$$R(x,\lambda) = Q(x,\lambda) \times (0,\lambda)$$

and

$$\beta(x,\lambda)^2 = \frac{\fint\!\!\fint_{R(x,\lambda)} |\nabla u(y,t) - \langle \partial_{n+1} u \rangle_{R(x,\lambda)} e_{n+1}|^2 \, dydt}{\fint\!\!\fint_{R(x,\lambda)} |\nabla u(y,t)|^2 \, dydt}$$

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Fix functions ψ and ϕ where $\phi \geq 1_{Q(0,1/2)}$ and $\phi, \psi \in C_c^\infty(Q(0,1))$. Then there exists $C \geq 1$ such that for all $(x, \lambda) \in \mathbb{R}^{n+1}_+$

$$\frac{\lambda |\nabla(\psi_\lambda * \omega)(x)|}{(\phi_\lambda * \omega)(x)} \leq C \Big(\beta(x, \lambda) + \alpha_A(x, \lambda) \Big)$$

where ω is the A-elliptic measure with pole at infinity.

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Remark:

- By a result of Fefferman–Kenig–Pipher, the Carleson norm of the left hand side being finite implies A_∞.
- By a result of Korey, the vanishing Carleson implies asymptotically flat A_∞.

Proof scheme III: Weights (B-E-S)

Theorem (Bortz–Egert–S 2022)

Let $n \geq 1$ and let $D \geq 1$. Then there exist constants $C, \varepsilon > 0$ such that the following holds. Let w be a weight with doubling constant D and let u_w be its heat extension. Let

$$d\mu(x,t) = |\nabla \log u_w(x,t^2)|^2 t \, dxdt.$$

- *If $\|\mu\|_{\mathcal{C}} < \varepsilon$, then $\log[w]_{A_\infty} \leq C\sqrt{\|\mu\|_{\mathcal{C}}}$.*
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Remark: A Littlewood–Paley decomposition argument allows to control the generic kernel by heat kernel.

Proof of the easy direction: $\log[w]_{A_\infty} \leq C\sqrt{\|\mu\|_{\mathcal{C}}}$

Key estimate: (write $W(x,r) = B(x,r) \times (r/2, r)$)

Lemma

Let w be a doubling weight. There exists a constant C depending only on dimension and the doubling constant such that for all $(x,r) \in \mathbb{R}^{n+1}_+$,

$$\frac{r^2|\Delta u(x,r^2)|}{u(x,r^2)} + \frac{r|\nabla u(x,r^2)|}{u(x,r^2)} \leq C\sqrt{\frac{1}{|\Delta(x,r)|}\iint_{W(x,r)} \frac{|\nabla u(y,t^2)|^2}{u(y,t^2)^2} t\, dydt}.$$

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This is standard interior estimate for heat equation plus

- parabolic **forward** Harnack due to positivity;
- parabolic **backward** Harnack due to doubling of initial data.

Computation

By scaling and translation, it suffices to prove the estimate for the unit ball $B = B(0,1)$.

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&= \Big(\log w_B - \sup_{x\in B} \log u(x,1)\Big) + \Big(\sup_{x\in B} \log u(x,1) - (\log u(\cdot,1))_B\Big) \\
&\qquad + \big((\log u(\cdot,1))_B - (\log w)_B\big) = \mathrm{I} + \mathrm{II} + \mathrm{III}
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\end{aligned}
$$

- III: FTC in t, heat equation, divergence theorem, key lemma.
- II: FTC in x, key lemma.

Bounding $\log w_B - \sup_{x\in B} \log u(x,1)$

It holds

$$\mathrm{I} \leq \log_+ \left(\frac{w_B - (u(\cdot,1))_B}{\sup_{x\in B} u(x,1)} + 1 \right)$$

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$$w_B - (u(1,\cdot))_B = \frac{1}{|B|} \int_B \int_0^1 t\Delta u(x,t^2)\, dxdt$$
$$\leq \frac{1}{|B|} \int_{\partial B} \int_0^1 t|\nabla u(x,t^2)|\, dxdt.$$

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The claim then follows from

$$\frac{\sup_{x\in B} u(x,t)}{\sup_{y\in B} u(y,1)} \leq \sup_{x\in B} \exp\left(- \int_t^1 \partial_s \log u(x,s)\, ds \right) \leq t^{-C\sqrt{\|\mu\|_{\mathcal{C}}}}.$$

Parabolic equations

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Divergence form parabolic equations

$$\begin{aligned} u_t - \operatorname{div} A\nabla u + B \cdot \nabla u = 0, &\quad \text{in } \mathbb{R} \times \mathbb{R}^{n+1}_+ \\ u = g, &\quad \text{on } \partial(\mathbb{R} \times \mathbb{R}^{n+1}_+) \end{aligned}$$

with $A : \mathbb{R} \times \mathbb{R}^{n+1}_+ \to \mathbb{R}^{n\times n}$ satisfying the conditions similar to as in the elliptic case and $B : \mathbb{R} \times \mathbb{R}^{n+1}_+ \to \mathbb{R}^n$ satisfying

$$z_{n+1}|B(t,z)| \leq \varepsilon$$

for a structural parameter $\varepsilon > 0$.

Let $W(z)$ be the Whitney type parabolic cylinder around z: for $z = (t, x, \lambda)$

$$W(z) = (t - \lambda^2, t + \lambda^2) \times Q(x, \lambda/2) \times (\lambda/2, \lambda).$$

Definition (Weak-DKP condition, parabolic)

Let $A : \mathbb{R} \times \mathbb{R}^{1+n}_+ \to \mathbb{R}^{n \times n}$ and $B : \mathbb{R} \times \mathbb{R}^{1+n}_+ \to \mathbb{R}^n$ be locally integrable functions. Define for $z \in R \times \mathbb{R}^{1+n}_+$

$$\alpha_A(z) = \left(\fint_{W(z)} |A(y) - A_0(z)|^2 \, dy \right)^{1/2}, \quad A_0(z) = \fint_{W(z)} A(y) \, dy$$

$$\alpha_B(z) = \left(\fint_{W(z)} |B(y)|^2 y_{n+2}^2 \, dy \right)^{1/2}, \quad \alpha_{A,B}(z) = \alpha_A(z) + \alpha_2(z).$$

Define $\mu_{A,B}(z) = \alpha_{A,B}(z)^2 z_{n+2}^{-1}$. We say that (A, B) satisfies a weak DKP-condition if $\|\mu_{A,B}\|_{\mathcal{C}(E)} < \infty$.

Theorem (Work in progress, Bortz–Egert–S)

Let M_0 be given. There exists $\varepsilon_0 > 0$, $\kappa_0 \geq 1$, $\delta_0 > 0$, $a \geq 1$ and $C \geq 1$ such that the following holds. Let $\delta \in (0, \delta_0)$, $\varepsilon \in (0, \varepsilon_0)$ and consider M_0-elliptic matrix and ε-small drift (previous slide).
Fix $(t_0, x_0, \lambda_0) \in \mathbb{R} \times \mathbb{R}^{1+n}_+$ and set $p_0 = a^+(t_0, x_0, \kappa_0\lambda_0)$. Let ω be the parabolic measure with pole at p_0. Denote

$$d\nu(t, x, \lambda) = \alpha_{A,B}(t, x, \lambda)^2 \frac{dtdxd\lambda}{\lambda},$$

where $\alpha_{A,B}$ is the DKP-quantity.

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where $\alpha_{A,B}$ is the DKP-quantity.
If $\|\nu\|_{\mathcal{C}} \leq \delta$, then $\omega \ll dtdx$ and denoting $k = \frac{d\omega}{dtdx}$, we have for all (t, x, λ) with $R(t, x, 2\lambda) \subset R(t_0, x_0, \lambda_0)$

$$\log \left(\fint\!\!\fint_{Q^{bdry}(t,x;\delta^a\lambda)} k(\tau, y)\, d\tau dy \right) - \fint\!\!\fint_{Q^{bdry}(t,x;\delta^a\lambda)} \log k(\tau, y)\, d\tau dy \leq C\sqrt{\delta}.$$

Thank you for your attention!

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