# Dimension-free Lp estimates for maximal Riesz transforms

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Based on joint work with Maciej Kucharski and Jacek Zienkiewicz

# Notation:

- $d$ - dimension of the space $\mathbb{R}^d$, $L^p = L^p(\mathbb{R}^d)$, $\|\cdot\|_p = \|\cdot\|_{L^p}$
- $k$ - fixed non-negative integer
- $\mathcal{H}_k$ - space of spherical harmonics of degree $k$ on $\mathbb{R}^d$
- For $P \in \mathcal{H}_k$ we define (Kernel of Riesz Transform)

$$\boxed{K_P(x) = \gamma_k \frac{P(x)}{|x|^{k+d}} \quad \text{with} \quad \gamma_k = \frac{\Gamma\left(\frac{k+d}{2}\right)}{\pi^{\frac{d}{2}}\,\Gamma\left(\frac{k}{2}\right)}}$$

- when $k=1$, $P(x) = x_j$, then $K_{x_j}$ - is the kernel of classical first order Riesz-transform $R_{x_j} = R_j$

# Definitions of Riesz transforms - RT

$$R_p^t f(x) = \int_{|y|>t} K_p(y) f(x-y)\,dy$$ (Truncated RT)

$$R_p f(x) = \lim_{t\to 0} R_p^t f(x)$$ (RT)

$$R_p^* f(x) = \sup_{t>0} |R_p^t f(x)|$$ (Maximal RT)

## A classical result (E.M. Stein 80')

For each $1 < p < \infty$ there exists $C(p)$ independent of dimension $d$ and such that

$$\left\| \left( \sum_{i=1}^{d} |R_{x_j} f|^2 \right)^{\frac{1}{2}} \right\|_p \leq C(p) \, \|f\|_p$$

Why care? Consequences:

$$\| |\nabla f| \|_p \leq C_p \, \| (-\Delta)^{\frac{1}{2}} f \|_p$$

$$\left\| \left( \sum_{j,k=1}^{d} |\partial_j \partial_k f|^2 \right)^{\frac{1}{2}} \right\|_p \leq D_p \, \| \Delta f \|_p$$

# Our goals

- Prove a similar result to Stein with $R_{x_j}$ replaced by $R_{x_j}^*$, e.g. $\boxed{\left\| \left( \sum |R_{x_j}^* f|^2 \right)^{\frac{1}{2}} \right\|_p \leq C_p \|f\|_p}$
- Control $\boxed{\|R_P^* f\|_p}$ in terms of $\boxed{\|R_P f\|_p}$ in a dimension-free fashion
- Dependence on $k$- degree of $P$ is allowed
- Get explicit dependence in terms of $p \in (1,\infty)$

## Previous results

- case $k=1$, $1<p<\infty$

$$\boxed{\|R^*_{x_j} f\|_p \leq C(p,d)\, \|R_{x_j} f\|_p, \qquad \text{Mateu, Verdera 2006}}$$

Here $C(p,d) \approx_p 4^d$

- case $k=1$, $p=2$

$$\boxed{\|R^*_{x_j} f\|_2 \leq 2\cdot 10^8\, \|R_{x_j} f\|_2 \qquad \text{Kucharski, Wróbel 2021}}$$

- general odd $k$, $P \in \mathcal{H}_k$, $1 < p < \infty$

$$\boxed{\|R_P^* f\|_p \le C(p,k,d)\, \|R_P f\|_p}$$

Mateu, Orobitg, Pérez, Verdera 2010

$C(p,k,d) \approx_{p,k} 4^d$ → general singular integrals with odd kernels

- general even $k$, $P \in \mathcal{H}_k$, $1 < p < \infty$

$$\boxed{\|R_P^* f\|_p \le C(p,k,d)\, \|R_P f\|}$$

Mateu, Orobitg, Verdera 2011

$C(p,k,d) \approx_{p,k} 4^d$ → general singular integrals with even kernels

## Our results: (Kucharski, W, Zienkiewicz 2022, 2023)

- For single Riesz transform

**THEOREM 1**

Let $k \in \mathbb{N}$. For $1 < p < \infty$ we have

$$\| R_p^* f \|_p \leq C(k) \left( \max\left(p, (p-1)^{-1}\right) \right)^{2+\frac{k}{2}} \| R_p f \|_p,$$

where $C(k)$ depends only on $k$ but not on the dimension $d$

— For vectors of Riesz transforms

**THEOREM 2**

Let $k \in \mathbb{N}$, and let $\mathcal{G}_k$ be a subset of $\mathcal{H}_k$.

Then for $1 < p < \infty$ we have

$$\left\| \left( \sum_{P \in \mathcal{G}_k} |R_P^* f|^2 \right)^{\frac{1}{2}} \right\|_p \leq C(k) \left( \max\left(p, (p-1)^{-1}\right) \right)^{3+\frac{k}{2}} \times$$

$$\times \left\| \left( \sum_{P \in \mathcal{G}_k} |R_P f|^2 \right)^{\frac{1}{2}} \right\|_p .$$

Here $C(k)$ depends only on $k$, and not on $d$

# Possible approaches – case $k=1$

- Cotlar's inequality

$$|R^*_{x_j} f(x)| \leq C(d) \left( M(R_{x_j} f)(x) + Mf(x) \right)$$

Problems: $C(d)$ depends on $d$. Extra term

$Mf(x)$ – Hardy–Littlewood maximal function

- Method of rotations as in Duoandikoetxea – Rubio de Francia

$$R^t_{x_j} f(x) \approx d^{\frac{1}{2}} \int_{S^{d-1}} H^t_\omega f(x)\, \omega_j \, d\omega$$

Is dimension-free & works for $\boxed{\|R^*_{x_j}\|_p \leq C_p \|f\|_p}$

we lose $R_{x_j} f(x)$

# Our approach – case $k=1$ – TWO STEPS

1. FACTORIZATION

$$R^t_{x_j} = M^t R_{x_j}$$

where $M^t = M^t_1$ – operators with radial kernel

2. METHOD OF ROTATIONS

$$R^t_{x_j} f(x) \approx \sqrt{d} \int_{S^{d-1}} \omega_j \, H^t_\omega f(x) \, d\omega$$

where $H^t_\omega$ – directional truncated Hilbert transform

# Sketch of TWO STEPS

## 1. FACTORIZATION

– (Mateu, Orobitg, Pérez, Verdera 2011) (implicitly) proved that

$$\boxed{R^t_{x_j} = M^t R_{x_j}, \quad j = 1, \dots, d}$$

$(M^t f)^\wedge(\xi) = m(t\xi)\hat{f}(\xi)$, $m$ - radial and bounded

$M^t$ - bounded on $L^p$, $1 < p < \infty$.

— Control of $R^*_{x_j}$ in terms, of $R_{x_j}$ is reduced to bounding $\boxed{M^* f(x) = \sup_{t>0} |M^t f(x)|}$

— For single Riesz transform we are left with

$$\boxed{\|M^* f\|_p \leq C\left(\max\left(p, (p-1)^{-1}\right)\right)^{\frac{5}{2}} \|f\|_p}$$

Where $1 < p < \infty$, and $C$ - is independent of the dimension $d$

— $M^t$ has a useful explicit formula

$$M^t R_{x_j} R_{x_j} = R_{x_j}^t R_{x_j}$$

$$M^t \sum_{j=1}^{d} (R_{x_j})^2 = \sum_{j=1}^{d} R_{x_j}^t R_{x_j}$$

$$M^t = - \sum_{j=1}^{d} R_{x_j}^t R_{x_j},$$

because $\sum_{j=1}^{d} (R_{x_j})^2 = - Id$

— caveat: this does not work for $k>1$.

- A curiosity (Liu, Melentijević, Zhu 2023)

$$\boxed{M^t f = \varphi_t * f}$$

where the kernel $\varphi_t$ is

- non-negative and radial
- $\|\varphi_t\|_{L^1} = 1$
- $\varphi_t(x)$ increases for $|x| < t$ and decreases for for $|x| > t$
- $\varphi_t(x) = \infty$ for $|x| = t$

## 2. METHOD OF ROTATIONS. (Sketch)

- We focus on the estimate for single Riesz transform

- The goal is to bound $\boxed{\|M^* f\|_p}$. We do not focus on dependence on p. Only on independence of d

- By method of rotations and factorization

$$\boxed{M^t f = -\sum_{j=1}^{d} R_j^t R_j f \approx -\sqrt{d} \int_{S^{d-1}} H_\omega^t \Big(\sum_j \omega_j R_j f\Big) d\omega}$$

– Minkowski's inequality gives

$$\boxed{\|M^* f\|_p \lesssim \sqrt{d} \int_{S^{d-1}} \|H_\omega^* (\sum_j \omega_j R_j f\|_p \, d\omega}$$

– Because $H_\omega^*$ is one-dimensional after Hölder and reverse Hölder on $S^{d-1}$ we get

$$\boxed{\|M^* f\|_p \le C_p \left\| \left(\sum_j |R_j f|^2\right)^{\frac{1}{2}} \right\|_p}$$

— By the classical result of Stein

$$\boxed{\left\| \left( \sum_j |R_j f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^d)} \le C_p \|f\|_{L^p(\mathbb{R}^d)}}$$

We are done

Remark: A similar, but more complicated argument works for the vectorial estimate. Extra ingredient: Khintchine inequality.

# CASE $k>1$, odd — THREE STEPS

1. FACTORIZATION, $\rho \in H_k$

$$R_\rho^t = M_k^t R_\rho$$

where $M^t = M_k^t$ – operators with radial kernel

2. AVERAGING

3. Only then METHOD OF ROTATIONS

# Extra ingredient for $\boxed{k>1}$ — AVERAGING

— Why is it needed?

Say $k=3$. Then

$$\boxed{-I = \sum_{i=1}^{d} \sum_{j=1}^{d} \sum_{l=1}^{d} R_{x_i}^2 R_{x_j}^2 R_{x_l}^2}$$

But a number of $R_{x_i} R_{x_j} R_{x_l}$ are not higher order Riesz-transforms...

For instance $R_{x_1}R_{x_1}R_{x_1}$ would correspond to $P(x)=x_1^3$

which is not in $\mathcal{H}_3$ (not harmonic)

— Moreover, the FACTORIZATION formula

$$\boxed{R^t_{x_i x_j x_\ell} = M_3^t\left(R_{x_i} R_{x_j} R_{x_\ell}\right)}$$

works only when $i, j, \ell$ — pairwise distinct

- How to remedy the problem?

- average over the $SO(d)$ group.

Denote $I = \{(i,j,\ell) \in \{1,\dots,d\}^3 : \text{pairwise distinct}\}$. Then

$$\boxed{I_d = C(d,k) \int_{SO(d)} \mathcal{C}_U \Big( \sum_{n \in I} \big( R_{x_{n_1} x_{n_2} x_{n_3}} \big)^2 \Big)(x) \, d\nu(U)}$$

where $C(d,k) \simeq 1$, uniformly in $d$, $d\nu$ - Haar measure on $SO(d)$, $\mathcal{C}_U$ - conjugation by $U \in SO(d)$

— This procedure in effect reduces our task to controlling the maximal function for

$$\boxed{R^t f = \sum_{n \in I} R^t_{x_{n_1} x_{n_2} x_{n_3}} R_{x_{n_1} x_{n_2} x_{n_3}}}$$

because $M^t$ has a radial symbol and

$$\boxed{M^t \left( \sum_{n \in I} \left( R_{x_{n_1} x_{n_2} x_{n_3}} \right)^2 \right) = \sum_{n \in I} M^t \left( R_{x_{n_1} x_{n_2} x_{n_3}} \right)^2 = R^t}$$

— In the case $k=1$ we had $R^t = M^t$

Case $k>1$ - even. FIVE STEPS

1. FACTORIZATION

2. AVERAGING

3. EXTENSION TO $\mathbb{C}^d$

4. COMPLEX METHOD OF ROTATIONS
   (IWANIEC AND MARTIN)

5. RESTRICTION TO $\mathbb{R}^d$

## Why are extra steps needed?

- (Real) method of rotations requires that the kernel $K^t_\rho$ or $K_\rho$ is odd

- When $k$ is even the kernel is even

- A curiosity: only easy case $k=2$, then

$$\boxed{M^t_2 = \text{Hardy-Littlewood averaging operator}}$$

# Extra ingredients for k-even

**STEP 3** | Extension to $\mathbb{C}^d$

- replace $K_p^t$ and $K_p$ – functions on $\mathbb{R}^d$

  with their appropriate complexifications

  $\tilde{K}_p^t$ and $\tilde{K}_p$ – functions on $\mathbb{C}^d \cong \mathbb{R}^{2d}$

- for instance

$$\tilde{K}_p(x+iy) = \tilde{\gamma}_k \frac{P(x+iy)}{|x+iy|^{2d+k}} \qquad \tilde{\gamma}_k = \frac{\Gamma\left(\frac{k}{2}+d\right)}{\pi^d \, \Gamma\left(\frac{k}{2}\right)}$$

## STEP 4. Complex method of rotations

$$\widehat{R}_p^t \tilde{f} = \widetilde{K}_p *_{\mathbb{R}^{2d}} \tilde{f} \approx d^{\frac{k}{2}} \int_{S^{2d-1}} P(\omega + i\sigma) H_{\zeta,k}^t \tilde{f} \, d\zeta,$$

- $\zeta = \omega + i\sigma \in \mathbb{C}^d$

- $d\zeta$ - normalized surface measure on $S^{2d-1}$

— $H^t_{\zeta,k}$ — $k$-th power of the complex Hilbert transform truncated at $t>0$

$$\boxed{H^t_{\zeta,k} f(z) = \int_{\mathbb{C}} 1_{|\lambda|\geq t} \left(\frac{\lambda}{|\lambda|}\right)^k \frac{1}{|\lambda|^2} f(z-\lambda\zeta)\, d\lambda}$$

— $H^t_{\zeta,k}$ is a two-dimensional truncated singular integral operator

## STEP 5 | Restriction to $\mathbb{R}^d$

— there is a standard procedure to go back to $\mathbb{R}^d$

→ integration of $\tilde{K}_p^t(x+iy)$ over $y \in \mathbb{R}^d$

— Problem: we do not get back to $K_p^t(x)$...

Instead we obtain a different kernel $\mathcal{K}_p^t(x)$

— Fortunately their difference can be estimated

## General singular integrals?

What about general singular integrals with even or odd kernels $\Omega$? This is more subtle ... If

$$T_\Omega f(x) = \int_{\mathbb{R}^d} \frac{\Omega\left(\frac{y}{|y|}\right)}{|y|^{d+1}} f(x-y)\,dy, \quad T_\Omega^* f(x) = \sup_{t>0} \left| \int_{|x-y|>t} \frac{\Omega\left(\frac{y}{|y|}\right)}{|y|^{d+1}} f(x-y)\,dy \right|$$

Then $\|T_\Omega^* f\|_2 \leq C \|T_\Omega f\|_2$ may only hold if $\Omega$ satisfies an algebraic condition related to its expansion in spherical harmonics (Mateu, Orobitg, Pérez, Verdera 2010, 2011)

Thank you for your attention!