Explicit Reciprocity Laws in Number Theory

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The Quadratic Reciprocity Law

ERICH HECKE (1923):

> *Modern number theory dates from the discovery of* **the reciprocity law.** *By its form it still belongs to the theory of rational numbers, as it can be formulated entirely as a simple relation between rational numbers; however its content points beyond the domain of rational numbers.*

EMMA LEHMER (1978):

> *... has been generalized over the years ... to the extent that it has become virtually unrecognizable.*

Gauß' Reciprocity Law

The diophantine equation

$$X^2 + pY = a$$

for $a \in \mathbb{Z}$ and an odd prime p with $(p, a) = 1$ has a solution in $\mathbb{Z}^2$ if and only if

$$X^2 = \bar{a} \in \mathbb{F}_p^\times$$

has a solution in $\mathbb{F}_p$, i.e., if a is a square there.

Adrien-Marie Legendre (1752-1833)

Carl Gustav Jacob Jacobi (1804-1851)

LEGENDRE/JACOBI-Symbol

$$\left(\frac{a}{p}\right) := \begin{cases} 1, & \bar{a} \in (\mathbb{F}_p^\times)^2; \\ -1, & \text{otherwise.} \end{cases}$$

EULER'S-criterion

$$\equiv a^{\frac{p-1}{2}} \mod p$$

because $\mathbb{F}_p^\times$ is cyclic of order $p-1$ and hence the following sequence is exact:

$$0 \longrightarrow (\mathbb{F}_p^\times)^2 \longrightarrow \mathbb{F}_p^\times \xrightarrow{\frac{p-1}{2}} \{1,-1\} \longrightarrow 0$$

where

$$\mu_2 = \{-1,1\} \subseteq \mathbb{F}_p^\times.$$

Johann Carl Friedrich Gauß (1777-1855)

Reciprocity Law I (GAUSS 1801) :

$l \neq p$ odd prime. Then:

$$\left(\frac{l}{p}\right) = (-1)^{\frac{l-1}{2}\frac{p-1}{2}}\left(\frac{p}{l}\right)$$

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Slogan:
l is a square modulo p if and only if p is a square modulo l - unless $l \equiv p \equiv 3 \mod 4$.

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Supplement: $\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}, \quad \left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}$

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Even more general the Reciprocity Law holds for odd, pairwise coprime natural numbers m, n instead of l, p.
An equivalent formulation: $l^* := (-1)^{\frac{l-1}{2}} l$:

$$\left(\frac{l^*}{p}\right) = \left(\frac{p}{l}\right).$$

One of more than 150 proofs

Easy calculation: $Fr_q(\sqrt{p^*}) = \left(\frac{p^*}{q}\right)\sqrt{p^*}$

Fr_q

$$\begin{array}{ccccccccc}
0 & \longrightarrow & G(\mathbb{Q}(\zeta_p)/\mathbb{Q}(\sqrt{p^*})) & \longrightarrow & G(\mathbb{Q}(\zeta_p)/\mathbb{Q}) & \longrightarrow & G(\mathbb{Q}(\sqrt{p^*})/\mathbb{Q}) & \longrightarrow & 0 \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
0 & \longrightarrow & (\mathbb{F}_p^\times)^2 & \longrightarrow & \mathbb{F}_p^\times & \xrightarrow{\left(\frac{\cdot}{p}\right)} & \{1,-1\} & \longrightarrow & 0
\end{array}$$

$\bar{q}$

$$\left(\frac{p^*}{q}\right) = 1 \Leftrightarrow (Fr_q)_{|\mathbb{Q}(\sqrt{p^*})} = \mathrm{id} \Leftrightarrow \bar{q} \in (\mathbb{F}_p^\times)^2 \Leftrightarrow \left(\frac{q}{p}\right) = 1$$

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$v \in \{p \| \text{prime}\} \cup \{\infty\}$		places of $\mathbb{Q}$.

Quadratic Hilbert symbol

David Hilbert (1862-1943)

$$\left(\frac{m,n}{v}\right) = \begin{cases} 1, & \text{if } mX^2 + nY^2 = Z^2 \text{ has non-trivial solution in } \mathbb{Q}_v; \\ -1, & \text{otherwise.} \end{cases}$$

defines symmetric, non-degenerate pairing

$$\left(\frac{-,-}{v}\right) : \mathbb{Q}_v^\times/(\mathbb{Q}_v^\times)^2 \times \mathbb{Q}_v^\times/(\mathbb{Q}_v^\times)^2 \to \mu_2.$$

Quadratic Hilbert symbol

HENSEL'S Lemma: Solvability of $X^2 = \overline{a} \in \mathbb{F}_p^\times \iff$
Solvability of $X^2 \equiv a \mod p^n,\ n \geq 1 \iff$
Solvability of $X^2 = a \in \mathbb{Z}_p$

implies

$$\left(\frac{p,q}{p}\right) = \left(\frac{q}{p}\right)$$

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The laws I and II are equivalent, for example II $\Rightarrow$ I:

$$1 = \prod_v \left(\frac{p,q}{v}\right) = \left(\frac{p,q}{2}\right)\left(\frac{p,q}{p}\right)\left(\frac{p,q}{q}\right) = (-1)^{\frac{q-1}{2}\frac{p-1}{2}}\left(\frac{q}{p}\right)\left(\frac{p}{q}\right)$$

Different concepts "Reciprocity Laws"

Gauß' Reciprocity Law is the beginning of **class field theory**, which classifies and describes all abelian extensions of global or local fields. The Artinian Reciprocity map can be considered as generalisation of it, for example in the case of local field extensions K/F we have

$$(-, K/F) : GL_1(F) = F^\times \to G(K/F)^{ab}.$$

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In this talk we focus rather on generalisations of the quadratic Hilbert symbol:

$\mu_n \subseteq F/\mathbb{Q}_p$ finite, L/F maximal abelian extension of exponent n. The *Kummer sequence*

$$1 \longrightarrow \mu_n \longrightarrow \bar{F}^\times \xrightarrow{\cdot^n} \bar{F}^\times \longrightarrow 1$$

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$$\begin{array}{ccccc}
G(L/F) & \times \mathrm{Hom}(G(L/F), \mu_n) & \xrightarrow{\text{Pontrjagin}} & \mu_n \\
\uparrow (-,L/F) & \uparrow \delta & & \| \\
F^\times/(F^\times)^n \quad \times & F^\times/(F^\times)^n & \xrightarrow{\left(\frac{\ ,\ }{F}\right)_n} & \mu_n \\
\downarrow \delta & \downarrow \delta & & \| \\
H^1(F,\mu_n) \quad \times & H^1(F,\mu_n) & \xrightarrow{\ \cup\ } & H^2(F,\mu_n^{\otimes 2}) \cong H^2(F,\mu_n)\otimes\mu_n \cong \mu_n
\end{array}$$

$$\left(\frac{a,b}{F}\right)_n := \frac{(a,L/K)(\sqrt[n]{b})}{\sqrt[n]{b}}$$

For $F = \mathbb{Q}_p$, $n = 2$ we have: $\left(\frac{\ ,\ }{\mathbb{Q}_p}\right)_2 = \left(\frac{\ ,\ }{p}\right)$.

Variants: Schmidt-Witt-Residue-Formula

$\mathrm{char}(F) = p$, e.g. $F = \mathbb{F}_p((Z))$. The *Artin-Schreier* sequence

$$0 \longrightarrow \mathbb{F}_p \longrightarrow F^{\mathrm{sep}} \xrightarrow{\wp} F^{\mathrm{sep}} \longrightarrow 0$$

with $\wp(x) := (Fr - \mathrm{id})(x) = x^p - x$ induces

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$$(\ ,\ ] : F^\times/(F^\times)^p \times F/\wp(F) \to \mathbb{F}_p.$$

$$(a, b] = \mathrm{Res}\left(b\frac{da}{a}\right)$$

with $\mathrm{Res}((\sum_i a_i Z^i)dZ) = a_{-1}$.

Variants: (Lubin-Tate) formal groups

$\mathcal{F}$ (Lubin-Tate) formal groups over $F/\mathbb{Q}_p$ attached to prime π. The sequence

$$0 \longrightarrow \mathcal{F}[\pi^n] \longrightarrow \mathcal{F}(\mathfrak{m}_{\bar{F}}) \xrightarrow{[\pi^n]} \mathcal{F}(\mathfrak{m}_{\bar{F}}) \longrightarrow 0$$

induces

$$(\ ,\ )_{\mathcal{F},n} : F^\times \times \mathcal{F}(\mathfrak{m}_F) \to \mathcal{F}[\pi^n].$$

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Let q be the cardinality of $O_F/\pi_F O_F$ and consider the unique decomposition

$$O_F^\times \cong \mu_{q-1} \times (1 + \pi_F O_F), \quad u \mapsto \omega(u)\langle u\rangle;$$

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For $a, b \in F^\times$ such that $\alpha = v_F(a)$, $\beta = v_F(b)$ it holds:

$$\left(\frac{a,b}{F}\right)_n = \omega\left((-1)^{\alpha\beta}\frac{b^\alpha}{a^\beta}\right)^{\frac{q-1}{n}}$$

Higher power residue symbol

In particular, for $a = \pi_L$ and $u \in O_F^\times$:

$\left(\frac{\pi_F, u}{F}\right)_n = \omega(u)^{\frac{q-1}{n}}$ is the root of unity $\alpha \in \mu_n$ determined by $\alpha \equiv u^{\frac{q-1}{n}} \mod \pi_F O_F$, i.e., we have:

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$$\left(\frac{\pi_F, u}{F}\right)_n = 1 \iff u \text{ is a } n\text{th power} \mod \pi_F O_F.$$

(Generalisation of Legendre symbols $\left(\frac{a}{p}\right)$!)

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$Tr = Tr_{K_n/\mathbb{Q}_p} = \sum_{\sigma\,\epsilon\,G(K_n/\mathbb{Q}_p)} \sigma$	trace

Artin-Hasse 1928

For $\beta \in 1 + \pi_n o_{K_n}$ it holds:

$$(\zeta_{p^n}, \beta)_{p^n} = \zeta_{p^n}^{\frac{1}{p^n} Tr(\log \beta)}$$

$$(\beta, \pi_n)_{p^n} = \zeta_{p^n}^{\frac{1}{p^n} Tr(\frac{\zeta_{p^n}}{\pi_n} \log \beta)}$$

Emil Artin (1898-1962),
Helmut Hasse (1898-1979)

Iwasawa 1968

For $\beta = (\beta_k) \in \varprojlim_{k, Norm} K_k^\times$,

$g_\beta \in \mathbb{Z}_p[[T]]$ with $g_\beta(\pi_n) = \beta_n$ and

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with invariant logarithmic derivation

$$D \log \beta = \left((1+T) \frac{g'_\beta(T)}{g_\beta(T)} \right)_{|T=\pi_n}.$$

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Cyclotomic setting:	$L = \mathbb{Q}_p$, $\pi = p$, $\mathcal{F} = \hat{\mathbb{G}}_m$

Wiles 1978

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Sir Andrew John Wiles (1953-)

Wiles 1978

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$$(\beta_n, \alpha)_{\mathcal{F},n} = [\frac{1}{\pi^n} Tr\,(\log_{\mathcal{F}}(\alpha) D \log g_\beta(\eta_n))]_{\mathcal{F}}(\eta_n)$$

with invariant logarithmic derivation $D \log g_\beta = \frac{1}{\log'_{\mathcal{F}}} \frac{g'_\beta}{g_\beta}$.

Iwasawa-cohomology

$K \subseteq K_1 \subseteq \cdots \subseteq K_n \subseteq \cdots \subseteq K_\infty = \bigcup K_n$ Galois tower with

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$$H^i_{Iw}(T) := \varprojlim_n H^i(K_n, T) \cong H^i(K, \Lambda(\Gamma) \otimes_{\mathbb{Z}_p} T) \quad \circlearrowleft \Lambda(\Gamma)$$

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$H^i_{Iw}(V) := H^i_{Iw}(T) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ in Iwasawa Main Conjecture

Kato 1991

The *equivariant Coates-Wiles homomorphisms* for $j, m \geq 1$ are defined by

$$\psi^j_{\mathrm{CW},m} : \varprojlim_n L_n^\times \to L_m; \; u \mapsto \frac{1}{j!\pi_L^{mj}} \left(\partial^j_{\mathrm{inv}} \log(g_{u,\eta})\right)\Big|_{Z=\eta_m}$$

with $\partial^j_{\mathrm{inv}} \log(g_{u,\eta}) = \partial^{j-1}_{\mathrm{inv}} \left(\partial_{\mathrm{inv}}(g_{u,\eta})/g_{u,\eta}\right)$.

Consider the KUMMER map

$$\varprojlim_n L_n^\times \xrightarrow{\kappa} H^1_{\mathrm{Iw}}(L_\infty/L, \mathbb{Z}_p(1)),$$

the *"twisting" map*, for $j \geq 1$,

$$\mathrm{tw}_j : H^1_{\mathrm{Iw}}(L_\infty/L, \mathbb{Z}_p(1)) \to H^1_{\mathrm{Iw}}(L_\infty/L, T^{\otimes(-j)}(1)),$$

induced by the cup product with $\eta^{\otimes(-j)} \in T^{\otimes(-j)}$,
the *projection=corestriction map*, for $m \geq 1$,

$$\mathrm{pr}_m : H^1_{\mathrm{Iw}}(L_\infty/L, -) \to H^1(L_m, -)$$

and the BLOCH-KATO **dual exponential map**

$$\exp_j^* := \exp^*_{L_m, V^{\otimes(-j)}(1)} : H^1(L_m, T^{\otimes(-j)}(1)) \xrightarrow{\exp_j^*} L_m.$$

We define the second map $\lambda_{m,j} : \varprojlim_n L_n^\times \to L_m$, as the composite

$$\begin{array}{ccc}
\varprojlim_n L_n^\times & & \\
\downarrow{\scriptstyle \kappa} & \searrow^{\lambda_{m,j}} & \\
H^1_{\mathrm{Iw}}(L_\infty/L, \mathbb{Z}_p(1)) & & \\
\downarrow{\scriptstyle \mathrm{tw}_j} & & \\
H^1_{\mathrm{Iw}}(L_\infty/L, T_\pi^{\otimes -j}(1)) & & \\
\downarrow{\scriptstyle \mathrm{pr_m}} & & \\
H^1(L_m, T_\pi^{\otimes -j}(1)) & \xrightarrow{\exp^*} & D^0_{\mathrm{dR},L_m}(V_\pi^{\otimes -j}(1)) \cong L_m,
\end{array}$$

Kato 1991

Then Kato's explicit reciprocity law for Lubin-Tate formal groups is stated as follows.

Theorem (Kato's explicit reciprocity law)

For any $j, m \geq 1$ *and* $u = (u_n)_n \in \varprojlim_n L_n^\times$, *we have*

$$\lambda_{m,j}(u) = j \cdot \psi^j_{\mathrm{CW},m}(u).$$

Kato gave a proof using syntomic cohomology.

Is there also a proof via (φ, Γ_L)**-modules?**

Further formulas by:

BRÜCKNER, G. HENNIART, KOLYVAGIN, KUMMER, VOSTOKOV, COLEMAN, SEN, DE SHALIT, FESENKO, BLOCH-KATO, BENOIS, ABRASHKIN, ...

Perrin-Riou's Reciprocity Law

Explicit computation of

Tate's local cup-product pairing
or
Iwasawa-cohomology pairing

by means of the Big Exponential map $\Omega_{V^*(1)}$ (and the regulator map $\mathcal{L}_V$) for

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V crystalline representation of $G_{\mathbb{Q}_p}$

Colmez 1998, Benois 1998, Kato-Kurihara-Tsuji (unpublished)

Cyclotomic setting: $D(\Gamma, \mathbb{Q}_p)$ distribution algebra

$$\begin{array}{ccccc}
H^1(K_n, V^*(1-j)) & \times & H^1(K_n, V(j)) & \xrightarrow{<,>_{Tate}} & H^2(K_n, \mathbb{Q}_p(1)) \cong \mathbb{Q}_p \\
\uparrow pr_n & & \uparrow pr_n & & \uparrow (-1)^j ev_{\chi_{trivial}} \\
D(\Gamma, \mathbb{Q}_p) \otimes_\Lambda H^1_{Iw}(V^*(1)) & \times & D(\Gamma, \mathbb{Q}_p) \otimes_\Lambda H^1_{Iw}(V) & \xrightarrow{\{,\}_{Iw}} & D(\Gamma, \mathbb{Q}_p) \\
\uparrow \Omega_{V^*(1)} & & \uparrow \Omega_V & & \| \\
D(\Gamma, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} D_{cris}(V^*(1)) & \times & D(\Gamma, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} D_{cris}(V) & \xrightarrow{[,]_{D_{cris}}} & D(\Gamma, \mathbb{Q}_p)
\end{array}$$

Adjunction of Big Exponential and regulator map

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(φ, Γ)-modules

$\mathbf{A}_L := \widehat{o_L[[Z]][\frac{1}{Z}]}$ p-adic completion with commuting actions by $\varphi = [\pi_L]^*$ and Γ_L via $[\chi_{LT}(\gamma)]^*$.

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$\Phi\Gamma(\mathbf{A}_L)$ category of (φ, Γ)-modules M consisting of finitely generated $\mathbf{A}_L$-modules with commuting (semi-linear) actions by some φ_M and by Γ_L.

ψ (almost) left inverse operator satisfying $\psi \circ \varphi_L = \frac{q}{\pi_L}$, with $q := \#(o_L/\pi_L o_L)$

SCHNEIDER-VENJAKOB: The above mentioned version of the Schmid-Witt-Formula for ramified Witt-vectors (of finite length) gives explicit determination of elements in Iwasawa-cohomology (in Lubin-Tate setting) as images under the Kummer map

$$\kappa : \varprojlim_{n,m} L_n^\times / L_n^{\times p^m} \xrightarrow{\cong} H^1_{Iw}(L_\infty/L, \mathbb{Z}_p(1)) \ ,$$

by means of (φ, Γ_L)-modules:

Explizit Reciprocity Formula I

$T^* = o_L\eta^*$ dual of Tate module $T := \varprojlim_n \mathcal{F}[\pi_L^n] = o_L\eta$,

$\tau := \chi_{cyc}\chi_{LT}^{-1}$

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$$\nabla : (\varprojlim_n L_n^\times) \otimes_{\mathbb{Z}} T^* \longrightarrow \mathbf{A}_L^{\psi=1}, u \otimes a\eta^* \longmapsto a\frac{\partial_{\mathrm{inv}}(g_{u,\eta})}{g_{u,\eta}}(\iota_{LT}(\eta)) .$$

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Theorem (*cycl:* Benois, Colmez | *LT:* Schneider-Venjakob)

The followiing diagram is commutative

$$\begin{array}{ccc} (\varprojlim_n L_n^\times) \otimes_{\mathbb{Z}} T^* & \xrightarrow{\kappa\otimes T^*} & H^1_{Iw}(L_\infty/L, o_L(\tau)) \\ & \searrow^{\nabla} & \cong \big\downarrow Exp^* \\ & & \mathbf{A}_L^{\psi=1} = D_{LT}(o_L)^{\psi=1} \end{array} \ .$$

BREAK

Colmez' abstract reciprocity law

$$V \leftrightarrow D = D(V) \in \Phi\Gamma(\mathbf{A}_{\mathbb{Q}_p})$$

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$$\mathbf{A}_{\mathbb{Q}_p}(\Gamma) := \widehat{\Lambda(\Gamma)[\frac{1}{\gamma-1}]} \quad p\text{-adic completion}$$

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The canonical pairing

$$\check{D} := Hom_{\mathbf{A}_{\mathbb{Q}_p}}(D, \Omega^1) \times D \to \Omega^1$$

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and is used by Colmez to study locally algebraic vectors in order to compare the p-adic with the classical local Langlands correspondence (for $GL_2(\mathbb{Q}_p)$)!

BREAK

The usual Robba ring

$L \subseteq K$ complete field, $\Omega \in K$

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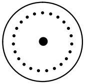

$\mathcal{R} \subseteq K[[Z, Z^{-1}]]$ analytic functions, which converge on $r \leq |Z| < 1$ for some $0 < r < 1$.

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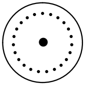

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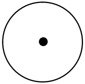

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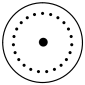

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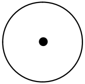

$\gamma \in \Gamma_L$, φ_L act on $\mathcal{R}, \mathcal{R}^+$ via $Z \mapsto [\chi_{LT}(\gamma)](Z)$ and $[\pi_L](Z)$, respectively.

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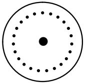

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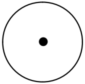

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Galois representations

$\mathrm{Rep}_{o_L,f}(G_L)$	finitely generated free o_L-modules with a continuous linear G_L-action
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$\Phi\Gamma^{an}_{\mathcal{R}_L}$	(φ_L, Γ_L)-modules with **L-linear** *Lie*(Γ_L)-action

Equivalence of categories (KISIN-REN/FONTAINE, BERGER)

$$Rep_L^{an}(G_L) \longleftrightarrow \Phi\Gamma_{\mathcal{R}_L}^{an,ét}$$
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Without ^{an} false for $L \neq \mathbb{Q}_p$!

A Robba ring version in the Lubin-Tate setting

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$M, \check{M} := \mathrm{Hom}_{\mathcal{R}}(M, \Omega^1_{\mathcal{R}}) \cong \mathrm{Hom}_{\mathcal{R}}(M, \mathcal{R})(\chi_{LT}) = D^{\dagger}_{\mathrm{rig}}(V^*(1))$ in $\Phi\Gamma^{an}(\mathcal{R})$ over the Robba ring $\mathcal{R}$

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BERGER's comparison isomorphism:

$$\mathrm{comp}_{\check{M}} : \mathcal{R}[\frac{1}{t_{LT}}] \otimes_{\mathcal{R}} \check{M} \xrightarrow{\cong} \mathcal{R}[\frac{1}{t_{LT}}] \otimes_L D_{cris,L}(V^*(1))$$

Abstract Reciprocity formula in Lubin-Tate setting

SCHNEIDER-VENJAKOB: Then

$$
\begin{array}{ccccccc}
\check{M}^{\psi_L=0} & \times & M^{\psi_L=0} & \xrightarrow{\{,\}'_{M,Iw}} & \mathcal{R}(\Gamma_L) & \xrightarrow[\cong]{\mathfrak{M}} & \mathcal{R}_L^{\psi=0} \\
\| & & \| & & & & \downarrow d \\
\check{M}^{\psi_L=0} & \times & M^{\psi_L=0} & \longrightarrow & & & (\Omega^1_{\mathcal{R}})^{\psi=0} \\
\updownarrow \mathrm{comp}_{\check{M}} & & \updownarrow \mathrm{comp}_{M} & & & & \updownarrow \mathrm{comp}_{\Omega^1} \\
\mathcal{R}^{\psi_L=0} \otimes_L D_{cris,L}(V^*(1)) & \times & \mathcal{R}^{\psi_L=0} \otimes_L D_{cris,L}(V(\tau^{-1})) & & \xrightarrow{[,]_{D_{cris,L}(V(\tau^{-1}))}} & & \mathcal{R}^{\psi_L=0} \otimes_L D_{cris,L}(L(\chi_{LT}))
\end{array}
$$

commutes up to inverting $t_{LT} := \log_{LT}$ as a consequence of ...

Serre duality: Comparing additive and multiplicative residuum maps

... the commutativity of

$$\begin{array}{ccc} \mathcal{R}_L(\Gamma_L) := \mathcal{R}_L(\mathfrak{X}^\times) & \xrightarrow[\cong]{\cdot d\log_{\mathfrak{X}^\times}} & \Omega^1_{\mathfrak{X}^\times} \\ \downarrow (-)(\mathrm{ev}_1 d\log_{\mathfrak{X}}) \cong & & \downarrow \mathrm{res}_{\mathfrak{X}^\times} \\ & & L \\ & & \uparrow \mathrm{res}_{\mathfrak{X}} \\ (\Omega^1_{\mathfrak{X}})^{\psi=0} & \xrightarrow{\mathrm{ev}_{-1}\cdot} & \Omega^1_{\mathfrak{X}}, \end{array}$$

with SCHNEIDER-TEITELBAUM's character varieties $\mathfrak{X}, \mathfrak{X}^\times$ for the groups $o_L, o_L^\times$ using their Fourier theory and Lubin-Tate isomorphism $\mathfrak{X} \cong \mathbb{B}$ over $\mathbb{C}_p$.

Reciprocity formula in the Lubin-Tate setting

$\Gamma_L := G(L_\infty/L)$, $D(\Gamma_L, \mathbb{C}_p)$ locally L-analytic distributions.

If $\mathrm{Fil}^{-1} D_{cris,L}(V^*(1)) = D_{cris,L}(V^*(1))$ and $D_{cris,L}(V^*(1))^{\varphi_L=\pi_L^{-1}} = D_{cris,L}(V^*(1))^{\varphi_L=1} = 0$, then the following diagram commutes:

$$\begin{array}{ccccc}
D^\dagger_{\mathrm{rig}}(V^*(1))^{\psi_L=\frac{q}{\pi_L}} & \times & D_{LT}(V(\tau^{-1}))^{\psi_L=1} & \xrightarrow{\{,\}_{Iw}} & D(\Gamma_L,\mathbb{C}_p) \\
\uparrow \Omega_{V^*(1),1} & & \downarrow \mathbf{L}_V & & \| \\
D(\Gamma_L,\mathbb{C}_p)\otimes_L D_{cris,L}(V^*(1)) & \times & D(\Gamma_L,\mathbb{C}_p)\otimes_L D_{cris,L}(V(\tau^{-1})) & \xrightarrow{[,]} & D(\Gamma_L,\mathbb{C}_p).
\end{array}$$

Consider the map

$$\Theta^*_{W,n} : K \otimes_L L_n \otimes_L D_{\mathrm{cris},L}(W) \to K[G_n] \otimes_L D_{\mathrm{cris},L}(W)$$

characterized by

$$(\mathrm{pr}_\rho \otimes \mathrm{id}) \circ \Theta^*_{W,n}(x) = \begin{cases} (1-\pi_L^{-1}\varphi_L^{-1})^{-1}(1-\frac{\pi_L}{q}\varphi_L)\mathrm{Tr}_{L_n/L}(x) & \text{if } a(\rho)=0, \\ \tau(\rho)^{-1}\pi_L^{a(\rho)}\varphi_L^{a(\rho)}\sum_{g\,\epsilon\, G_{a(\rho)}}\rho(g)g^{-1}(\mathrm{Tr}_{L_n/L_{a(\rho)}}(x)) & \text{if } a(\rho)\geq 1 \end{cases}$$

for a character ρ of $G_n := \Gamma_L/\Gamma_n$ and $x \,\epsilon\, L_n \otimes_L D_{\mathrm{cris},L}(W)$ and with $\mathrm{pr}_\rho : K[G_n] \to K$ the ring homomorphism being induced by ρ.

Let $y_{\chi_{\mathrm{LT}}^{-j},n}$ denote the image of $y \in H^1_{\mathrm{Iw}}(L_\infty/L, T)$ under

$$H^1_{\mathrm{Iw}}(T) \xrightarrow{\mathrm{tw}_j} H^1_{\mathrm{Iw}}(T(\chi_{\mathrm{LT}}^{-j})) \xrightarrow{\mathrm{cor}} H^1(L_n, T(\chi_{\mathrm{LT}}^{-j})) \to H^1(L_n, V(\chi_{\mathrm{LT}}^{-j})).$$

The map

$$\widetilde{\exp}^*_{L_n,V(\chi_{\mathrm{LT}}^{-j}),\mathrm{id}} : H^1(L_n, V(\chi_{\mathrm{LT}}^{-j})) \to L_n \otimes_L D_{\mathrm{cris},L}(V(\tau^{-1}\chi_{\mathrm{LT}}^{-j}))$$

satisfies $\widetilde{\exp}^*_{L_n,V(\chi_{\mathrm{LT}}^{-j}),\mathrm{id}} \otimes \mathbf{d}_1 = \exp^*_{L_n,V(\chi_{\mathrm{LT}}^{-j})}$ and the map

$$\widetilde{\log}_{L_n,V(\chi_{\mathrm{LT}}^{-j}),\mathrm{id}} : H^1_f(L_n, V(\chi_{\mathrm{LT}}^{-j})) \to L_n \otimes_L D_{\mathrm{cris},L}(V(\tau^{-1}\chi_{\mathrm{LT}}^{-j}))$$

satisfies $\widetilde{\log}_{L_n,V(\chi_{\mathrm{LT}}^{-j}),\mathrm{id}} \otimes \mathbf{d}_1 = \log_{L_n,V(\chi_{\mathrm{LT}}^{-j})}$. $\quad (D_{\mathrm{cris},L}(V(\tau)) = L\mathbf{d}_1.)$

The following result generalizes results by LOEFFLER-ZERBES from the cyclotomic case :

Theorem (SCHNEIDER-V., 2023, SANO-V., 2025)

Assume $V^*(1) \in \mathrm{Rep}_L^{\mathrm{cris,an}}(G_L)$ *with* $\mathrm{Fil}^{-1} D_{\mathrm{cris},L}(V^*(1)) = D_{\mathrm{cris},L}(V^*(1))$ *and* $D_{\mathrm{cris},L}(V^*(1))^{\varphi_L=\pi_L^{-1}} = D_{\mathrm{cris},L}(V^*(1))^{\varphi_L=1} = 0$. *Then it holds for* $y \in H^1_{\mathrm{Iw}}(L_\infty/L, T)$ *and* $j \in \mathbb{Z}$ *that*

$$\Omega^j \mathbf{L}_V(y)(\chi_{\mathrm{LT}}^j, n) =$$
$$\begin{cases} j!\,\Theta^*_{V(\tau^{-1}\chi_{\mathrm{LT}}^{-j}),n}\left(\widetilde{\exp}^*_{L_n, V(\chi_{\mathrm{LT}}^{-j}),\mathrm{id}}(y_{\chi_{\mathrm{LT}}^{-j},n})\right) \otimes e_j & \text{if } j \geq 0, \\ \dfrac{(-1)^{j+1}}{(-1-j)!}\Theta^*_{V(\tau^{-1}\chi_{\mathrm{LT}}^{-j}),n}\left(\widetilde{\log}_{L_n, V(\chi_{\mathrm{LT}}^{-j}),\mathrm{id}}(y_{\chi_{\mathrm{LT}}^{-j},n})\right) \otimes e_j & \text{if } j \leq -1, \end{cases}$$

if $1 - \pi_L^{-1}\varphi_L^{-1}, 1 - \frac{\pi_L}{q}\varphi_L$ *are invertible on* $D_{\mathrm{cris},L}(V(\tau^{-1}\chi_{\mathrm{LT}}^{-j}))$.

Corollary (Sano-V.,2025)

Let $j \in \mathbb{Z}$ and let ρ be a finite order character of Γ_L of conductor $n = a(\rho)$, i.e., ρ factorizes over G_n, but not over G_{n-1}. Then for any $y \in H^1_{\mathrm{Iw}}(L_\infty/L, T)$ we have

- *if $j \geq 0$,* $\Omega^j \mathbf{L}_V(y)(\rho\chi_{\mathrm{LT}}^j) =$

$$j! \begin{cases} (1-\pi_L^{-1-j}\varphi_L^{-1})^{-1}(1-\frac{\pi_L^{j+1}}{q}\varphi_L)\left(\widetilde{\exp}^*_{L,V(\chi_{\mathrm{LT}}^{-j}),\mathrm{id}}(y_{\chi_{\mathrm{LT}}^{-j}}) \otimes e_j\right), & \text{if } n=0,\\ \pi_L^{n(1+j)}\tau(\rho)^{-1}\varphi_L^n \widetilde{\exp}^*_{L,V(\chi_{\mathrm{LT}}^{-j}\rho^*),\mathrm{id}}(y_{\chi_{\mathrm{LT}}^{-j}\rho^*,n}) \otimes e_j, & \text{if } n>0.\end{cases}$$

- *if $j \leq -1$,* $\Omega^j \mathbf{L}_V(y)(\rho\chi_{\mathrm{LT}}^j) =$

$$\frac{(-1)^{j+1}}{(-1-j)!} \begin{cases} (1-\pi_L^{-1-j}\varphi_L^{-1})^{-1}(1-\frac{\pi_L^{j+1}}{q}\varphi_L)\left(\widetilde{\log}_{L,V(\chi_{\mathrm{LT}}^{-j}),\mathrm{id}}(y_{\chi_{\mathrm{LT}}^{-j}}) \otimes e_j\right), & \text{if } n=0,\\ \pi_L^{n(1+j)}\tau(\rho)^{-1}\varphi_L^n \widetilde{\log}_{L,V(\chi_{\mathrm{LT}}^{-j}\rho^*),\mathrm{id}}(\sigma^{-1}(y_{\chi_{\mathrm{LT}}^{-j}\rho^*,n})) \otimes e_j, & \text{if } n>0.\end{cases}$$

How can one prove Kato's ERL via (φ, Γ_L)-modules?

Define

$$t_{\mathrm{LT}} := \log_{\mathrm{LT}}(Z) \in L[[Z]]$$

and, for $m \geq 1$,

$$\varphi^{-m} = \varphi_L^{-m} : \mathbf{A}_L^{\psi=1} \to L_m[[t_{\mathrm{LT}}]];$$
$$f \mapsto f\left(\eta_m +_{\mathrm{LT}} \exp_{\mathrm{LT}}\left(\frac{t_{\mathrm{LT}}}{\pi_L^m}\right)\right)$$

Then, for $j, m \geq 1$, we define an *evaluation map*

$$\mathrm{ev}_{m,j} : \mathbf{A}_L^{\psi=1} \to L_m$$
$$f \mapsto \text{coefficient of } t_{\mathrm{LT}}^{j-1} \text{ in } \pi_L^{-m}\varphi^{-m}(f).$$

By explicit calculation:

$$\mathrm{ev}_{m,j}(f) = \frac{1}{(j-1)!\pi_L^{mj}}\left(\partial_{\mathrm{inv}}^{j-1} f\right)|_{Z=\eta_m}.$$

Theorem (I Sano-V.,2025)

For any $m \geq 1$ and $j \geq 1$ such that $\pi_L^j \neq q$, the following diagram is commutative:

$$\begin{array}{ccc}
H^1_{\mathrm{Iw}}(L_\infty/L, T^{\otimes(-1)}(1)) & \xrightarrow{\mathrm{Exp}^*} & \mathbf{A}_L^{\psi=1} \\
\downarrow \mathrm{tw}_{j-1} & & \\
H^1_{\mathrm{Iw}}(L_\infty/L, T^{\otimes(-j)}(1)) & & \downarrow \mathrm{ev}_{m,j} \\
\downarrow \mathrm{pr}_m & & \\
H^1(L_m, T^{\otimes(-j)}(1)) & \xrightarrow[\exp_j^*]{} & L_m.
\end{array}$$

In the *cyclotomic case*, i.e.,

$$L = \mathbb{Q}_p, \ \pi_L = p, \ \mathcal{F} = \widehat{\mathbb{G}}_m, \text{ and } T = \mathbb{Z}_p(1),$$

this Theorem is proved by CHERBONNIER-COLMEZ:

$$p^{-m}\varphi^{-m}(\mathrm{Exp}^*(\mu)) = \sum_{j \in \mathbb{Z}} \exp^*_{L_m,\mathbb{Q}_p(1-j)} \left(\int_{\Gamma_{L_m}} \chi_{\mathrm{cyc}}(x)^{1-j} \mu(x) \right).$$

Theorem (Colmez' reciprocity law in the Lubin-Tate setting)

Assume $D_{\mathrm{cris},L}(V(\tau^{-1}))^{\varphi_L=1} = D_{\mathrm{cris},L}(V(\tau^{-1}))^{\varphi_L=\pi_L^{-1}} = 0$. *Then, for all* $j \geq 1$ *such that the operators* $1 - \pi_L^{-1-j}\varphi_L^{-1}, 1 - \frac{\pi_L^{j+1}}{q}\varphi_L$ *are invertible on* $D_{\mathrm{cris},L}(V(\tau^{-1}))$*, the following diagram commutes:*

$$\begin{array}{ccc}
H^1_{\mathrm{Iw}}(L_\infty/L, T) & \xrightarrow{\ \mathrm{Exp}^*\ } & D_{\mathrm{LT}}(T(\tau^{-1}))^{\psi_L=1} \\
 & & \downarrow{\scriptstyle \pi_L^{-m}\varphi_L^{-m}\otimes \mathbf{d}_1} \\
\downarrow{\scriptstyle \mathrm{pr}_{m,-j}} & & L_m((t_{\mathrm{LT}})) \otimes_L D_{\mathrm{cris}}(V) \\
 & & \downarrow{\scriptstyle c_{t_{\mathrm{LT}}^j} \otimes (\mathrm{id} \otimes e_{-j})} \\
H^1(L_m, V(\chi_{\mathrm{LT}}^{-j})) & \xrightarrow[\ \exp^*_{L_m, V(\chi_{\mathrm{LT}}^{-j})}\ ]{} & L_m \otimes_L D_{\mathrm{cris}}(V(\chi_{\mathrm{LT}}^{-j})).
\end{array}$$

In other words, in $L_m((t_{\mathrm{LT}})) \otimes_L D_{\mathrm{dR}}(V) \subseteq B_{\mathrm{dR}} \otimes_L V$ we have

$$\pi_L^{-m}\varphi_L^{-m}(\mathrm{Exp}^*) \otimes \mathbf{d}_1 = \sum_{j\geq 1} \exp^*_{L_m, V(\chi_{\mathrm{LT}}^{-j})} \circ \mathrm{pr}_{m,-j} = \sum_{j\,\epsilon\,\mathbb{Z}} \exp^*_{L_m, V(\chi_{\mathrm{LT}}^{-j})} \circ \mathrm{pr}_{m,-j},$$

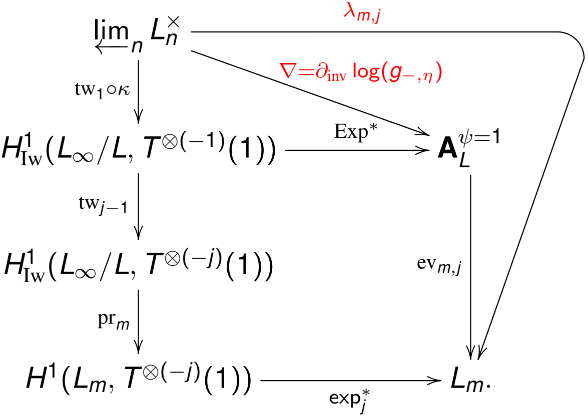
if the above conditions hold for all $j \geq 1$, i.e., if φ_L acting on $D_{\mathrm{cris},L}(V(\tau^{-1}))$ does not have any eigenvalue in $\pi_L^{\mathbb{N}_0} \cup q\pi_L^{\mathbb{N}_0}$; to this end we identify the target $L_m \otimes_L D_{\mathrm{cris}}(V(\chi_{\mathrm{LT}}^{-j}))$ with $L_m t_{\mathrm{LT}}^j \otimes_L D_{\mathrm{cris},L}(V) \subseteq L_m((t_{\mathrm{LT}})) \otimes_L D_{\mathrm{dR}}(V)$.

Note that $c_{t^i}(\pi_L^{-m}\varphi_L^{-m}(\mathrm{Exp}^*) \otimes \mathbf{d}_1) = 0$ for $i \leq 0$, because $\mathrm{Fil}^0 D_{\mathrm{dR}}(V) = 0$ as the Hodge-Tate weights of V are at least 1.

Back to Kato's ERL

$$\begin{array}{ccc}
\varprojlim_n L_n^\times & \xrightarrow{\quad\lambda_{m,j}\quad} & \\
\big\downarrow{\scriptstyle \mathrm{tw}_1\circ\kappa} & & \\
H^1_{\mathrm{Iw}}(L_\infty/L, T^{\otimes(-1)}(1)) & \xrightarrow{\ \mathrm{Exp}^*\ } & \mathbf{A}_L^{\psi=1} \\
\big\downarrow{\scriptstyle \mathrm{tw}_{j-1}} & & \big\downarrow{\scriptstyle \mathrm{ev}_{m,j}} \\
H^1_{\mathrm{Iw}}(L_\infty/L, T^{\otimes(-j)}(1)) & & \\
\big\downarrow{\scriptstyle \mathrm{pr}_m} & & \\
H^1(L_m, T^{\otimes(-j)}(1)) & \xrightarrow[\ \exp_j^*\ ]{} & L_m.
\end{array}$$

Back to Kato's ERL



Proof of Kato's ERL by above Theorem (Sano/V.).

Theorem 5 implies that $\lambda_{m,j}$ coincides with the composition

$$\lambda_{m,j} : \varprojlim_n L_n^\times \xrightarrow{\kappa} H^1_{\mathrm{Iw}}(L_\infty/L, \mathbb{Z}_p(1)) \xrightarrow{\mathrm{tw}_1} H^1_{\mathrm{Iw}}(L_\infty/L, T^{\otimes(-1)}(1))$$
$$\xrightarrow{\mathrm{Exp}^*} \mathbf{A}_L^{\psi=1} \xrightarrow{\mathrm{ev}_{m,j}} L_m.$$

By Theorem 2 and the explicit description (50) of the map $\mathrm{ev}_{m,j}$, we have for any $u \in \varprojlim_n L_n^\times$

$$\lambda_{m,j}(u) = \mathrm{ev}_{m,j}(\partial_{\mathrm{inv}} \log(g_{u,\eta})) = \frac{1}{(j-1)!\pi_L^{mj}} \left(\partial_{\mathrm{inv}}^j \log(g_{u,\eta})\right)\Big|_{Z=\eta_m}$$
$$= j \cdot \psi^j_{\mathrm{CW},m}(u).$$

□

How to proof Theorem I, 5?

How to proof Theorem I, 5?

It is equivalent to the commutativity of the following two diagrams for any $j, m \geq 1$:

Easy to check:

$$\begin{array}{ccc}
H^1_{\mathrm{Iw}}(L_\infty/L, T_\pi^{\otimes -1}(1)) & \xrightarrow{\mathrm{tw}_{j-1}} & H^1_{\mathrm{Iw}}(L_\infty/L, T_\pi^{\otimes -j}(1)) \\
\Big\downarrow {\scriptstyle \mathrm{Exp}^*} & & \Big\downarrow {\scriptstyle \mathrm{Exp}^*} \\
\mathbf{A}_L^{\psi=1} & \xrightarrow{\otimes \eta^{\otimes 1-j}} & \mathbf{A}_L^{\psi=1}(\chi_{\mathrm{LT}}^{1-j}) \\
\Big\downarrow {\scriptstyle \pi_L^{-m}\varphi_L^{-m}\otimes \mathbf{d}_1} & & \Big\downarrow {\scriptstyle \pi_L^{-m}\varphi_L^{-m}\otimes \mathbf{d}_1} \\
L_m((t_{\mathrm{LT}})) \otimes D_{\mathrm{cris}}(L(\tau)) & \xrightarrow{\cdot t_{\mathrm{LT}}^{1-j}\otimes e_{1-j}} & L_m((t_{\mathrm{LT}})) \otimes D_{\mathrm{cris}}(L(\chi_{\mathrm{LT}}^{-j})(1)) \\
\Big\downarrow {\scriptstyle c_{t_{\mathrm{LT}}^{j-1}}\otimes \mathrm{id}} & & \Big\downarrow {\scriptstyle c_{t_{\mathrm{LT}}^{0}}\otimes \mathrm{id}} \\
L_m \otimes D_{\mathrm{cris}}(L(\tau)) & \xrightarrow{\mathrm{id}\otimes e_{1-j}} & L_m \otimes D_{\mathrm{cris}}(L(\chi_{\mathrm{LT}}^{-j})(1)),
\end{array}$$

where $c_{t_{\mathrm{LT}}^{j-1}} =$ "coefficient of t_{LT}^{j-1}", note that $\mathrm{ev}_{m,j} = c_{t_{\mathrm{LT}}^{j-1}} \circ \pi_L^{-m}\varphi_L^{-m}$

The second diagram is the following:

$$
\begin{array}{ccc}
H^1_{\mathrm{Iw}}(L_\infty/L, T_\pi^{\otimes -j}(1)) & \xrightarrow{\mathrm{Exp}^*} & \mathbf{A}_L^{\psi=1}(\chi_{\mathrm{LT}}^{1-j}) \\
 & & \Big\downarrow{\scriptstyle \pi_L^{-m}\varphi_L^{-m}\otimes \mathbf{d}_1} \\
\Big\downarrow{\scriptstyle \mathrm{pr}_m} & & L_m((t_{\mathrm{LT}})) \otimes_L D_{\mathrm{cris}}(L(\chi_{\mathrm{LT}}^{-j})(1)) \\
 & & \Big\downarrow{\scriptstyle c_{t^0_{\mathrm{LT}}}\otimes \mathrm{id}} \\
H^1(L_m, T_\pi^{\otimes -j}(1)) & \xrightarrow{\exp_j^*} & L_m \otimes_L D_{\mathrm{cris}}(L(\chi_{\mathrm{LT}}^{-j})(1)).
\end{array}
$$

Indeed:

$$
\begin{array}{ccccc}
H^1_{\mathrm{Iw}}(L_\infty/L, T_\pi^{\otimes -1}(1)) & \xrightarrow{\mathrm{tw}_{j-1}} & H^1_{\mathrm{Iw}}(L_\infty/L, T_\pi^{\otimes -j}(1)) & \xrightarrow[\mathrm{pr}_m]{} & H^1(L_m, T_\pi^{\otimes -j}(1)) \\
\downarrow{\scriptstyle \mathrm{Exp}^*} & & \downarrow{\scriptstyle \mathrm{Exp}^*} & & \\
\mathbf{A}_L^{\psi=1} & \text{1st diag} & \mathbf{A}_L^{\psi=1}(\chi_{\mathrm{LT}}^{1-j}) & \text{2nd diag} & \swarrow{\scriptstyle \exp_j^*} \\
\downarrow{\scriptstyle \mathrm{ev}_{m,j}\otimes \mathbf{d}_1} & \left(c_{t^0_{\mathrm{LT}}} \circ \pi_L^{-m}\varphi_L^{-m}\right)\otimes \mathbf{d}_1 & \downarrow & & \\
L_m \otimes D_{\mathrm{cris}}(L(\tau)) & \xrightarrow{\mathrm{id}\otimes e_{1-j}} & L_m \otimes D_{\mathrm{cris}}(L(\chi_{\mathrm{LT}}^{-j})(1)), & &
\end{array}
$$

Instead of proving the commutativity of the *second diagram* directly we extend it to the following larger diagram, in which the upper line is just the regulator map $\mathbf{L}_{L(\chi_{\mathrm{LT}}^{-j})(1)}$ by SCHNEIDER-V.

$$\begin{array}{ccccc}
H^1_{\mathrm{Iw}}(L_\infty/L, T_\pi^{\otimes -j}(1)) & \xrightarrow{\mathrm{Exp}^*} & \mathbf{A}_L^{\psi=1}(\chi_{\mathrm{LT}}^{1-j}) & \xrightarrow{\Xi_{L(\chi_{\mathrm{LT}}^{1-j})}} & \frac{1}{\Gamma'_{L(\chi_{\mathrm{LT}}^{j+1})}} D(\Gamma_L, K) \otimes_L D_{\mathrm{cris}}(L(\chi_{\mathrm{LT}}^{-j})(1)) \\
\downarrow{\scriptstyle \mathrm{pr}_m} & & \downarrow{\scriptstyle \pi_L^{-m}\varphi_L^{-m}\otimes \mathbf{d}_1} & & \downarrow{\scriptstyle \mathrm{pr}_{\Gamma_m}\otimes \mathrm{id}} \\
 & & L_m((t_{\mathrm{LT}})) \otimes D_{\mathrm{cris}}(L(\chi_{\mathrm{LT}}^{-j})(1)) & & \\
 & & \downarrow{\scriptstyle c_{t^0_{\mathrm{LT}}}\otimes \mathrm{id}} & & \\
H^1(L_m, T_\pi^{\otimes -j}(1)) & \xrightarrow{\exp_j^*} & L_m \otimes D_{\mathrm{cris}}(L(\chi_{\mathrm{LT}}^{-j})(1)) & \xrightarrow{\Theta_{L(\chi_{\mathrm{LT}}^{1-j}),D,m}} & K[\Gamma_L/\Gamma_m] \otimes_L D_{\mathrm{cris}}(L(\chi_{\mathrm{LT}}^{-j})(1)).
\end{array}$$

and where the right inner diagram commutes by construction and $\Theta_{L(\chi_{\mathrm{LT}}^{1-j}),D,m}$ is a bijection. This **reduces the problem to** proving the commutativity of the outer diagram, i.e., to the

equivariant interpolation formula 3.

Many thanks

for your attention!