

L^p -Heisenberg–Pauli–Weyl Uncertainty Inequalities on Heisenberg Groups

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(Joint work with Pritam Ganguly)

Classical Heisenberg–Pauli–Weyl uncertainty inequality

- 1 The **Heisenberg uncertainty principle** in quantum mechanics states that the position and momentum of a given particle cannot be accurately determined simultaneously.
- 2 Pauli and Weyl provided a more rigorous mathematical foundation.
- 3 This result is later known as the **Heisenberg–Pauli–Weyl (HPW) uncertainty inequality**, which, in its most general form, states that for every $\alpha, \beta > 0$ and $f \in L^2(\mathbb{R}^n)$,

$$\|f\|_{L^2(\mathbb{R}^n)}^{\alpha+\beta} \lesssim_{\alpha,\beta,n} \left(\int_{\mathbb{R}^n} \|x\|^{2\alpha} |f(x)|^2 dx \right)^{\frac{\beta}{2}} \left(\int_{\mathbb{R}^n} \|\xi\|^{2\beta} |\widehat{f}(\xi)|^2 d\xi \right)^{\frac{\alpha}{2}}.$$

- 4 The function and its Fourier transform cannot be sharply localised simultaneously.

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More general formulation

- 1 Let Δ be the Laplacian on \mathbb{R}^n . One can define the operator $(-\Delta)^s$, $s > 0$ as follows:

$$\widehat{(-\Delta)^s g}(\xi) = \|\xi\|^{2s} \widehat{g}(\xi), \quad \xi \in \mathbb{R}^n, \quad g \in C_0^\infty(\mathbb{R}^n).$$

- 2 Using Plancherel formula, the HPW inequality can be reformulated as

$$\|f\|_{L^2(\mathbb{R}^n)}^{\alpha+\beta} \lesssim_{\alpha,\beta,n} \|\|\cdot\|^\alpha f\|_{L^2(\mathbb{R}^n)}^\beta \|(-\Delta)^{\frac{\beta}{2}} f\|_{L^2(\mathbb{R}^n)}^\alpha.$$

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Some L^2 analogues

- 1 Thangavelu was the first to establish an analogue of the HPW inequality for $\alpha = \beta = 1$, replacing the Laplacian with **Hermite and special Hermite operators**, as well as the **sub-Laplacian on Heisenberg groups** (Proc. Indian Acad. Sci. Math. Sci., 1990).
- 2 Xiao–He proved the inequality for all $\alpha, \beta > 0$ on Heisenberg groups (Proc. Indian Acad. Sci. Math. Sci., 2012).
- 3 Ray established the inequality on **two-step nilpotent Lie groups** for $\alpha = \beta = 1$, where the Laplacian is replaced by the sub-Laplacian of the group (Proc. Indian Acad. Sci. Math. Sci., 2001).
- 4 Ciatti–Ricci–Sundari extended these results to a broader setting of **Lie groups with polynomial volume growth**, replacing the Laplacian with a **Hörmander-type sub-Laplacian** (Adv. Math., 2007).
- 5 Certain **Metric measure spaces (X, d, μ)** and positive self-adjoint operators L on $L^2(X)$ generating an ultracontractive semigroup (Ciatti et al., Adv. Math., 2007; Martini, Math. Z., 2010).

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L^p -HPW inequality

- ① Assume that $\beta, \delta \in (0, \infty)$, $p, q, r \in [1, \infty)$, and that

$$\frac{\alpha + \beta}{p} = \frac{\alpha}{r} + \frac{\beta}{q}.$$

Then, there exists a constant $C > 0$ such that for any $f \in C_c^\infty(G)$, the following holds:

$$\|f\|_{L^p(\mathbb{R}^n)}^{\alpha+\beta} \lesssim_{\alpha,\beta,n} \|\|\cdot\|^\alpha f\|_{L^q(\mathbb{R}^n)}^\beta \|(-\Delta)^{\frac{\beta}{2}} f\|_{L^r(\mathbb{R}^n)}^\alpha.$$

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Incorporating Fourier transform

- 1 Hausdorff–Young inequality: $\|\hat{f}\|_{L^{p'}(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}$, $p \in [1, 2]$.
- 2 $\| |\cdot|^\beta \hat{f} \|_{L^{p'}(\mathbb{R}^n)} \lesssim_p \| (-\Delta)^{\frac{\beta}{2}} f \|_{L^p(\mathbb{R}^n)}$.
- 3 Can the HPW inequality be sharpened by incorporating the Fourier transform?
- 4 Let $1 \leq p < 2$. Suppose $\alpha, \beta \in (0, \infty)$ are s.t. $\beta > n(1/p - 1/2)$.
Then

$$\|f\|_{L^p(\mathbb{R}^n)}^{\alpha+\beta} \leq C(\alpha, \beta, n, p) \| |\cdot|^\alpha f \|_{L^p(\mathbb{R}^n)}^\beta \| |\cdot|^\beta \hat{f} \|_{L^{p'}(\mathbb{R}^n)}^\alpha.$$

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Harmonic analysis on the Heisenberg group

- 1 Consider the Heisenberg group $\mathbb{H}^n := \mathbb{C}^n \times \mathbb{R}$ equipped with the group law

$$(z, t) \cdot (w, s) := \left(z + w, t + s + \frac{1}{2} \Im(z\bar{w}) \right).$$

This is a **two-step nilpotent Lie group** where the Lebesgue measure $dzdt$ plays the role of Haar measure.

- 2 The **group Fourier transform** of $f \in L^1(\mathbb{H}^n)$ is an **operator valued function** defined on non-zero reals by

$$\widehat{f}(\lambda) := \int_{\mathbb{H}^n} f(z, t) \pi_\lambda(z, t) dz dt$$

where π_λ 's are Schrödinger representation of \mathbb{H}^n given by

$$\pi_\lambda(z, t) \varphi(\xi) = e^{i\lambda t} e^{i\lambda(x \cdot \xi + \frac{1}{2}x \cdot y)} \varphi(\xi + y),$$

where $z = x + iy$ and $\varphi \in L^2(\mathbb{R}^n)$.

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① $\hat{f}(\lambda)$ is a bounded linear operator on $L^2(\mathbb{R}^n)$ and $\|\hat{f}(\lambda)\|_{op} \leq \|f\|_1$.

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sub-Laplacian

- 1 We may consider elements of \mathfrak{h}_n as left-invariant differential operators acting on $C^\infty(H^n)$.
- 2 Consider the following basis of \mathfrak{h}_n :

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- 3 The sub-Laplacian on H^n is defined by

$$\mathcal{L} = -\sum_{j=1}^n (X_j^2 + Y_j^2).$$

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Spectral decomposition of $H(\lambda)$

① $h_k(t) = (2^k \sqrt{\pi} k!)^{-\frac{1}{2}} H_k(t) e^{-t^2/2},$

② $H_k(t) = (-1)^k \frac{d^k}{dt^k} (e^{-t^2}) e^{t^2}, t \in \mathbb{R}, k = 0, 1, 2, \dots.$

- ③ For $\lambda \in \mathbb{R}^*$, the scaled higher dimensional Hermite functions are defined as

$$\Phi_\alpha^\lambda(x) := |\lambda|^{n/4} \prod_{j=1}^n h_{\alpha_j}(\sqrt{|\lambda|} x_j), \quad \alpha \in (\mathbb{N} \cup \{0\})^n, x \in \mathbb{R}^n.$$

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Dilations and metric on H^n

- ① On H^n , we have **non-isotropic dilation**:

$$\delta_r(z, t) = (rz, r^2t), (z, t) \in H^n, r > 0.$$

- ② A **homogeneous norm** on H^n known as the Koranyi norm is given by

$$|(z, t)| = (|z|^4 + 16t^2)^{1/4}.$$

- ③ Note that $|\delta_r(z, t)| = r|(z, t)|$.

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L^p -HPW inequality on the Heisenberg group

Theorem 1 (Ganguly-S., 2025).

Suppose $p \in [1, 2)$, $\gamma > 0$, and $\beta > (2n + 2)(1/p - 1/2)$. Then we have the following.

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Two estimates

$$\begin{aligned} & \int_{\mathbb{R}^*} \sum_{\alpha: (2|\alpha|+n)|\lambda| \geq c} \|\hat{f}(\lambda)\Phi_\alpha^\lambda\|_{L^2(\mathbb{R}^n)}^2 |\lambda|^n d\lambda \\ &= \int_{\mathbb{R}^*} \sum_{\alpha: (2|\alpha|+n)|\lambda| \geq c} (2|\alpha|+n)|\lambda|^{-\beta} \|\hat{f}(\lambda)H(\lambda)^{\beta/2}\Phi_\alpha^\lambda\|_{L^2(\mathbb{R}^n)}^2 |\lambda|^n d\lambda \\ &\leq \|\hat{f}(\lambda)H(\lambda)^{\beta/2}\|_{op}^2 \int_{\mathbb{R}^*} \sum_{\alpha: (2|\alpha|+n)|\lambda| \geq c} (2|\alpha|+n)|\lambda|^{-\beta} |\lambda|^n d\lambda \leq C_1 c^{n-\beta+1}. \end{aligned}$$

Using Plancherel's formula, we get

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As $\beta > n + 1$, we can choose c large and $a > 0$ accordingly so that last quantity is positive. So,

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But $\|\hat{f}(\lambda)\|_{op} \leq 1$. This implies that

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We now choose $c_1 > 0$ such that

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Final step

Observe that

$$\begin{aligned} & \int_{\mathbb{R}^*} \sum_{\alpha: c_1 \leq (2|\alpha|+n)|\lambda| \leq c} \|\hat{f}(\lambda)\Phi_\alpha^\lambda\|_{L^2(\mathbb{R}^n)} |\lambda|^n d\lambda \\ &= \int_{\mathbb{R}^*} \sum_{\alpha: c_1 \leq (2|\alpha|+n)|\lambda| \leq c} ((2|\alpha|+n)|\lambda|)^{-\beta/2} \|\hat{f}(\lambda)H(\lambda)^{\beta/2}\Phi_\alpha^\lambda\|_{L^2(\mathbb{R}^n)} |\lambda|^n d\lambda \\ &\leq c_1^{-\beta/2} \int_{\mathbb{R}^*} \sum_{\alpha: c_1 \leq (2|\alpha|+n)|\lambda| \leq c} \|\hat{f}(\lambda)H(\lambda)^{\beta/2}\|_{\text{op}} |\lambda|^n d\lambda \\ &\leq C_2 \sup_{\lambda \in \mathbb{R}^*} \|\hat{f}(\lambda)H(\lambda)^{\beta/2}\|_{\text{op}}. \end{aligned}$$

- ① The proof for $p \in (1, 2)$ adapts the L^1 approach, but the greater complexity of the **Schatten p norm** requires a more delicate analysis.
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Thank You