Coulomb gas droplets, quadrature domains, and Schwarz reflections dynamics

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Quadrature domains in statistical physics

Motivation from physics: 2D Coulomb gas ensembles

- Consider N electrons placed in the complex plane at points $\{z_j\}_{j=1}^N$, influenced by a strong external (magnetic/electrostatic) field with uniform density.

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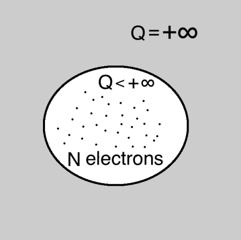

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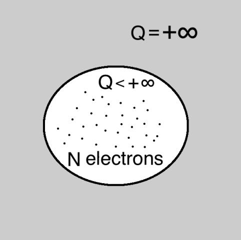

- The combined energy resulting from particle interaction and external potential is:

$$\mathcal{E}_Q(z_1, \cdots, z_N) = \sum_{i \neq j} \ln |z_i - z_j|^{-1} + N \sum_{j=1}^{N} Q(z_j).$$

Large N behavior of 2D Coulomb gas ensembles

- It is more likely to find configurations of electrons with low energy. This leads to the following joint density of states

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- We are interested in the limiting behavior of the point process as the number of electrons grows to infinity.
- In the limit, the electrons condensate on a compact set T, and they are distributed according to the measure $\mu := \Delta Q \, \mathbf{1}_T \, dA$. (Wiegmann, Zabrodin, Elbau, Felder, Hedenmalm, Makarov, et al.)

Random normal matrix ensemble

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- In many physically interesting cases, the potential Q satisfies certain physical and algebraic properties.
- The set T (on which eigenvalues/electrons condensate) is then called an *algebraic droplet* of Q.

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Definition (Quadrature Domains – Aharonov, Shapiro)

A domain $\Omega \subsetneq \hat{\mathbb{C}}$ (with $\infty \notin \partial\Omega$ and $\mathrm{int}(\overline{\Omega}) = \Omega$) is called a *quadrature domain* if there exists a continuous function $\sigma : \overline{\Omega} \to \hat{\mathbb{C}}$ satisfying the following two properties:

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- The map σ is called the *Schwarz reflection map* of Ω.
- Thus, the study of algebraic droplets is related to the study of quadrature domains.
- Lee and Makarov used iteration of Schwarz reflection maps to study the topology of quadrature domains.

Classical perspective: quadrature identity

- A domain $\Omega \subsetneq \widehat{\mathbb{C}}$ (with $\infty \notin \partial\Omega$ and $\mathrm{int}(\overline{\Omega}) = \Omega$) is a quadrature domain $\iff$
  There exists a rational map R_Ω with all poles inside Ω such that

$$\int_\Omega \phi dA = \frac{1}{2i} \oint_{\partial\Omega} \phi(z) R_\Omega(z) dz \quad \left(= \sum c_k \phi^{(n_k)}(a_k)\right)$$

  for all $\phi \in H(\Omega) \cap C(\overline{\Omega})$ (if $\infty \in \Omega$, one also requires $\phi(\infty) = 0$).

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- The rational map R_Ω and the Schwarz reflection map σ have the same (finite) set of poles.

Algebraic properties, and complexity

Simply connected quadrature domains

- A simply connected domain $\Omega \subsetneq \hat{\mathbb{C}}$ with $\infty \notin \partial\Omega$ and $\mathrm{int}(\overline{\Omega}) = \Omega$ is a quadrature domain if and only if the Riemann map $\phi : \mathbb{D} \to \Omega$ extends as a rational map of $\widehat{\mathbb{C}}$.

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$$\begin{tikzcd} \overline{\mathbb{D}} \arrow[r, "\varphi"] \arrow[d, "1/\overline{z}"] & \overline{\Omega} \arrow[d, "\sigma"] \\ \hat{\mathbb{C}} \setminus \mathbb{D} \arrow[r, "\varphi"] & \hat{\mathbb{C}} \end{tikzcd}$$

- The rational map ϕ semi-conjugates the reflection map $1/\overline{z}$ of $\mathbb{D}$ to the Schwarz reflection map σ of Ω.

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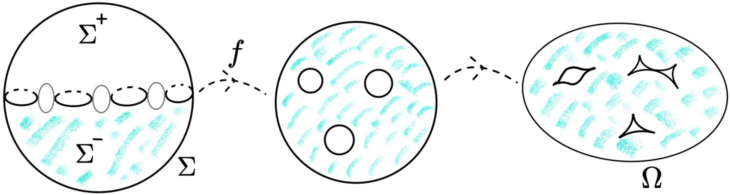
Theorem (Gustafsson, 1983)

There exist a genus $k-1$ compact Riemann surface Σ, an anti-conformal involution ι of Σ, and a meromorphic map $f : \Sigma \to \widehat{\mathbb{C}}$ such that

1. $\mathrm{Fix}(\iota)$ *is a disjoint union of k circles;*
2. $\Sigma \setminus \mathrm{Fix}(\iota) = \Sigma^+ \sqcup \Sigma_-$, *where $\Sigma^\pm$ are connected;*
3. $f : \Sigma^- \to \Omega$ *is a conformal isomorphism; and*
4. $\sigma \equiv f \circ \iota \circ (f|_{\Sigma^-})^{-1}$.



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- Can the topology + geometry of a quadrature domain Ω be controlled by d_f?

Topology/geometry of quadrature domains via dynamics

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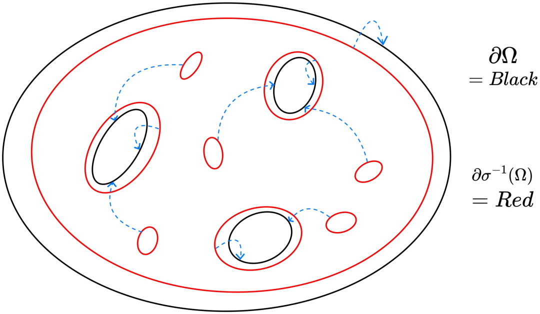
  2. Obtain the desired bound by studying antiholomorphic dynamics of Schwarz reflection maps.

Key step of Lee–Makarov proof

- Look at the degree $\approx d_f$ branched covering $\sigma : \sigma^{-1}(\Omega) \to \Omega$; i.e., forget the action of σ over Ω^c.



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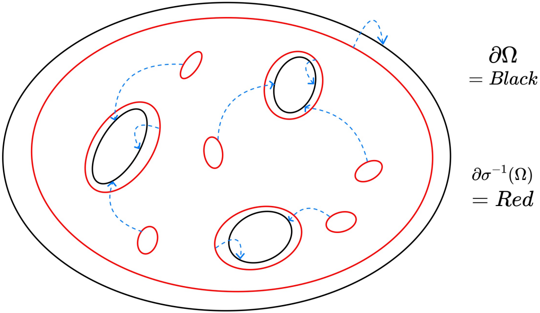

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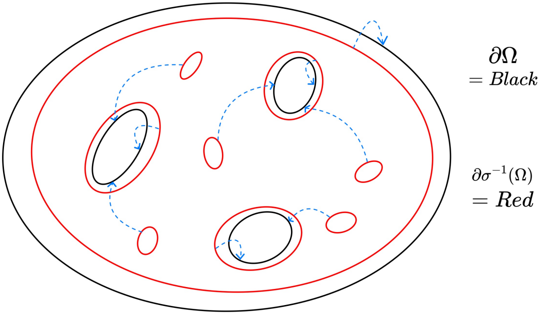

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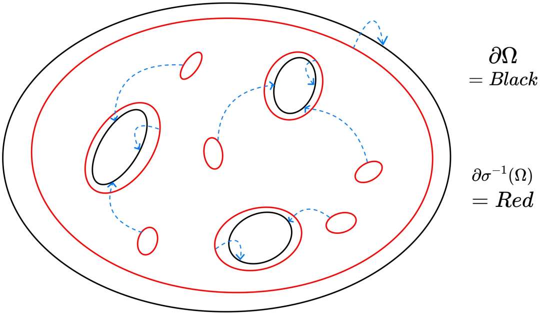

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- Promote $\widetilde{\sigma}$ to a holomorphic (rational) map using quasiconformal methods; smoothness of $\partial\Omega$ is essential for this step.
- A degree $\approx d_f$ rational map has at most $O(d_f)$ attracting fixed points.

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- The harder part: Use the dynamics of the Schwarz reflection σ to assign a critical point of f to each component of Ω^c and each double point on $\partial\Omega$.
- The bound on connectivity and double points is sharp.

Key lemma

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Let T be a component of Ω^c with m double points. Then, at least $(m+3)$ critical values of f lie in $\bigcup_{j\geq 0} \sigma^{-j}(T)$.

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- While the Lee–Makarov proof ‘forgets’ the action of σ over Ω^c, we analyze the set of points escaping to Ω^c to prove the key lemma.

Proof of main theorem assuming the lemma:

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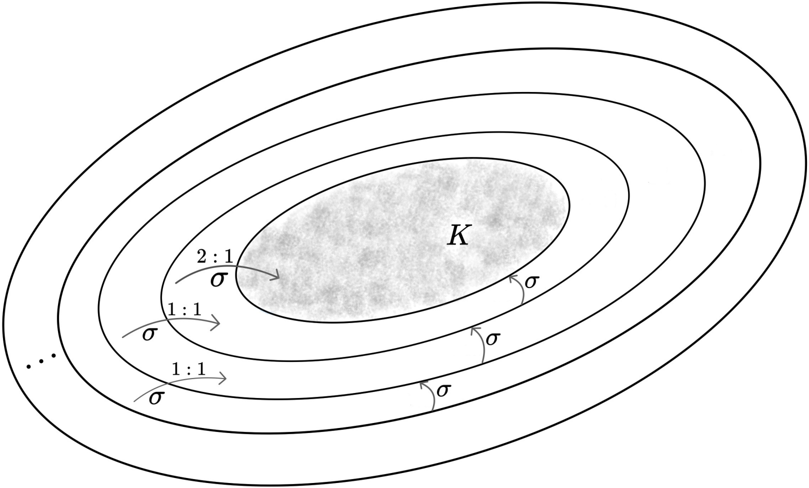
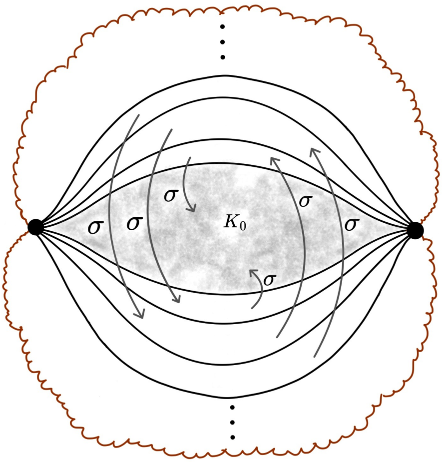
- By Riemann-Hurwitz, f has $(2d_f + 2k - 4)$ critical points.
- Hence, $3k + \#\ \mathrm{Double\ points} \leq 2d_f + 2k - 4$,
$\implies k + \#\ \mathrm{Double\ points} \leq 2d_f - 4$.

The non-singular case



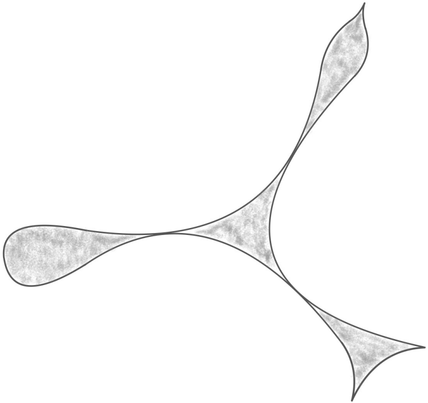
- Dynamics of σ + moduli of annuli argument.

The case of two cusps



- Dynamics of σ + contraction of hyperbolic metric + parabolic dynamics at cusps.

The case of double points



- Dynamics of σ + hyperbolic contraction + parabolic dynamics at cusps + (easy) graph theory.

Microscopic limits and universality

Universality

- Linear statistics: For a test function g, let
  $\mathrm{trace}_n(g) : \mathbb{C}^n \to \mathbb{R}, \quad (\lambda)_{j=1}^n \to \sum_{j=1}^n g(\lambda);$
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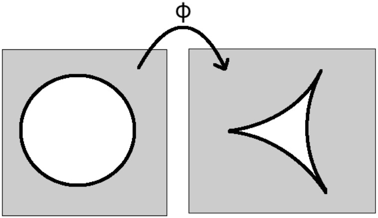
- Universality classes near cusps and double points?

Schwarz dynamics from a different viewpoint

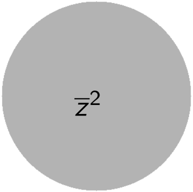
The deltoid reflection map

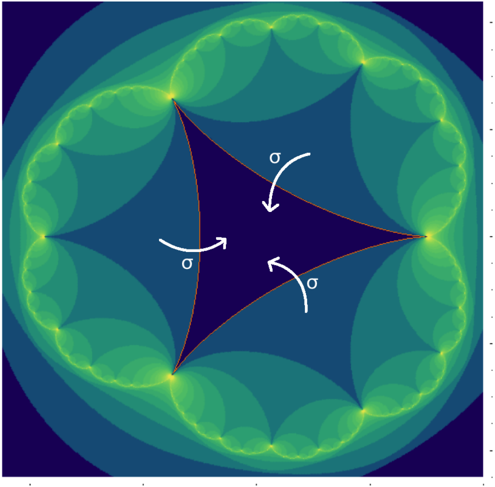
- For the cubic potential $Q(z) = |z|^2 - \mathrm{Re}(z^3)$, the deltoid appears as a "maximal" algebraic droplet.
- The complement of the deltoid has a Riemann map $\phi(z) = z + \frac{1}{2z^2}$, so it is a quadrature domain.

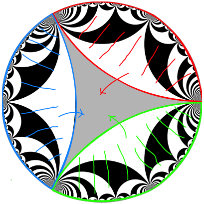


- The corresponding Schwarz reflection map is unicritical, and has a super-attracting fixed point at ∞.

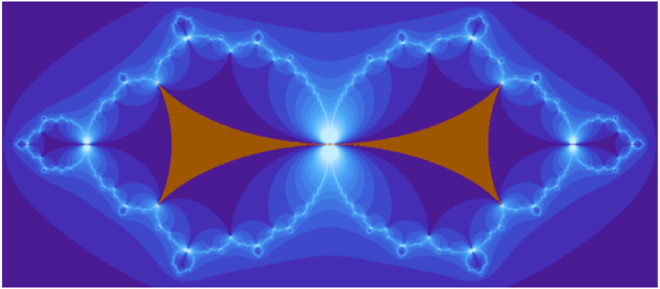
Deltoid Reflection as a mating

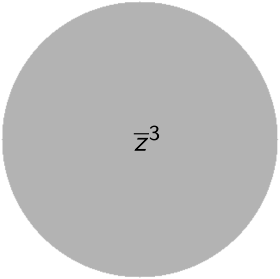


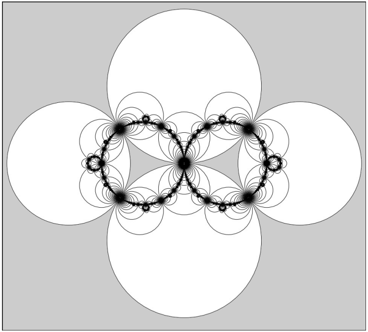




(Lee–Lyubich–Makarov–M)




$\bar{z}^3$



A general combination theorem

Theorem (Luo–Lyubich–M)

Let f be a 'generic' degree d anti-polynomial with connected Julia set. Then, there exists a Schwarz reflection map, unique up to Möbius conjugacy, that is a conformal mating between f and the Nielsen map of an ideal $(d+1)-$gon reflection group.

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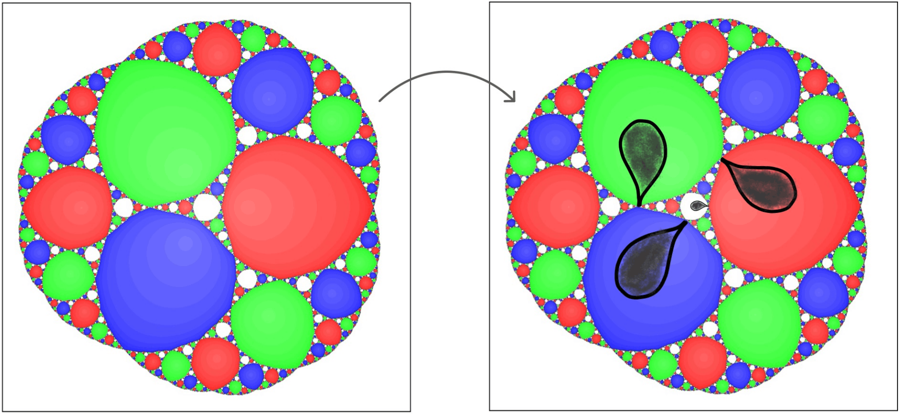
- Requires new surgery/uniformization techniques.
- The same surgery machinery yields sharpness of our upper bounds.

Constructing droplets with prescribed topology and singularities

David surgery



(Byun–M–Rashmita–Yoo)

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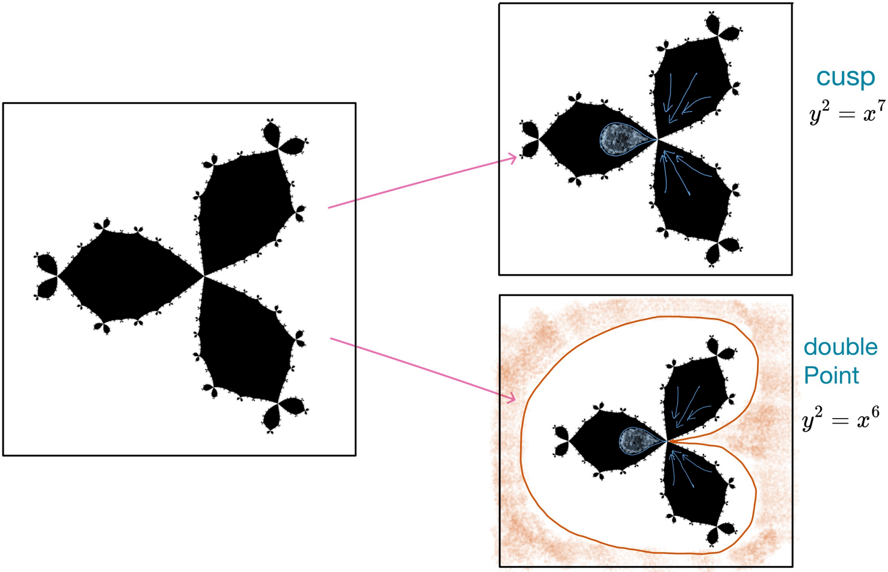

(Byun–M–Rashmita–Yoo)

Thank you!