

Symplectic geometry, curves, and counting tropically

Sushmita Venugopalan

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Table of Contents

- 1 The symplectic form
- 2 Holomorphic curves in symplectic geometry
- 3 Mikhalkin's tropical graphs
- 4 Single cut
- 5 Multiple cut
- 6 Mikhalkin-like curves in almost torics

Motivations from Hamiltonian mechanics: A particle on a line

- The motion of a particle on a line is completely described by its position $q(t)$ and momentum $p(t)$.
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- This curve $(p(t), q(t))$ is the solution of an ODE whose initial value $(p(0), q(0))$ is specified.
- The map

$$\phi_t : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad (p(0), q(0)) \mapsto (p(t), q(t))$$

is the **time evolution** of the system.

A particle on a line

- It was known in the nineteenth century that, if the force field is conservative, then the evolution map preserves area in the p - q plane.
- The area form $dq \wedge dp$ is a conserved quantity of the motion.

Hamiltonian mechanics

- The total energy, or **Hamiltonian**, of a particle is a function of its position $q \in \mathbb{R}^n$ and momentum $p \in \mathbb{R}^n$, and is given by

$$H : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}, \quad H(p, q) = \frac{|p|^2}{2m} + V(q).$$

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Hamilton's equations

$$\frac{d}{dt}q_i = \frac{\partial H}{\partial p_i}, \quad \frac{d}{dt}p_i = -\frac{\partial H}{\partial q_i}, \quad i = 1, \dots, n.$$

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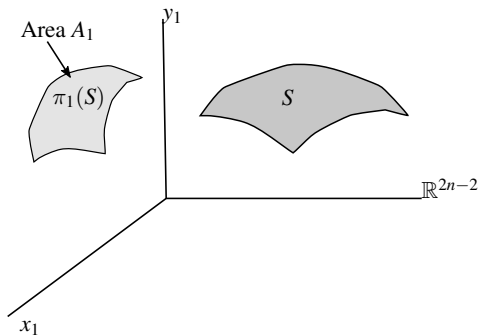
- The quantity $\sum_i dq_i \wedge dp_i$ is conserved by the motion.
- This is the **symplectic form** on $\mathbb{R}^{2n}$.

Symplectic form on $\mathbb{R}^{2n}$

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- The ω -area is $\omega(S) := \sum_{i=1}^n A_i$, where A_i is the area of the projection of S to the (x_i, y_i) -planes.

Symplectic form

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- That is, at any point $m \in M$,

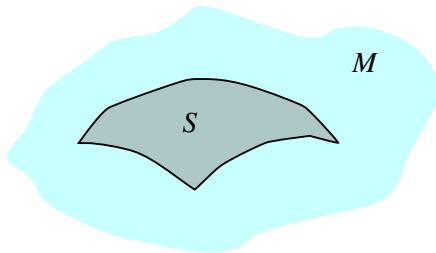
$$\omega_m : T_m M \times T_m M \rightarrow \mathbb{R}$$

is a bilinear skew-symmetric non-degenerate pairing on the tangent space $T_m M$.

Symplectic form

- A two-form ω is an area function on surfaces in M : given a surface $S \subset M$, we get an ω -area

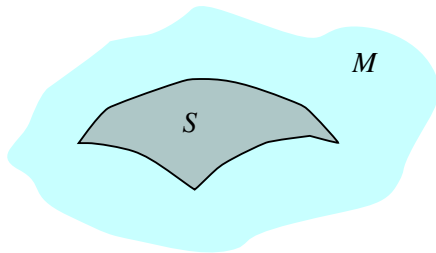
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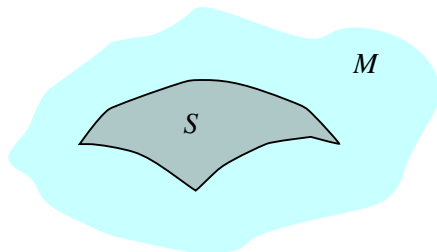


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Symplectic form

- A two-form ω is an area function on surfaces in M : given a surface $S \subset M$, we get an ω -area

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- How to think of this? **Darboux's theorem** : The neighbourhood of any point in a symplectic manifold looks exactly like $\mathbb{R}^{2n}$ with standard symplectic form.

Examples of symplectic manifolds

- Cotangent bundles: For any smooth manifold X , T^*X is a symplectic manifold.
- Complex projective space with Fubini-Study form $\mathbb{P}^n$.
- Any non-singular projective variety.
- Examples : Torus $\mathbb{T}^{2n} := \mathbb{R}^{2n}/\mathbb{Z}^{2n}$, two-sphere $\mathbb{S}^2$.

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- Examples : Torus $\mathbb{T}^{2n} := \mathbb{R}^{2n}/\mathbb{Z}^{2n}$, two-sphere $\mathbb{S}^2$.
- Non-examples: Non-orientable manifolds, $\mathbb{S}^{2n}$ for any $n \geq 2$.

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- Symplectic geometry displays a mix of flexibility (lack of structure) and rigidity (structure).
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- Contrast this with Riemannian manifolds.
- A Riemannian manifold comes with an inner product on the tangent space at any point. Thus, on a Riemannian manifold, one can measure length, angle, surface area, volume.
- Riemannian geometry is more rigid than symplectic geometry : to every point, one can associate a 'curvature' .

The two-dimensional case

The 2-balls are embedded in Euclidean 3-space. Both inherit an area form and a Riemannian metric from the ambient Euclidean space.



They are the ‘same’ as symplectic manifolds because they have the same area. But they are not the same Riemannian manifolds.

Symplectomorphisms

- A smooth invertible map $\phi : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ is a **symplectomorphism** if it preserves symplectic area.
- That is, for any surface S in $\mathbb{R}^{2n}$, $\omega(S) = \omega(\phi(S))$.

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- Example :

$$(q_1, p_1, \dots, q_n, p_n) \mapsto (2q_1, \frac{1}{2}p_1, q_2, p_2, \dots, q_n, p_n).$$

- Another example :

$$(q_1, p_1, \dots, q_n, p_n) \mapsto (q_2, p_2, q_1, p_1, \dots, q_n, p_n).$$

Symplectic invariants?

- Volume is an invariant of a symplectic manifold. The symplectic form gives a volume form

$$\text{vol} := \omega^n := \omega \wedge \cdots \wedge \omega.$$

- The rigidity question: Is symplectic structure much stronger than a volume form?

Another version of the rigidity question

Can any volume preserving map be approximated by symplectomorphisms?

Gromov's non-squeezing theorem (1985)

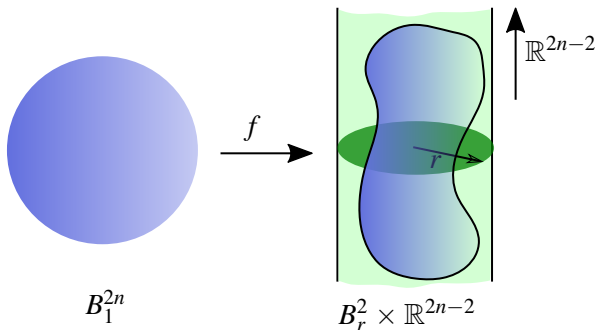
Gromov's non-squeezing theorem

A unit ball in the symplectic vector space $(\mathbb{R}^{2n}, \omega)$ cannot be mapped by a symplectomorphism into any cylinder $B_r^2 \times \mathbb{R}^{2n-2}$ which is narrower than the ball (i.e. $r < 1$).

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Can a symplectic camel pass through the eye of a needle?

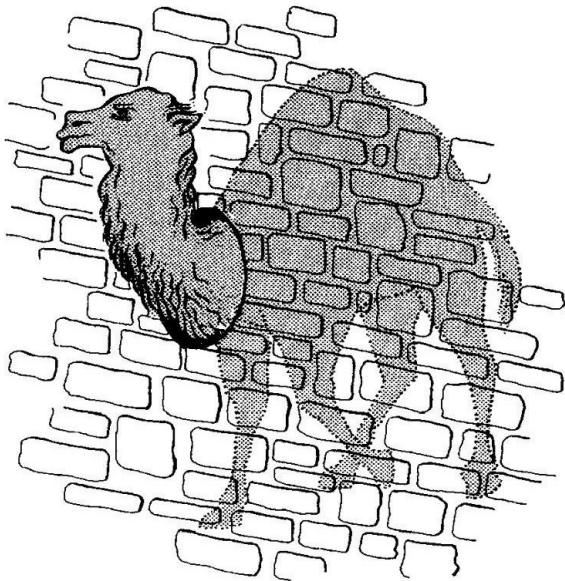


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Complex structure on a vector space

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- If the vector space V has a symplectic form ω , J is **compatible** with ω if
 - 1 $\omega(v, Jv) > 0$ for all non-zero vectors $v \in V$,
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- The pairing

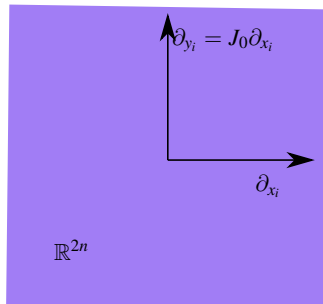
$$(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}, \quad (v, w) \mapsto \omega(v, Jw)$$

is a positive definite inner product.

- A symplectic vector space has a compatible J . The set of compatible J s is contractible.

Complex structure on a vector space

The standard complex structure on $\mathbb{R}^{2n}$ is compatible with the standard symplectic form $\omega_0 = \sum_i dx_i \wedge dy_i$.



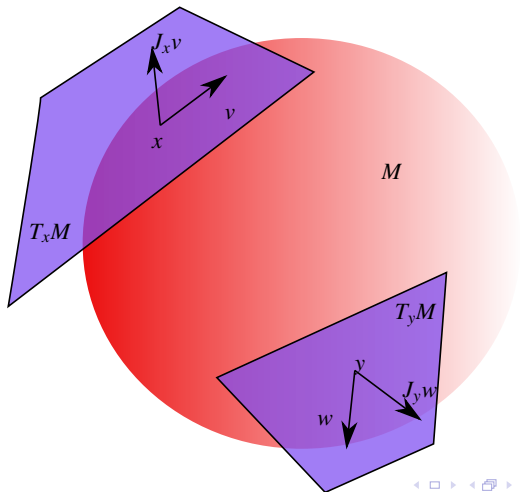
$$J_0 \partial_{x_i} = \partial_{y_i}$$

$$J_0 \partial_{y_i} = -\partial_{x_i}$$

$$\omega_0(\partial_{x_i}, J_0 \partial_{x_i}) = \omega_0(\partial_{x_i}, \partial_{y_i}) > 0$$

Almost complex structure on a manifold

On a symplectic manifold (M, ω) , an **almost complex structure** consists of a compatible complex structure $J_x : T_x M \rightarrow T_x M$ on every tangent space $(T_x M, \omega_x)$.



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- But there are many choices for a compatible almost complex structure.
- The space of ω -compatible almost complex structures $\mathcal{J}(M, \omega)$ is connected : so any two elements J_0 and J_1 can be connected by a path $\{J_t\}_{t \in [0,1]}$ in $\mathcal{J}(M, \omega)$.

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- In fact $\mathcal{J}(M, \omega)$ is contractible, but that is not relevant for us now.

J -holomorphic curves

- In (M, ω, J) , a J -holomorphic map

$$u : \Sigma \rightarrow M$$

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- For a J -holomorphic map, the ω -area is

$$A(u) := \int_{\Sigma} u^* \omega = \int_{\Sigma} \omega(\partial_s u, \partial_t u) \geq 0,$$

and is zero iff u is constant.

J-holomorphic curves, a technical result

Theorem

Let $A \in H_2(M)$ be such that there is no sphere class $B \in H_2(M)$ satisfying

$$0 < \omega(B) < \omega(A).$$

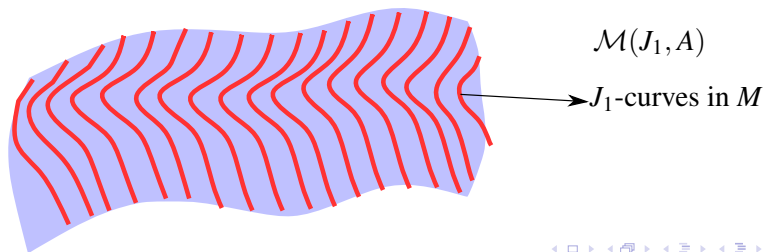
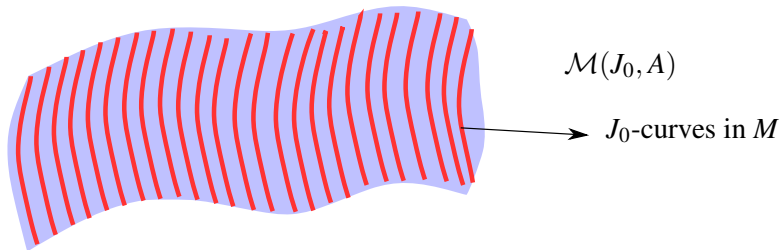
Then the moduli space

$$\mathcal{M}(A, J)$$

of J -holomorphic maps of class A is compact. Furthermore, if J_0, J_1 are ω -compatible, $\mathcal{M}(A, J_0)$ and $\mathcal{M}(A, J_1)$ are cobordant.

J -holomorphic curves

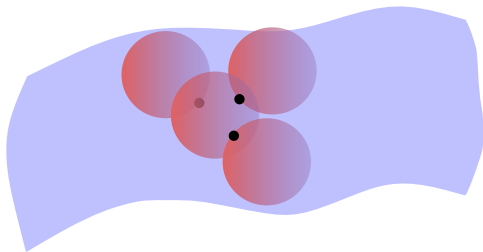
$J_0, J_1 : \omega$ -compatible almost complex structures.



Aside: Gromov compactness

Theorem

Let J be an ω -compatible almost complex structure on M . A sequence of J -holomorphic spheres $u_\nu : \mathbb{S}^2 \rightarrow M$ in the class $A \in H_2(M)$ has a subsequence whose **Gromov limit** is a map from a nodal curve to M .



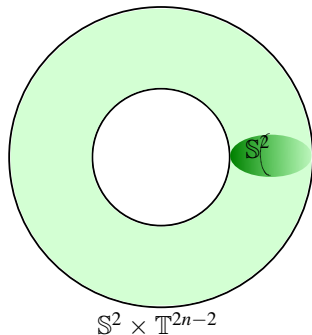
J -holomorphic curves in a product

- We will be interested in $M = \mathbb{S}^2 \times \mathbb{T}^{2n-2}$ with a product symplectic structure.
- For the product almost complex structure

$$J_{std} := J_{\mathbb{S}^2} \times J_{\mathbb{T}^2},$$

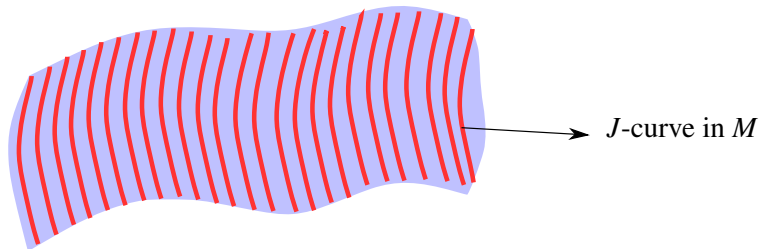
through any point $p \in M$, there is one J -curve in the homology class $[\mathbb{S}^2 \times \{point\}]$.

- For any J , the “count” of J -curves in the class $[\mathbb{S}^2 \times \{point\}]$ through a fixed point p is 1.



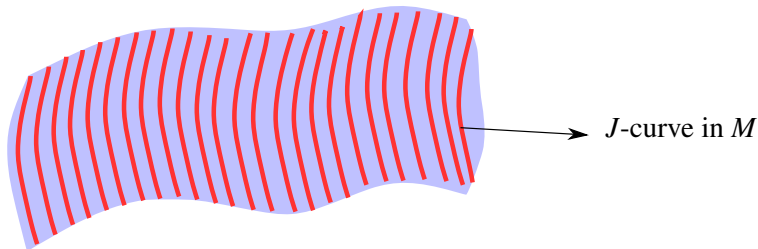
J -holomorphic curves

We continue with $M = \mathbb{S}^2 \times \mathbb{T}^{2n-2}$. For any ω -compatible almost complex structure, and any fixed point $p \in M$, the count of J -curves through p with homology class $[\mathbb{S}^2 \times \text{point}] \in H_2(M, \mathbb{Z})$ is 1. This is an example of a **Gromov-Witten invariant**.



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In general Gromov-Witten invariants are counts of curves satisfying certain constraints, such as passing through some fixed points or cycles.

Gromov-Witten invariant

Wikipedia : *Gromov-Witten invariants are rational numbers that, in certain situations, count pseudoholomorphic curves meeting prescribed conditions in a given symplectic manifold.*

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Kontsevich produced a recursive formula for the number of degree d rational curves in $\mathbb{P}^2$ passing through $3d - 1$ fixed generic points.

Back to Gromov's non-squeezing theorem

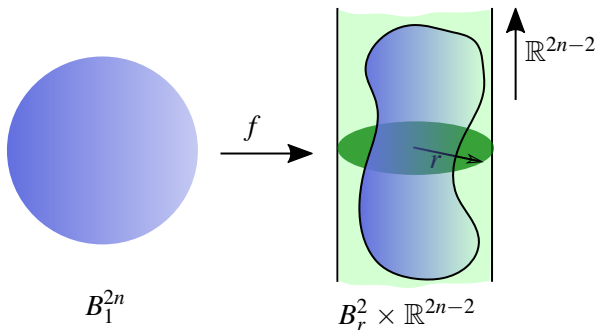
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Proof of the non-squeezing theorem

Assume the contrapositive. Suppose for some $r < 1$ there is a symplectic embedding

$$f : B_1^{2n} \rightarrow B_r^2 \times \mathbb{R}^{2n-2}.$$

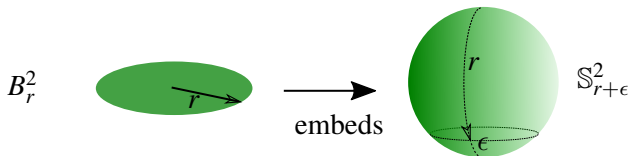


Proof of the non-squeezing theorem

We change the target space to a compact symplectic manifold. Firstly there is a symplectic embedding

$$B_r^2 \rightarrow \mathbb{S}_{r+\epsilon}^2,$$

where $\mathbb{S}_{r+\epsilon}^2$ has area $\pi(r+\epsilon)^2$, and $r < r+\epsilon < 1$.

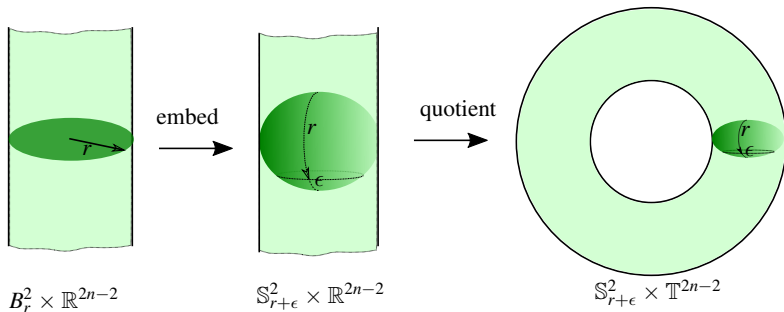


Proof of the non-squeezing theorem

Quotient the factor $\mathbb{R}^{2n-2}$ by a lattice $a\mathbb{Z}^{n-2}$:

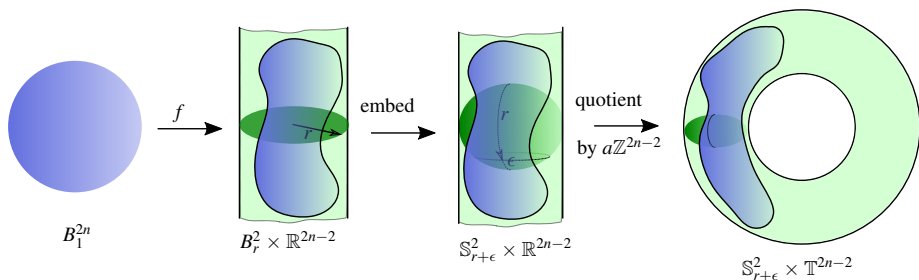
$$\mathbb{R}^{2n-2}/a\mathbb{Z}^{2n-2} \simeq \mathbb{T}^{2n-2},$$

where a is chosen to be large.



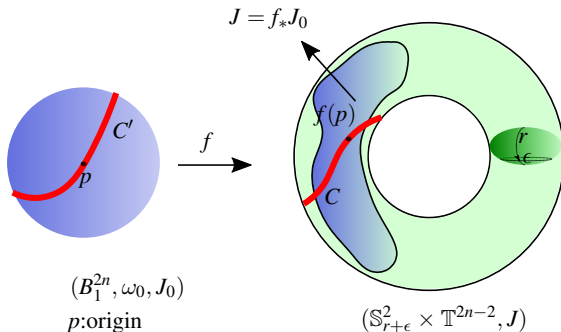
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The constant a is chosen so that the composed map is an embedding.



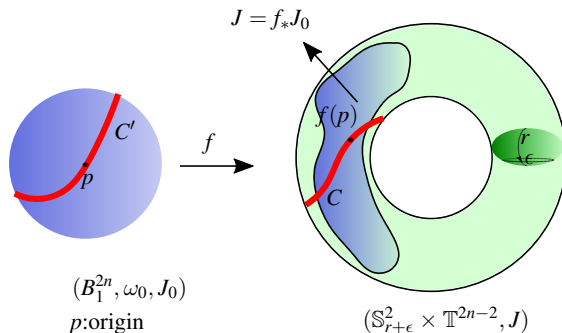
Proof of the non-squeezing theorem

By our earlier result, for any compatible almost complex structure J , there is a J -curve C of homology class $[\mathbb{S}^2] \times pt$ through the point $f(p)$.



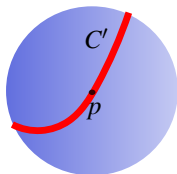
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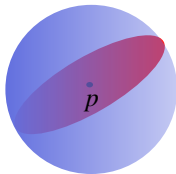


Let $C' := f^{-1}(C)$.

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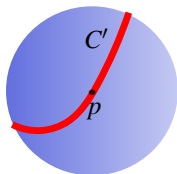


$(B_1^{2n}, \omega_0, J_0)$
 p :origin

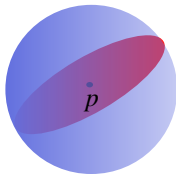


- Holomorphic curves are minimal surfaces for the metric $\omega(\cdot, J\cdot)$.
- Thus C' is a minimal surface for the standard Euclidean metric $\omega_0(\cdot, J_0\cdot)$.

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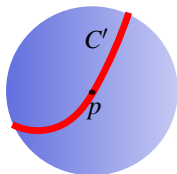


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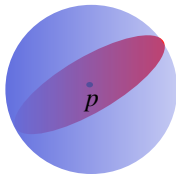
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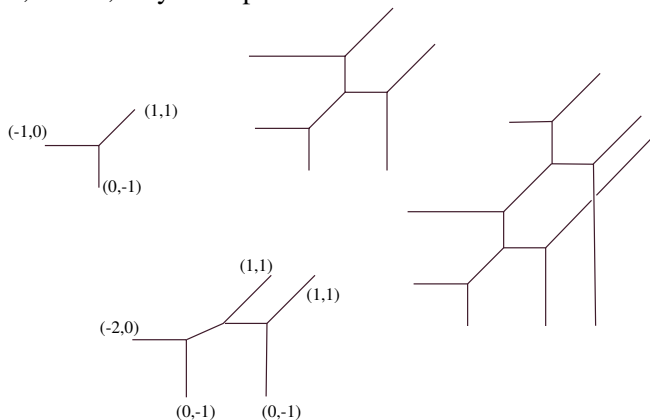
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- Therefore,
 $\pi \leq \text{Area}(C') < \pi(r + \epsilon)^2$
which contradicts $r + \epsilon < 1$.

Table of Contents

- 1 The symplectic form
- 2 Holomorphic curves in symplectic geometry
- 3 Mikhalkin's tropical graphs**
- 4 Single cut
- 5 Multiple cut
- 6 Mikhalkin-like curves in almost torics

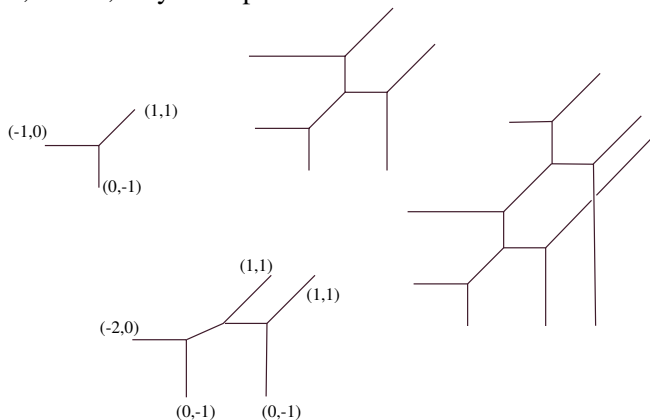
Tropical curves in $\mathbb{R}^2$

- A **tropical curve** is a graph in $\mathbb{R}^2$ whose edges have a direction in $\mathbb{Z}^2$; and at all vertices, the directions of the incident edges satisfy a **balancing condition**, that is, they add up to 0.



Tropical curves in $\mathbb{R}^2$

- A **tropical curve** is a graph in $\mathbb{R}^2$ whose edges have a direction in $\mathbb{Z}^2$; and at all vertices, the directions of the incident edges satisfy a **balancing condition**, that is, they add up to 0.



- Edges may be incident on a single vertex. These edges are infinite, and they have directions in $\mathbb{Z}_{>0}\{(0,-1), (-1,0), (1,1)\}$.

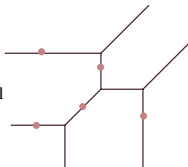
Tropical curves in $\mathbb{R}^2$

- For any integer $d \geq 1$, $3d - 1$ points determine a discrete set of tropical curves of degree d .

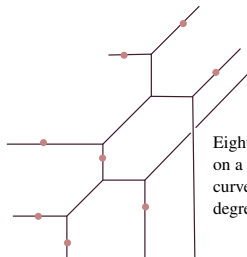
Two points
on a tropical
line



Five points
on a tropical
curve of
degree 2



Eight points
on a tropical
curve of
degree 3



Tropical curves in $\mathbb{R}^2$

- Let N_d be the number of rational curves of degree d in $\mathbb{P}^2$ passing through $3d - 1$ points $x_1, \dots, x_{3d-1}$ in general position, i.e.

$$N_d := \{u : \mathbb{P}^1 \rightarrow \mathbb{P}^2 \text{ of degree } d : x_1, \dots, x_{3d-1} \in u(\mathbb{P}^1)\} / \text{Aut}(\mathbb{P}^1).$$

Tropical curves in $\mathbb{R}^2$

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- $N_1 = N_2 = 1$ (Elementary), $N_3 = 12$ (1848), N_4 (1873).
- Kontsevich (1994) : recursive formula to compute N_d .

Tropical curves in $\mathbb{R}^2$

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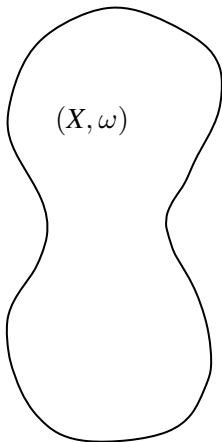
$$N_d := \{u : \mathbb{P}^1 \rightarrow \mathbb{P}^2 \text{ of degree } d : x_1, \dots, x_{3d-1} \in u(\mathbb{P}^1)\} / \text{Aut}(\mathbb{P}^1).$$

- $N_1 = N_2 = 1$ (Elementary), $N_3 = 12$ (1848), N_4 (1873).
- Kontsevich (1994) : recursive formula to compute N_d .
- (Mikhalkin, 2004): The number of degree d tropical curves through $3d - 1$ points is equal to N_d , which is the number of rational curves of degree d in $\mathbb{P}^2$.

Table of Contents

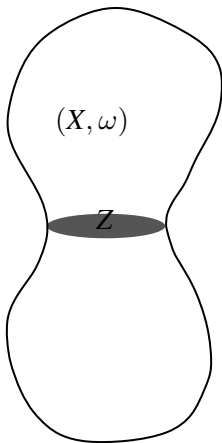
- 1 The symplectic form
- 2 Holomorphic curves in symplectic geometry
- 3 Mikhalkin's tropical graphs
- 4 Single cut**
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Symplectic cut



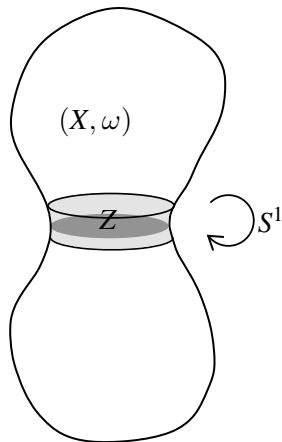
Given a symplectic manifold (X, ω) ,

Symplectic cut



a separating hypersurface $Z \subset X$,

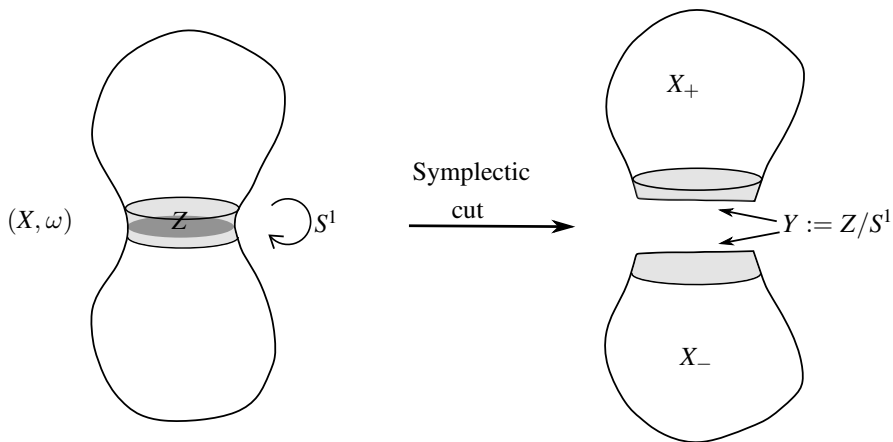
Symplectic cut



and a Hamiltonian S^1 -action in a neighborhood of Z ,

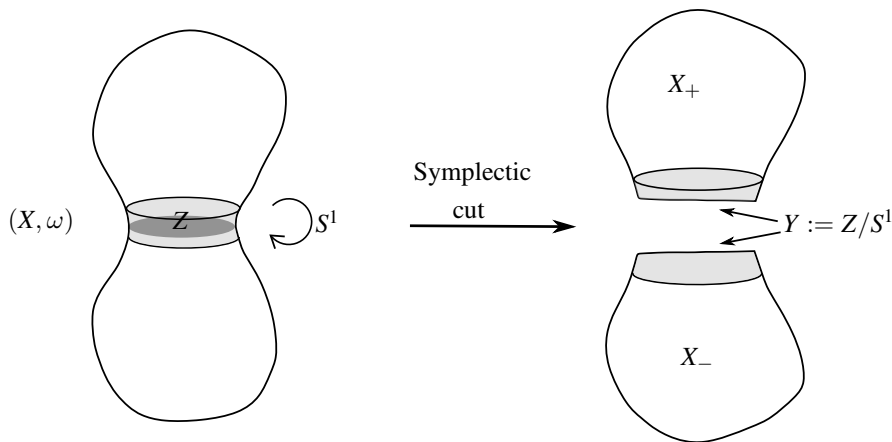
Symplectic cut

a symplectic cut produces two symplectic manifolds X_+ , X_- .



Symplectic cut

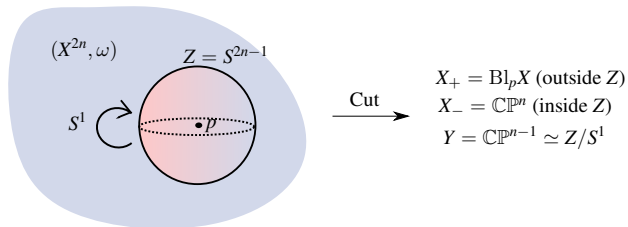
a symplectic cut produces two symplectic manifolds X_+ , X_- .



Both X_+ , X_- contain $Y := Z/S^1$ as a **relative divisor**.

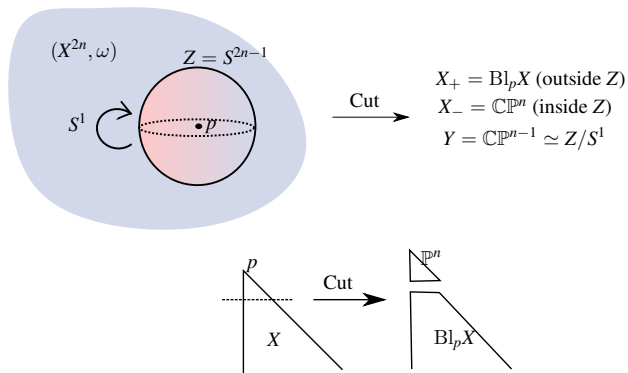
Symplectic cut : examples

- Blowing up a point.



Symplectic cut : examples

- Blowing up a point.



Symplectic cut : examples

- $X = \text{Bl}_p \mathbb{P}^2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$.
- $X_+ = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$, $X_- = \mathbb{P}^1 \times \mathbb{P}^1$, $Y = \mathbb{P}^1$ with trivial normal bundle.

Cutting preserves curve counts

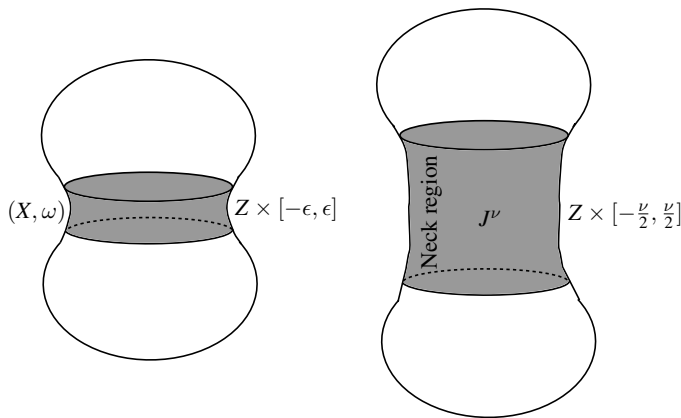
- Let $\mathcal{M}(X, \beta)$ be a zero dimensional moduli space of pseudoholomorphic spheres of type $\beta \in H_2(X)$ (possibly with point constraints),
- and let $\mathcal{M}(\mathcal{X}, \beta)$ be the moduli space of broken spheres in the same class.

Theorem (Ionel-Parker, Hofer et al, and others)

$$\#\mathcal{M}(X, \beta) = \#\mathcal{M}(\mathcal{X}, \beta).$$

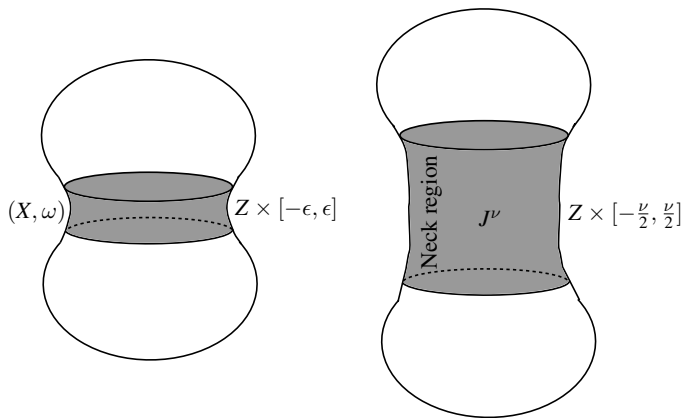
Idea of proof : stretching necks

For any $\nu > 0$, we equip the symplectic manifold (X, ω) with a tamed almost structure J^ν so that (X, J^ν) has a **neck** of length ν .



Idea of proof : stretching necks

For any $\nu > 0$, we equip the symplectic manifold (X, ω) with a tamed almost structure J^ν so that (X, J^ν) has a **neck** of length ν .



The neck region is a fibration $Z \times [-\frac{\nu}{2}, \frac{\nu}{2}] \rightarrow Y$, and the fibers are holomorphic cylinders $S^1 \times [-\frac{\nu}{2}, \frac{\nu}{2}]$.

Convergence to broken maps

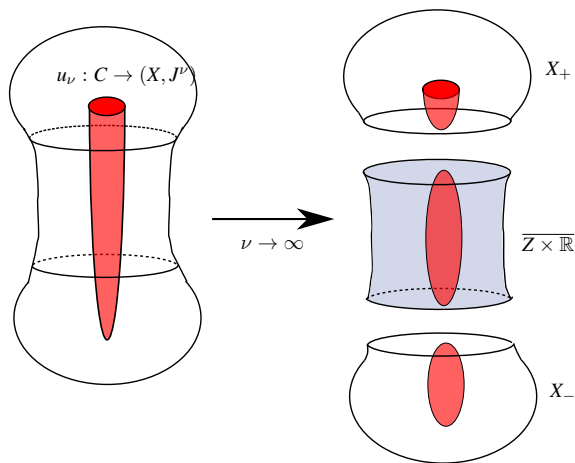
- For any ν_1, ν_2 , the moduli spaces of spheres $\#\mathcal{M}(X, \beta, J^{\nu_1})$ and $\#\mathcal{M}(X, \beta, J^{\nu_2})$ are cobordant, and so, $\#\mathcal{M}(X, \beta, J^\nu)$ is ν -independent.

Convergence to broken maps

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- A sequence of maps $u_\nu \in \mathcal{M}(X, \beta, J^\nu)$ converges, after passing to a subsequence, to a broken map in $\mathcal{M}(\mathcal{X}, \beta)$, and therefore, we obtain

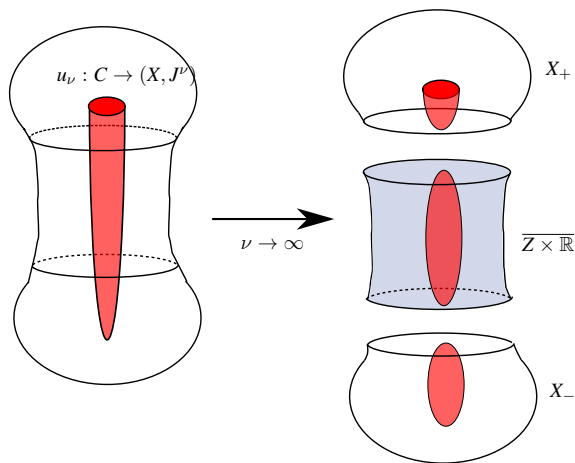
$$\#\mathcal{M}(X, \beta, J^\nu) = \#\mathcal{M}(\mathcal{X}, \beta).$$

Convergence to broken maps



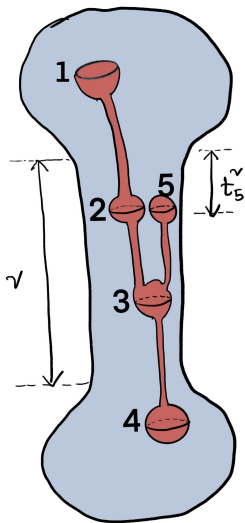
In the limit some curve components collapse into $\overline{Z \times \mathbb{R}}$ which is a thickening of the relative divisor.

Convergence to broken maps



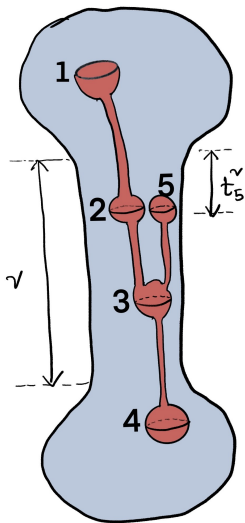
In the limit some curve components collapse into $\overline{Z \times \mathbb{R}}$ which is a thickening of the relative divisor. Thus the target space of a broken map is the **broken manifold** $\mathcal{X} = X_+ \cup \overline{Z \times \mathbb{R}} \cup X_-$.

Convergence to broken maps



- A converging sequence of maps $u_\nu : C \rightarrow X^\nu$ consists of pockets of high area separated by long cylinders :

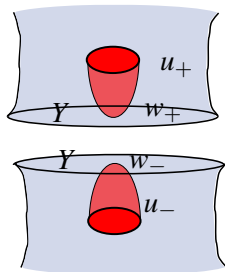
Convergence to broken maps



- A converging sequence of maps $u_\nu : C \rightarrow X^\nu$ consists of pockets of high area separated by long cylinders :
- A pocket of high area converges modulo a translation in the target space.
For example $e^{t_5^\nu} u_\nu$ converges to a limit component of the broken map.

Broken maps

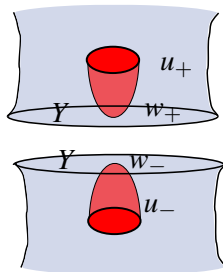
$Y := Z/S^1$
is the relative
divisor



- A **broken map** consists of components in X_+ , X_- and the neck piece $\overline{Z \times \mathbb{R}}$ satisfying the following matching condition at nodes:

Broken maps

$Y := Z/S^1$
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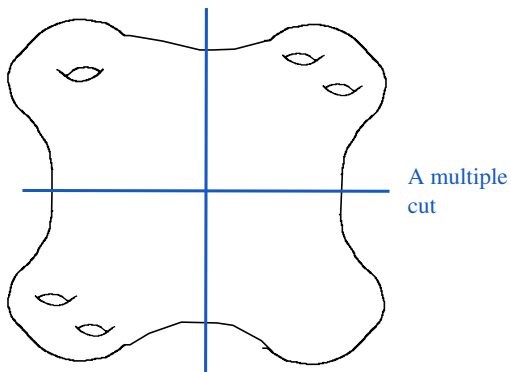


- A **broken map** consists of components in X_+ , X_- and the neck piece $\overline{Z \times \mathbb{R}}$ satisfying the following matching condition at nodes:
- $u_+(w_+)$, $u_-(w_-)$ are the same points in the relative divisor Y ,
- The intersection multiplicities of u_+ , u_- with Y at the nodal point are equal.

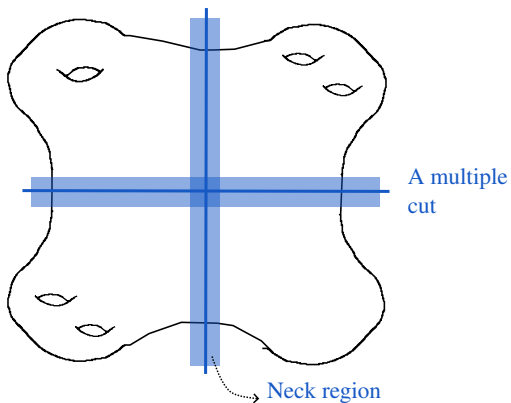
Table of Contents

- 1 The symplectic form
- 2 Holomorphic curves in symplectic geometry
- 3 Mikhalkin's tropical graphs
- 4 Single cut
- 5 Multiple cut**
- 6 Mikhalkin-like curves in almost torics

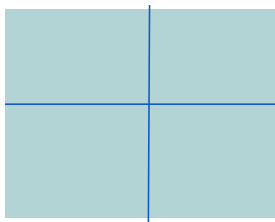
Multiple cut



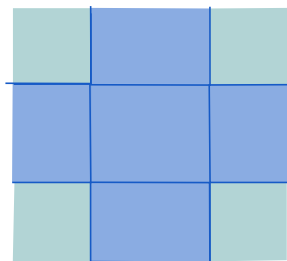
Multiple cut



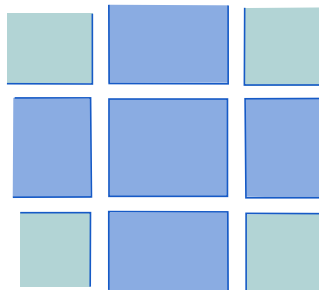
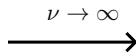
Multiple cut



Multiple cut on a manifold



Neck-stretched manifold (X, J_ν)

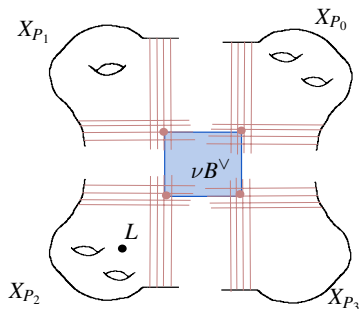


Broken manifold

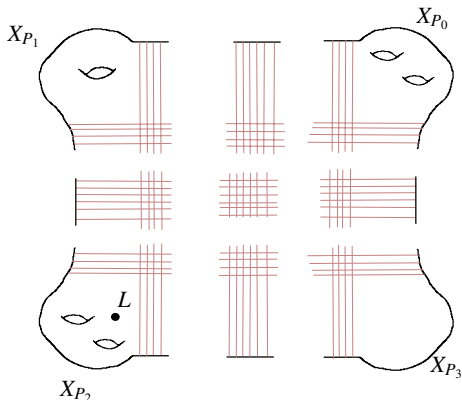
Neck-stretching, multiple cut



Dual complex

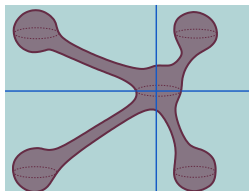


Neck-stretched manifold, $\nu \in \mathbb{R}$ large



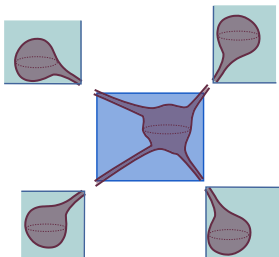
Broken manifold

Neck-stretching, multiple cut

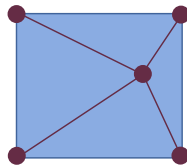


A converging sequence
of maps $\{u_\nu\}_\nu$
in a neck-stretched
manifold

$$\nu \rightarrow \infty$$



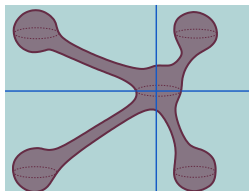
The limit broken map u



Tropical graph of u

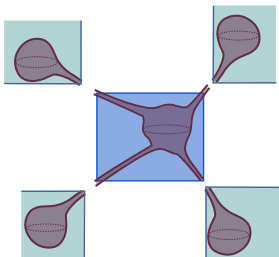
The tropical graph lives in the dual complex $B^\vee$.

Neck-stretching, multiple cut

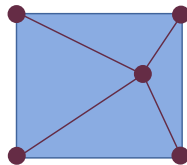


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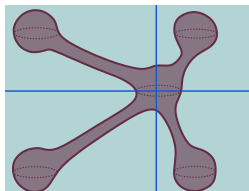
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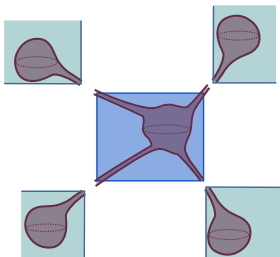
The tropical graph lives in the dual complex $B^\vee$. The edge directions represent intersection multiplicities at the node, and lie in $t\mathbb{Z}$.

Neck-stretching, multiple cut

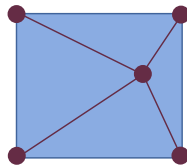


A converging sequence
of maps $\{u_\nu\}_\nu$
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$\nu \rightarrow \infty$
 $\longrightarrow$



The limit broken map u



Tropical graph of u

The tropical graph lives in the dual complex $B^\vee$. The edge directions represent intersection multiplicities at the node, and lie in $\mathbb{t}\mathbb{Z}$.

Result (Brett Parker, V-Woodward)

Let $\mathcal{X}$ be a broken manifold produced by a multiple cut. For a sphere type β (with point constraints), $\#\mathcal{M}(X, \beta) = \#\mathcal{M}(\mathcal{X}, \beta)$.

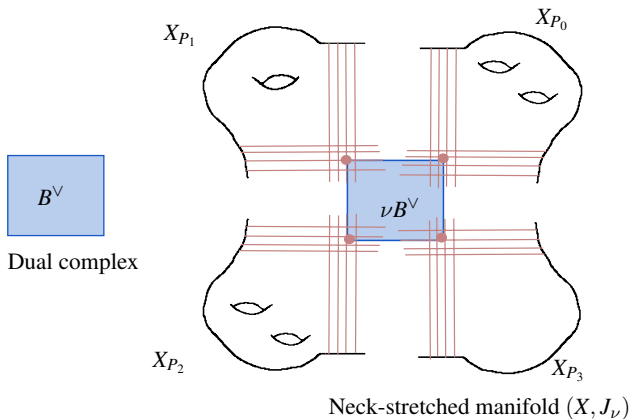
Broken map

A broken map consists of a

- a **map part** u which is a collection of holomorphic maps in the pieces of the broken manifold $\mathcal{X}$ satisfying a matching condition at nodes,
- and a **tropical part** which is a tropical graph Γ underlying u in the dual complex $B^\vee$.

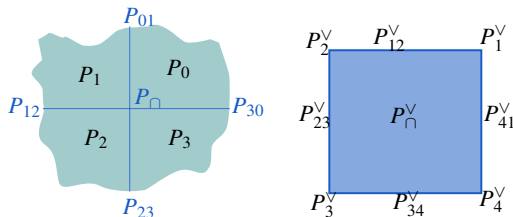
Dual complex

- A dual complex encodes the proportion in which the neck is stretched in different directions. Example:



Dual complex

- The dual complex consists of a complementary dimensional polytope denoted $P^\vee$ for every P in the polyhedral decomposition underlying the multiple cut. Example, continued:

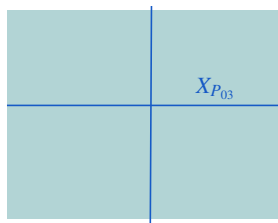


The case of toric surfaces

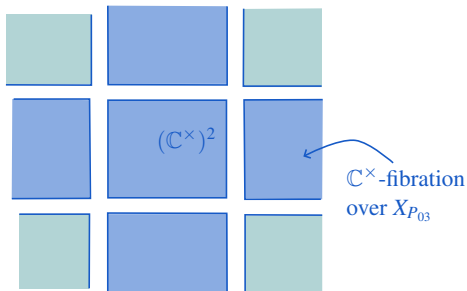
- A toric surface has a **moment map** $\Phi : X \rightarrow \mathbb{R}^2$ whose image is a convex polytope $\Delta \subset \mathbb{R}^2$, and $\Phi|_{\Phi^{-1}(\Delta^\circ)}$ is a fibration by Lagrangian tori.
- We assume cuts are toric, that is, they are along level sets of components of the moment map.

The case of toric surfaces

- The tropical graph determines the map, provided there are no map components in pieces X_P , $\text{codim}(P) = 0$.
- The other pieces of the broken manifold are toric compactifications of $(\mathbb{C}^\times)^2$ and the map intersects the toric boundary at isolated points.



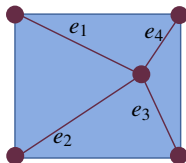
Multiple cut on a manifold



Broken manifold

The case of toric surfaces

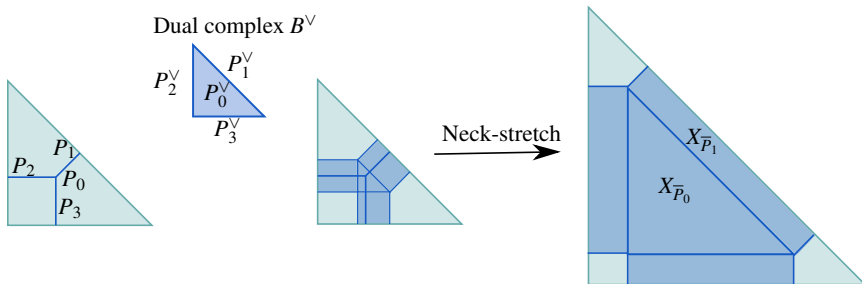
- The tropical graph satisfies a balancing condition: on ‘good’ vertices the directions of edges add to zero.



$$\begin{aligned}\mathcal{T}(e_i) &\in \mathbb{Z}^2 \\ \sum_i \mathcal{T}(e_i) &= 0\end{aligned}$$

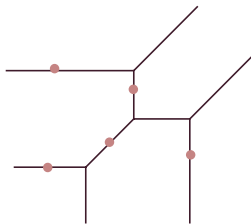
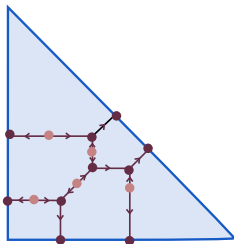
Example of $\mathbb{P}^2$

- Consider the following multiple cut on $\mathbb{P}^2$.



Example of $\mathbb{P}^2$

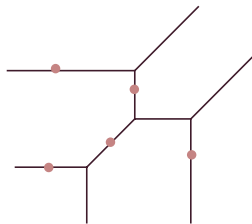
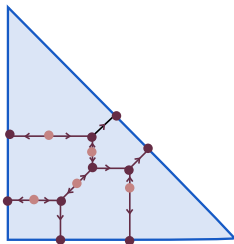
- Given a set of five points in the P_0 -neck region, a degree 2 broken map has the following tropical graph (left). The broken map can be recovered from the graph.



- Thus, counting curves in $\mathbb{P}^2$ is equivalent to counting tropical graphs.

Example of $\mathbb{P}^2$

- Given a set of five points in the P_0 -neck region, a degree 2 broken map has the following tropical graph (left). The broken map can be recovered from the graph.



- Thus, counting curves in $\mathbb{P}^2$ is equivalent to counting tropical graphs.
- The tropical graph corresponds to a Mikhalkin tropical curve in $\mathbb{R}^2$ (right).

Example of $\mathbb{P}^2$

- We have thus reproved the following result of Mikhalkin (2004).

Result

The number of degree d rational curves in $\mathbb{P}^2$ through $3d - 1$ generic points is equal to the number of tropical curves in $\mathbb{R}^2$ through $3d - 1$ generic points.

Table of Contents

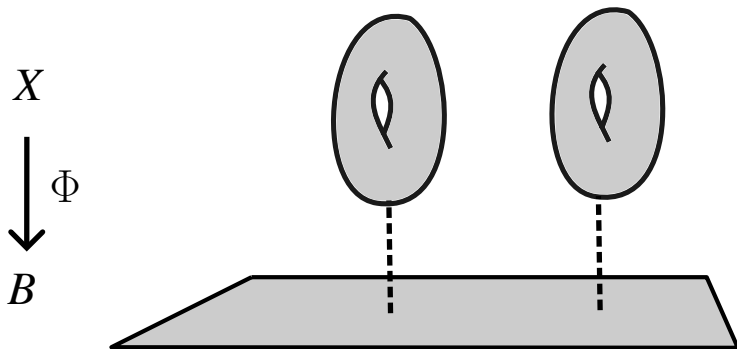
- 1 The symplectic form
- 2 Holomorphic curves in symplectic geometry
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Almost toric manifolds

- A **toric manifold** has a moment map $\Phi : X \rightarrow B$ where $B \subset \mathfrak{t}^\vee$ is a polytope. Assuming compact fibers, on the interior $B^\circ \subset B$, Φ is a Lagrangian fibration,

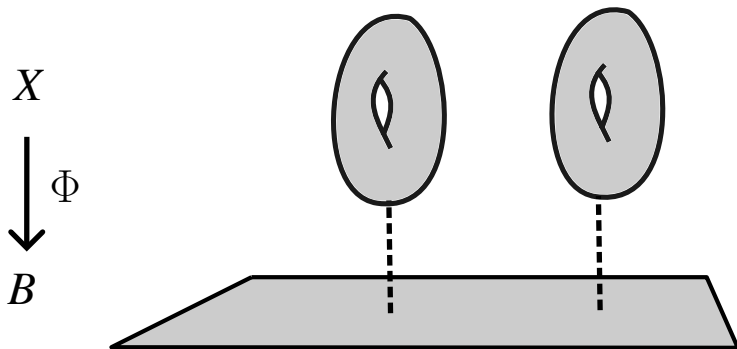
Almost toric manifolds

- A **toric manifold** has a moment map $\Phi : X \rightarrow B$ where $B \subset \mathfrak{t}^V$ is a polytope. Assuming compact fibers, on the interior $B^\circ \subset B$, Φ is a Lagrangian fibration, and there is a trivialization $B^\circ \times T$.



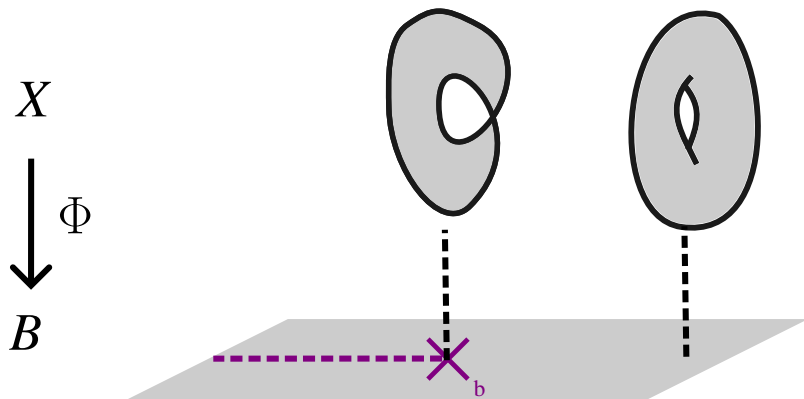
Almost toric manifolds

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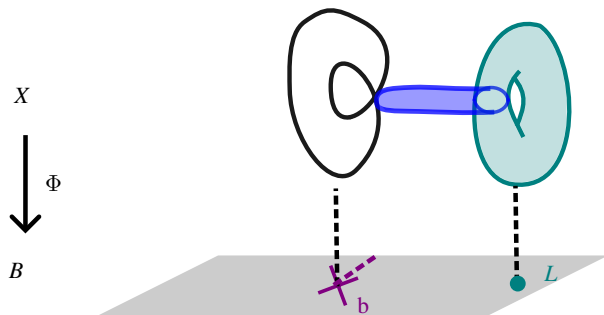
- We assume $\dim(X) = 4$, $\dim(T) = 2$.

Almost toric manifold



- An **almost toric manifold** is also a Lagrangian fibration $\Phi : X \rightarrow B$ with the additional feature that the map Φ has **focus-focus** singularities.
- For a focus-focus singular value b , the fiber $\Phi^{-1}(b)$ is a pinched torus. The monodromy $A_b : T^2 \rightarrow T^2$ around the singular fiber $\Phi^{-1}(b)$ is given by a shear map $\hat{A}_b \in GL(2, \mathbb{Z})$.

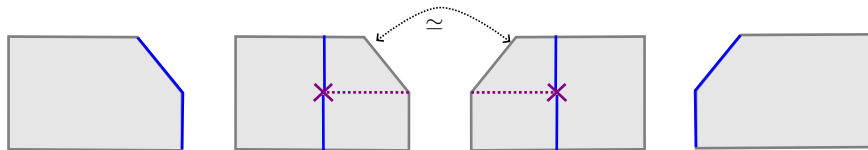
Holomorphic curves in almost toric manifolds



- Assuming a ‘standard complex structure’, there is an exceptional holomorphic curve in X that projects to a ray ending in the focus-focus singular value b (Auroux), and that is perpendicular to the branch cut.

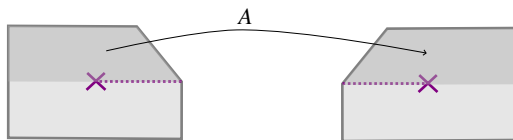
Example, almost toric manifold

Consider $X = \text{Bl}_{\text{point}}(\mathbb{P}^1 \times \mathbb{P}^1)$, which is a Lefschetz fibration over $\mathbb{P}^1$ with regular fiber $\mathbb{P}^1$, and one singular fiber (in blue). The following are different almost toric fibrations of X .



Example, almost toric manifold

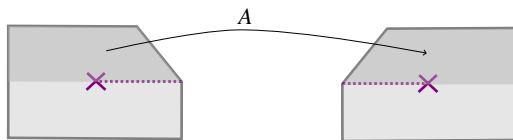
The middle two are the same fibration $\Phi : X \rightarrow B$, drawn using different branch cuts.



Here $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

Example, almost toric manifold

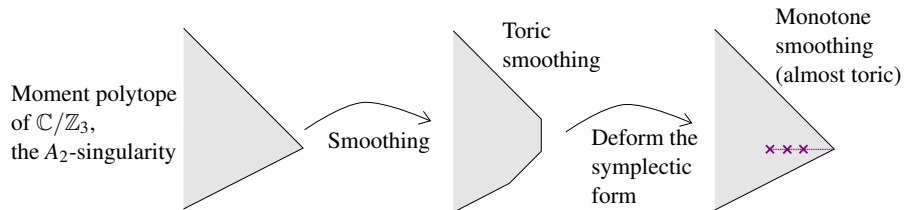
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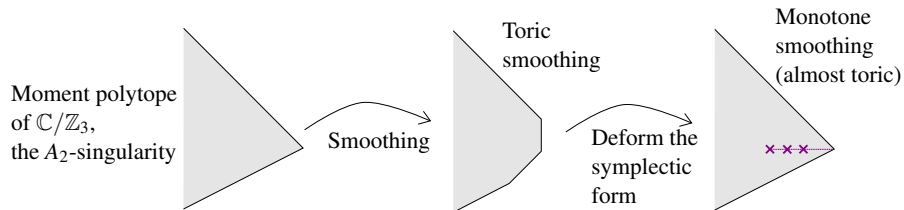
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The space B is a **singular affine integral manifold**. That is, in the complement of singularities, B is covered by charts to $\mathfrak{t}^\vee \simeq \mathbb{R}^2$ and the transition functions are of the form $\phi(x) = Ax + b$, where $b \in \mathbb{R}^2$, $A \in GL(2, \mathbb{Z})$.

Example, almost toric manifold, A_n -singularity

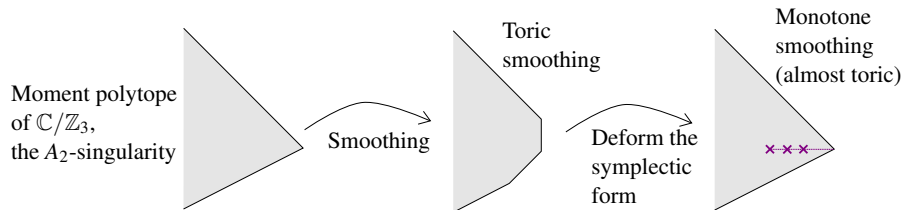


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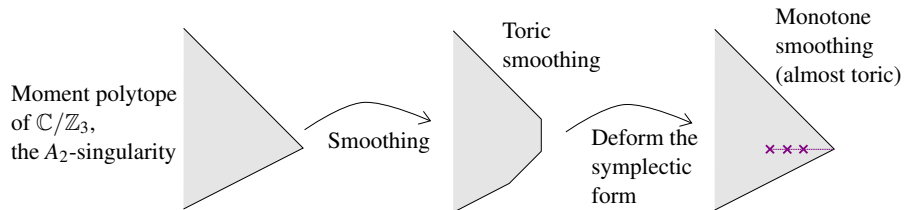
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Example, almost toric manifold, A_n -singularity



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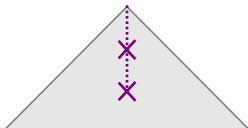
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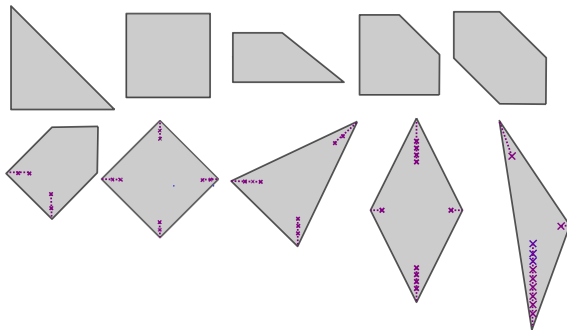
- The **toric smoothing** has two holomorphic/symplectic (-2) -spheres.
- In the **monotone smoothing**, these spheres are Lagrangian.
- In the smoothing of the **A_n -singularity** $\mathbb{C}^2/\mathbb{Z}_{n+1}$, there are n such spheres, and n focus-focus singularities in the almost toric fibration.

Example, almost toric manifold, T^*S^2

Almost toric fibration on T^*S^2 , which is also a smoothing of an A_1 -singularity.



Note: the toric divisor is smooth.



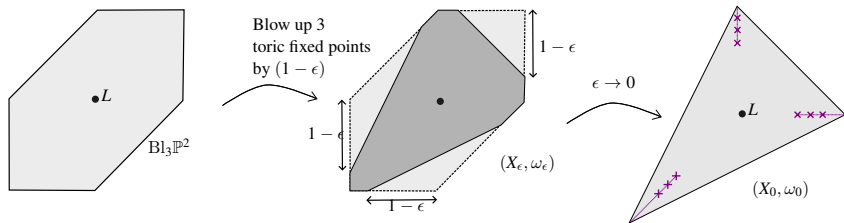
- The del Pezzo surfaces are

$$\mathrm{Bl}_k \mathbb{P}^2, \quad 0 \leq k \leq 8, \quad \mathbb{P}^1 \times \mathbb{P}^1.$$

Del Pezzos possess almost toric structures (Vianna).

- The neighborhoods of vertices are $\mathbb{Z}_p$ -quotients of smoothings A_n -singularities. These are smoothings of **cyclic quotient T -singularities**.

Example, almost toric fibration on $\mathrm{Bl}_6\mathbb{P}^2$



Disk potential

- Let $L \subset (X, \omega)$ be a Lagrangian brane. The **potential** $W_{X,L}$ is the signed count of pseudoholomorphic disks bounding L that satisfies a point constraint on the boundary.

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$$c_1(TX) = \tau[\omega]$$

for some $\tau > 0$. Del pezzos are the only compact monotone symplectic four-manifolds.

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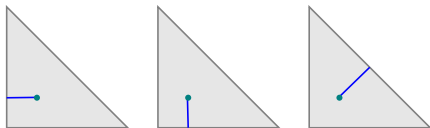
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- Each of the almost toric del Pezzos has a monotone torus fiber.
- In both the smooth manifold X and broken manifold $\mathcal{X}$, a monotone Lagrangian $L \subset X$ has the same potential, i.e. $W_{X,L} = W_{\mathcal{X},L}$.

Disk potential of the Clifford torus in $\mathbb{P}^2$

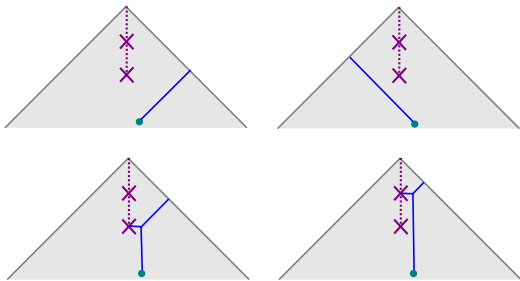
The potential is a signed count of disks with Maslov index 2. For the Clifford torus in $\mathbb{P}^2$, there are 3 such disks:



(This is a cartoon diagram.)

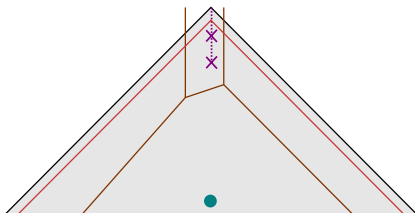
Disk potential, smoothening of A_1 -singularity

Consider a monotone Lagrangian torus L in the smoothening of the A_1 -singularity. There are 4 disks of Maslov index 2.

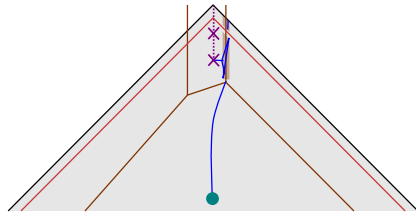


(This is a cartoon diagram.)

Multiple cut on smoothening of A_1 -singularity

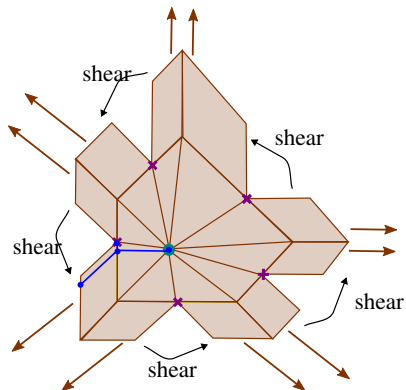
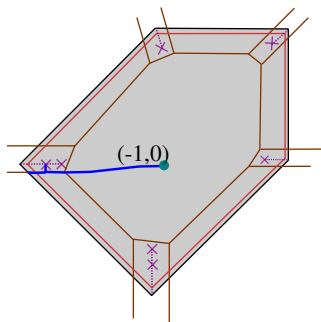


A multiple cut

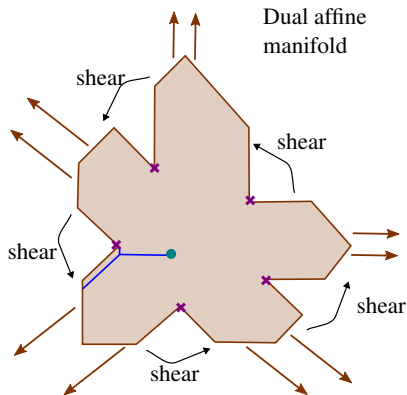
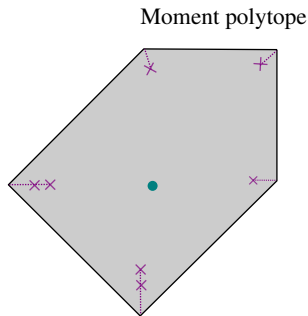


A broken disk

Multiple cut on $\text{Bl}_4\mathbb{P}^2$



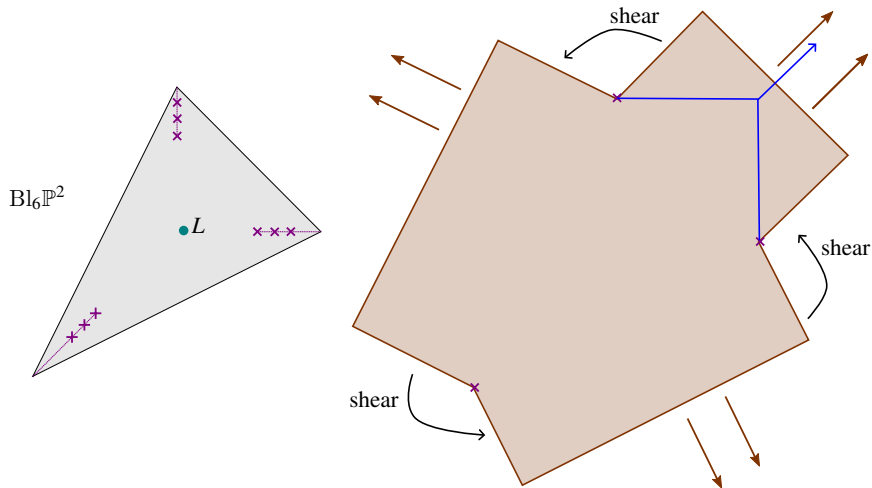
Tropical disk in the dual complex of $Bl_4\mathbb{P}^2$



Result

In a monotone almost toric fibration, disks of Maslov index 2 correspond to rigid tropical graphs in the dual complex $B^\vee$ whose paths originate from L and singular points in $B^\vee$.

Tropical sphere in the dual complex of $\mathrm{Bl}_6\mathbb{P}^2$



One of the 27 lines in the cubic.

Thank you

Based on joint work with Chris Woodward:
Tropical Fukaya Algebras (EMS Monographs in Mathematics, 2025),
Tropical disk potentials for almost toric manifolds (arXiv:2604.03161).