Row-Factorization matrices and generic ideals

Kriti Goel
Early Career Fellow
Indian Institute of Technology Gandhinagar, India
(joint work with Om Prakash Bhardwaj and Indranath Sengupta)

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Numerical semigroups

- Let $n_1, \ldots, n_s \in \mathbb{N}$ such that $\gcd(n_1, \ldots, n_s) = 1$. Then an additive submonoid

$$H = \langle n_1, \ldots, n_s \rangle = \left\{ \sum_{i=1}^{s} a_i n_i \mid a_1, \ldots, a_s \in \mathbb{N} \right\}$$

  is called a numerical semigroup. (i.e., $\mathbb{N} \setminus H$ is finite)
- Frobenius number: $\mathrm{F}(H) = \max \{\mathbb{N} \setminus H\}$.
- Pseudo-Frobenius number: $f \notin H$ such that $f + h \in H$, for all $h \in H \setminus \{0\}$.
- The set of pseudo-Frobenius numbers of H is denoted by $\mathrm{PF}(H)$.
- H is symmetric $\iff \mathrm{PF}(H) = \{\mathrm{F}(H)\}$.
- Let k be a field. Then $k[H] = k[t^h \mid h \in H]$ is the semigroup ring of H.
- (Kunz, 1970) H is symmetric if and only if $k[H]$ is a Gorenstein ring.

Row-Factorization matrices

- The set of pseudo-Frobenius numbers of H,

$$\mathrm{PF}(H) = \{f \notin H \mid f + n_i \in H, \text{ for all } i = 1, \ldots, s\}.$$

- (Moscariello, 2016) Let $f \in \mathrm{PF}(H)$. An $s \times s$ matrix $M = (m_{ij})$ is a row-factorization (RF) matrix of f if for all $i = 1, \ldots, s$,

$$\sum_{j=1}^{s} m_{ij}\, n_j = f,$$

where $m_{ii} = -1$ and $m_{ij} \in \mathbb{N}$ for all $i \neq j$, $i = 1, \ldots, s$.

- RF-matrices, in general, are not unique.
- **Theorem.** Cohen-Macaulay type of an almost Gorenstein monomial curve in $\mathbb{A}^4$ is at most 3. In other words, for an almost symmetric numerical semigroup H generated by 4 elements, $|\,\mathrm{PF}(H)| \leq 3$.
- Almost symmetric: If for any $f \in \mathrm{PF}(H) \setminus \{\mathrm{F}(H)\}$, $\mathrm{F}(H) - f \in \mathrm{PF}(H)$.

Example

- Let $H = \langle 5, 6, 9 \rangle$. Then $\mathbb{N} \setminus H = \{1, 2, 3, 4, 7, 8, 13\}$ and $\mathrm{PF}(H) = \{13\}$.

$$13 + 5 = 3(6) + 0(9) \qquad 13 + 5 = 0(6) + 2(9)$$
$$13 + 6 = 2(5) + 1(9)$$
$$13 + 9 = 2(5) + 2(6)$$

$$\mathrm{RF}(13) = \begin{bmatrix} -1 & 3 & 0 \\ 2 & -1 & 1 \\ 2 & 2 & -1 \end{bmatrix}, \qquad \mathrm{RF}(13) = \begin{bmatrix} -1 & 0 & 2 \\ 2 & -1 & 1 \\ 2 & 2 & -1 \end{bmatrix}$$

Almost arithmetic sequence

- Let $m_0, m_1, \ldots, m_p \in \mathbb{N}$ be a strictly increasing arithmetic sequence and let $n \in \mathbb{N}$ such that $\gcd(m_0, \ldots, m_p, n) = 1$. Also, assume that $\{m_0, \ldots, m_p, n\}$ is a minimal generating set for the numerical semigroup H.
- (Patil-Singh, 1990) Studied this class of numerical semigroups.
- (Patil, 1993) Gave minimal generating set of the defining ideal.
- (Patil-Sengupta, 1999) Gave a complete description of pseudo-Frobenius numbers in the above setup.
- (Bhardwaj-G-Sengupta, 2021) Give a description of the row-factorization matrices.

Generic toric ideals

- Let $H = \langle n_1, \ldots, n_s \rangle$. The semigroup ring $k[H] \simeq k[x_1, \ldots, x_s]/I_H$, where I_H is called the defining ideal of H.
- I_H is a toric ideal, generated by binomials.
- (I. Peeva-B. Sturmfels, 1998) If a toric ideal has a minimal generating set consisting of binomials with full support, then it is called generic.
- **Theorem.** If I_H is a generic toric ideal, then the ring $k[H]$ is Golod and so the Poincaré series of the residue field is rational.
- **Theorem.** (K. Eto, 2020) I_H is generic if and only if for each $f \in \mathrm{PF}(H)$, $\mathrm{RF}(f) = (a_{i,j})$ is unique and $a_{i,j} \neq a_{i',j}$ if $i \neq i'$.
- **Example.** Let $H = \langle 5, 6, 9 \rangle$. Then

$$k[H] = k[t^5, t^6, t^9] = \frac{k[x, y, z]}{(y^3 - z^2, x^3 - yz)}.$$

Generic toric ideals

- **Theorem.** (Bhardwaj-G-Sengupta, 2021) Let H be a numerical semigroup minimally generated by an almost arithmetic sequence, i.e., $H = \langle m_0, \ldots, m_p, n \rangle$, where $m_i = m_0 + id$ for $i \in [1, p]$ and $\gcd(m_0, n, d) = 1$.

  (i) If $p = 0$, then I_H is generic.
  (ii) If $p = 1$, and if $W \neq \emptyset$, $\mu > 0$, then I_H is generic. Otherwise, it is never generic.
  (iii) If $p > 1$, then I_H is not generic.

- **Theorem.** (Bhardwaj-G-Sengupta, 2021) Let H be a complete intersection numerical semigroup with embedding dimension at least 3. Then I_H is not generic.

RF-relations

- Let H be a numerical semigroup and let $f \in \mathrm{PF}(H)$. Let $\delta_1, \ldots, \delta_s$ denote the row vectors of $\mathrm{RF}(f)$. Set $\delta_{(ij)} = \delta_j - \delta_i$, for all $1 \leq i < j \leq s$.
- Then $\phi_{ij} = \mathbf{x}^{\delta^+_{(ij)}} - \mathbf{x}^{\delta^-_{(ij)}} \in I_H$ for all $i < j$. We call ϕ_{ij} an $\mathrm{RF}(f)$-relation.
- We call a binomial relation $\phi \in I_H$ an RF-relation if it is an $\mathrm{RF}(f)$-relation for some $f \in \mathrm{PF}(H)$.
- Herzog-Watanabe raised the following question:

  Question: When is I_H minimally generated by RF-relations?

- **Theorem.** (Bhardwaj-G-Sengupta, 2021) Let $H = \langle m_0, m_1, \ldots, m_p, n\rangle$ be a symmetric numerical semigroup generated by an almost arithmetic sequence, where $p = 2$ or 3. Then I_H has a minimal generating set consisting of RF-relations.

Affine semigroups in $\mathbb{N}^d$

- Let S be a finitely generated submonoid of $\mathbb{N}^d$, say generated by $n_1, \dots, n_s \subseteq \mathbb{N}^d$. Such submonoids are called affine semigroups.
- Let $G(S) \subseteq \mathbb{Z}^d$ denote the group generated by S. Set
$$\Gamma(S) := (G(S) \setminus S) \cap \mathbb{N}^d.$$
- The set of pseudo-Frobenius elements of S,
$$\mathrm{PF}(S) = \{f \in \Gamma(S) \mid f + n_j \in S,\ \forall\, j \in [1, s]\}.$$
- S has maximal projective dimension (MPD) if $\mathrm{pdim}_R\, k[S] = s - 1$. Equivalently, $\mathrm{depth}_R\, k[S] = 1$.
- (J. I. García-García et. al., 2020) S is a MPD-semigroup $\iff \mathrm{PF}(S) \neq \emptyset$.
- $|\,\mathrm{PF}(S)| < \infty$ and $\mathrm{type}(S) = |\,\mathrm{PF}(S)|$ is the type of S.
- The definition of row-factorization matrices holds.

Example

- Let $S = \langle (0,1), (3,0), (4,0), (1,4), (5,0), (2,7) \rangle$. Then, for $R = k[x_1, \ldots, x_6]$, we have a minimal free resolution

$$0 \to R(16,15) \oplus R(17,18) \to R^{12} \to R^{27} \to R^{28} \to R^{12} \to R \to k[S] \to 0.$$

$$\mathrm{PF}(S) = \{(16,15) - (15,12) = (1,3),\ (17,18) - (15,12) = (2,6)\}.$$

$$\mathrm{RF}(1,3) = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 0 & 0 \\ 3 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 3 & 2 & 0 & 0 & -1 & 0 \\ 10 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

Frobenius element

- We define the (set of) Frobenius elements of S by

$$\mathrm{F}(S) = \{f \in \Gamma(S) \mid f = \max_{\prec} \Gamma(S), \text{ with respect to some term order } \prec\}.$$

- If $|\,\mathrm{PF}(S)| = 1$ and Frobenius element exists, then S is called a symmetric semigroup.
- The semigroup $S_1 = \langle(0,1),(3,0),(5,0),(1,3),(2,3)\rangle$ is a symmetric semigroup as $\mathrm{PF}(S_1) = \{(7,2)\}$ and $(7,2) = \max_{\prec} \Gamma(S_1)$, where $\prec$ is a graded lexicographic order.
- Let $S_2 = \langle n_1, n_2, n_3, n_4\rangle$ where, for $h \geq 2$, $n_1 = (2h-1)2h$, $n_2 = (2h-1)(2h+1)$, $n_3 = 2h(2h+1)$ and $n_4 = 2h(2h+1)+2h-1$. Let $\bar{S}_2$ denote the projective closure of the monomial curves. Then

$$\bar{S} = \langle(0,n_4),(n_1,n_4-n_1),(n_2,n_4-n_2),(n_3,n_4-n_3),(n_4,0)\rangle.$$

  Then $\mathrm{PF}(\bar{S}) = \{f = (16h^3-6h+1, 8h^2-6h+1)\}$ but f is not a Frobenius element.

RF-matrices and generic toric ideals

- **Theorem.** (Bhardwaj-G-Sengupta) Let S be a MPD-semigroup. If I_S is generic, then for each $f \in \mathrm{PF}(S)$, $\mathrm{RF}(f) = (a_{i,j})$ is unique and $a_{i,j} \neq a_{i',j}$ if $i \neq i'$.
- Let $\mathrm{PF}'(S) = \mathrm{PF}(S) \setminus \{\mathrm{F}(S)\}$ and $\mathrm{PF}'(S) \neq \emptyset$. For any $g \in \mathrm{PF}'(S)$, if $\mathrm{F}(S) - g \in \mathrm{PF}'(S)$, we say that S is almost symmetric.
- **Theorem.** (Bhardwaj-G-Sengupta) Let $n \geq 4$ and $S = \langle a_1, \ldots, a_n \rangle$ be an almost symmetric MPD semigroup. Then I_S is not generic.

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Thank You