$\mathbb{A}^1$-fibrations on Affine Varieties

R. V. Gurjar

Department of Mathematics, IIT Bombay

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Introduction

▷ For simplicity we will only consider algebraic varieties over $\mathbb{C}$, the field of complex numbers.

Definition ($\mathbb{A}^1$-fibration)

A morphism $f : W \to V$ on an affine variety W to an algebraic variety V is called an $\mathbb{A}^1$***-fibration*** *if*

1. *f is dominant; and*
2. *a general fiber of f is isomorphic to the affine line $\mathbb{A}^1$.*

Generalities about log Kodaira dimension $\bar{\kappa} = -\infty$

Let X be a smooth affine variety.

▷ We can embed $X \subset Y$, where Y is a smooth projective completion such that $D := Y - X$ is a simple normal crossing (SNC) divisor.

• (SNC divisors when $\dim Y = 2$) : A divisor D of Y is an SNC divisor if all the irreducible components of D are smooth, any two irreducible components of D meet transversally and no three of them meet in a point.

▷ Let K_Y be the canonical divisor of Y. If the linear system

$$|n(K_Y + D)| = H^0\left(Y, n(K_Y + D)\right) = (0),$$

for all $n \geq 1$, we say that the ***log Kodaira dimension*** $\bar{\kappa}(X) = -\infty$.

▷ This was defined by S. Iitaka.

Basic properties of $\bar{\kappa}$

Two very basic properties of $\bar{\kappa}$ are the following (the proofs are not difficult).

1. Let $\varphi : X \to Z$ be a dominant morphism between smooth varieties such that a general fiber of φ is a finite set (i.e., φ is quasi-finite). If $\bar{\kappa}(X) = -\infty$, then $\bar{\kappa}(Z) = -\infty$.

2. Let $\varphi : X \to Z$ be a finite unramified morphism between smooth affine varieties. Then $\bar{\kappa}(X) = -\infty$ if and only if $\bar{\kappa}(Z) = -\infty$.

A result due to Fujita–Miyanishi–Sugie

T. Fujita–M. Miyanishi–T. Sugie proved the following fundamental result in 1979.

Theorem 1

A smooth affine surface X has $\bar{\kappa}(X) = -\infty$ if and only if X has an $\mathbb{A}^1$-fibration $f : X \to C$ onto a smooth algebraic curve C (which can be a projective curve).

As an easy consequence of the above theorem we get:

Theorem 2 (Cancellation Theorem for $\mathbb{C}^2$)

If $X \times \mathbb{C} \cong \mathbb{C}^3$, then $X \cong \mathbb{C}^2$.

Examples of $\mathbb{A}^1$-fibrations

▷ It is well-known that the existence of a locally nilpotent derivation action on an affine domain R over $\mathbb{C}$ is equivalent to the action of the affine algebraic group $\mathbb{G}_a := (\mathbb{C}, +)$ on $\operatorname{Spec} R$.

▷ A rich source of examples of $\mathbb{A}^1$-fibrations is the following.

Example 1

Let δ be a non-trivial locally nilpotent derivation acting on the coordinate ring $\Gamma(X, \mathcal{O}_X)$ of a smooth affine variety X such that the ring of constants $\Gamma(X, \mathcal{O}_X)^\delta := \ker(\delta)$ is a finitely generated $\mathbb{C}$-algebra. Let Y be the affine variety corresponding to the ring of constants. Then the induced morphism $\eta : X \to Y$ is an $\mathbb{A}^1$-fibration.

▷ In fact, understanding these examples is very important in Affine Algebraic Geometry.

A result due to Białynicki-Birula

Notation : For a complex analytic variety X, $H_c^i(X)$ denotes the ***cohomology group with compact support*** of X with coefficient group $\mathbb{Z}$.

Białynicki-Birula proved the following result.

Theorem 3

If $\mathbb{G}_a$ acts on an irreducible smooth affine variety X then

$$H_c^i(X) \cong H_c^i(X^{\mathbb{G}_a}) \quad \textit{for } i = 0, 1.$$

Here, $X^{\mathbb{G}_a}$ is the closed subvariety of $\mathbb{G}_a$-fixed points of X.

▷ This is one illustration of the usefulness of topological methods in Affine Algebraic Geometry.

Singular fibers of an $\mathbb{A}^1$-fibration

Let $f : X \to C$ be an $\mathbb{A}^1$-fibration on a smooth affine surface X. Then X has a smooth completion Y such that f extends to a $\mathbb{P}^1$-fibration $f' : Y \to \overline{C}$, where $\overline{C}$ is a smooth completion of C.

▷ A fiber of f' which has at least two irreducible components is called a ***singular fiber*** of f'.

▷ A very useful property of a singular fiber F_0 of f' is:

1. F_0 is connected.

2. F_0 contains at least one (-1)-curve (i.e., a smooth projective rational curve B such that the self-intersection $(B^2) = -1$).

   - If an irreducible component of F_0 is a (-1)-curve and occurse with multiplicity 1 in F_0, then F_0 contains another (-1)-curve.

3. By successively contracting (-1)-curves in F_0 we can blow-down F_0 to a regular fiber on a smooth projective surface $\overline{Y}$.

Using this result the following can be proved.

Theorem 4

If $f : X \to C$ is an $\mathbb{A}^1$-fibration on a smooth affine surface then every fiber of f is a disjoint union of curves isomorphic to $\mathbb{A}^1$, possibly occuring with multiplicities in the fiber.

Connection with locally nilpotent derivations

Theorem 5

Let $f : X \to C$ be an $\mathbb{A}^1$-fibration on a normal affine surface. Suppose C is affine. Then there is an action of a locally nilpotent $\mathbb{C}$-derivation δ on $\Gamma(X, \mathcal{O}_X)$ such that the kernel of δ is $\Gamma(C, \mathcal{O}_C)$.

▷ This can be generalized to the case when $\dim X \geq 3$, but we omit the statement.

▷ Converse of the above result is also true.

$\mathbb{A}^1$-fibrations on singular surfaces

Let $f : X \to C$ be an $\mathbb{A}^1$-fibration on a normal affine surface.

▷ M. Miyanishi proved most of the assertions in the following theorem.

Theorem 6

1. *Every singular fiber of f is a disjoint union of irreducible curves isomorphic to $\mathbb{A}^1$.*
2. *X has atmost cyclic quotient singular points.*
3. *If an irreducible component Δ of a singular fiber occurs with multiplicity 1 in the fiber, then Δ does not contain any singular point of X.*

Some remarks

▷ I gave a simple proof of the previous theorem using topological ideas and proved the smoothness of any irreducible component of a singular fiber.

▷ There are some results of the above type when X is non-normal. We will not mention them here for lack of time.

Higher dimensions

▷ There exist 3-dimensional smooth affine varieties X such that $\bar{\kappa}(X) = -\infty$, but X has no $\mathbb{A}^1$-fibration.

Example 2 (Dubouloz–Kishimoto)

There exists a smooth cubic hypersurface $S \subset \mathbb{P}^3$ such that $\bar{\kappa}(\mathbb{P}^3 - S) = -\infty$, but $\mathbb{P}^3 - S$ has no $\mathbb{A}^1$-fibration.

This is closely related to the following result:

Result 7

$\mathbb{P}^4$ *has smooth cubic 3-folds T which are unirational but not rational.*

Singular fibers

Now we assume $\dim X \geq 3$ and X has an $\mathbb{A}^1$-fibration $f : X \to Y$ onto a normal algebraic variety Y of dimension $\dim X - 1$.

▷ As a sample, we state the following result.

Theorem 8

Let δ be a locally nilpotent derivation on smooth affine 3-fold X and $Y := X/\!\!/\mathbb{G}_a$, where $\mathbb{G}_a$ is the affine algebraic group $(\mathbb{C}, +)$ acting on X corresponding to δ and Y corresponds to the ring of constants $\Gamma(X, \mathcal{O}_X)^\delta := \ker(\delta)$. Let C_0 be a 1-dimensional irreducible component of a fiber F_0 of the quotient morphism $\pi : X \to Y$. Then $C_0 \cong \mathbb{A}^1$.

▷ The proof is somewhat non-trivial.

Many locally nilpotent derivations

Let X be a smooth affine variety, and $\delta_1, \delta_2, \ldots, \delta_r$ "independent" locally nilpotent derivations on $\Gamma(X, \mathcal{O}_X)$.

▷ The case when $r = \dim X$ is of particular interest. Even in this case, when $\dim X > 2$, the structure of X is not well-understood.

Open problems

We end with two unsolved problems.

Question 1

Let X be a smooth affine surface such that $H_i(X;\mathbb{Z}) = 0$ for all $i > 0$. Let δ be a non-trivial locally nilpotent derivation on $\Gamma(X, \mathcal{O}_X)$.

Is $\Gamma(X, \mathcal{O}_X)^\delta := \ker(\delta)$ a regular affine domain?

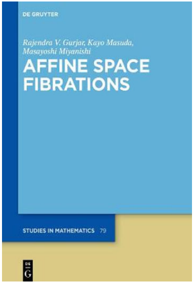
▷ It can be proved that X has at most one singular point and it is of E_8 type.

Question 2

Let X be a smooth affine contractible 3-fold. If X has 3 independent locally nilpotent derivations, is $X \cong \mathbb{C}^3$?

Suggestive book on affine space fibrations

In this lecture I have just scratched the surface of the theory of $\mathbb{A}^1$-fibrations.



- For more details, please see the recently published book *Affine Space Fibrations* by M. Miyanishi–K. Masuda–R.V. Gurjar. It is published by *de Gruyter* of Germany.

- The book also deals with $\mathbb{A}^2$-fibrations.

- This book is based largely on the research works of M. Koras, K. Masuda, M. Miyanishi, P. Russell, and R.V. Gurjar over the past thirty five years.

THANK YOU