The Waring rank of binary binomial forms

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Waring rank

- Let $S := \mathbb{C}[x_0, \ldots, x_n]$ and F be a homogeneous polynomial in S of degree d. It is well-known that there exist linear forms L_i where $r \leq \binom{n+d}{n}$ such that $F = L_1^d + \cdots + L_r^d$.

Waring rank of F

$\mathrm{rk}(F) := \min\{r : F = \sum_{i=1}^{r} L_i^d,\ L_i \in S_1\}$

Example:

$$xy = \frac{1}{4}(x+y)^2 - \frac{1}{4}(x-y)^2$$

Question: Can we write $xy = L^2$ for some linear form L in $\mathbb{C}[x,y]$?

Easy: No !

Hence $\mathrm{rk}(xy) = 2$.

Waring rank of quadratic forms

- Quadratic Form: Let $F = \sum_{1 \leq i,j \leq n} a_{ij} x_i x_j = \mathbf{x}^T A \mathbf{x}$ where A is a symmetric matrix and $\mathbf{x} = (x_0, x_1, \ldots, x_n)^T$
- Diagonalizing the symmetric matrix A we can write $F = y_1^2 + \cdots + y_r^2$ for suitable linear forms $y_i \in S_1$ and $r = \text{rank}(A)$.
- Hence $\text{rk}(F) = \text{rank}(A)$

Waring Problem

The series of problems which ask for information on minimal Waring expansions for forms of degree d is usually called Waring problem for forms.

Why the name Waring problem ?

Lagrange (1770): Every positive integer can be written as the sum of four squares
Waring's conjecture($\sim$ 1770): For all $k \in \mathbb{N}$, there exists a $g(k)$ such that every $a \in \mathbb{N}$ can be written as the sum of at most $g(k)$ kth powers of positive integers.

Hilbert (1909): $g(k)$ exists for every k

$g(k) := \min\{s :$
every integer can be written as the sum of s kth powers$\}$

$G(k) := \min\{s :$
every sufficiently large integer can be written as the sum of s kth powers$\}$

Generic rank

Generic rank

$G(n, d) :=$ rank of the generic degree d form in S

Theorem (Alexander-Hirschowitz'90)

$$G(n+1, d) = \left\lceil \frac{\binom{n+d}{d}}{n+1} \right\rceil \textit{ except } (n, d) = (n, 2), (2, 4), (3, 4), (4, 3), (4, 4).$$

This does not (much) help to compute the rank of a given specific form!

Example

$G(2, 3) = 2$, but $\mathrm{rk}((x_1^3 + x_2^3)) = 2$ and $\mathrm{rk}(x_1 x_2^2) = 3$.

Apolarity

$T = \mathbb{C}[X_0, \ldots, X_n]$. Consider S as a T-module by means of the apolar action:

$$X_0^{a_0} \ldots X_n^{a_n} \circ F := \left(\frac{\partial^{a_0}}{\partial x_0^{a_0}} \cdots \frac{\partial^{a_n}}{\partial x_n^{a_n}} \right) (F)$$

Perp/Apolar ideal

$F^\perp := \{\partial \in T : \partial \circ F = 0\}$ is an ideal of T

- $X_1^2 \circ x_1^2 x_2 = 2x_2$
- $(x_0 x_1)^\perp = (X_0^2, X_1^2)$

Fact: $S/F^\perp$ is an Artinian Gorenstein ring.

Apolarity lemma

$F = L_1^d + \cdots + L_r^d$ if and only if $I(X) \subseteq F^\perp$ where $X = \{[L_1], \ldots, [L_r]\} \subseteq \mathbb{P}^n_{\mathbb{C}}$ is a set of r distinct points.

Sylvester's algorithm

Let $S = \mathbb{C}[x, y]$ and $F \in S_d$.

Recall: $G(2, d) = \lceil \frac{d+1}{2} \rceil$

Fact: $F^\perp = (g_1, g_2)$ where $\text{degree}\, g_1 + \text{degree}\, g_2 = d + 2$.

Sylvester's algorithm

Assume that $\text{degree}\, g_1 \leq \text{degree}\, g_2$. Then

$$\text{rk}(F) = \begin{cases} \text{degree}\, g_1 & \text{if } g_1 \text{ is square-free} \\ \text{degree}\, g_2 & \text{if } g_1 \text{ is not square-free} \end{cases}$$

Example

Let $F = x^a y^b$ where $1 \leq a \leq b$. Then $F^\perp = (X^{a+1}, Y^{b+1})$. Thus $\text{rk}(x^a y^b) = b + 1$.

Carlini-Catalisano-Geramita (2012): Waring rank of monomials in any number of variables

What if F is a binomial ?

Strassen's Conjecture: If $F_1, \ldots, F_m$ are forms in distinct set of variables, then $\mathrm{rk}(F_1 + \cdots + F_m) = \mathrm{rk}(F_1) + \cdots + \mathrm{rk}(F_m)$

Carlini-Catalisano-Geramita (2012): Strassen's conjecture is true if F_i are monomials

Remarks:

- $\mathrm{rk}(x^d + y^d) = 2$
- Let $M_1,\ M_2$ be distinct monomials in $\mathbb{C}[x, y]$. It is easy to see that $\mathrm{rk}(M_1 + M_2) \leq \mathrm{rk}(M_1) + \mathrm{rk}(M_2)$. But the actual rank could be very less.
- For instance, $F = x^2y^3 + x^3y^2$. Here, $\mathrm{rk}(x^2y^3) = \mathrm{rk}(x^3y^2) = 4$. But, $\mathrm{rk}(F) = 3$.

Example

Let $F = x^r y^r (y + x) = x^r y^{r+1} + x^{r+1} y^r$. Then

$$F^{\perp} = (g_1 := X^{r+1} - X^r Y + \cdots + (-1)^{r+1} Y^{r+1}, Y^{r+2}).$$

Notice that g_1 is square-free. Thus $\mathrm{rk}(F) = r + 1$.

Example

Let $F = xy^4 + x^2 y^3$. Then $F^{\perp} = (x^3, g_2)$ where degree $g_2 = 4$. Here $\mathrm{rk}(F) = 4$.

Result

Theorem

Let $F = ax^r y^{s+\alpha} + bx^{r+\alpha} y^s$ *be a binomial form where* $a, b \neq 0$, $0 \leq r \leq s$ *and* $\alpha \geq 1$.

(1) *If* $s \geq r + \alpha$, *then* $\text{rk } F = s + 1$.

(2) *Suppose that* $0 \leq r \leq s < r + \alpha$. *Set* $\delta := r + \alpha - s$. *Then*

$$\text{rk } F = \begin{cases} s+2 & \text{if } r \equiv 0 \mod \alpha \text{ where } \delta \geq 2 \text{ and } r = s \\ r+\alpha-j & \text{if } r \equiv j \mod \alpha \text{ where } 1 \leq j < \lceil \frac{\delta-1}{2} \rceil, \text{ OR } j = 0 \text{ and } r < \\ s+j+1 & \text{if } r \equiv j \mod \alpha \text{ when } \delta \text{ is odd and } j = \frac{\delta-1}{2} \\ s+j & \text{if } r \equiv \delta - j \mod \alpha \text{ where } 1 < j \leq \lceil \frac{\delta-1}{2} \rceil \\ s+1 & \text{if } r \equiv j \mod \alpha \text{ where } \max\{\delta-1, 1\} \leq j \leq \min\{\alpha-1, \\ r+\alpha+1 & \text{if } r \equiv j \mod \alpha \text{ where } \delta+1 \leq j \leq \alpha-1 \end{cases}$$

In particular, the Waring rank of F *is independent of* a *and* b.

Sketch of Proof

- We use Sylvester's algorithm to compute $\mathrm{rk}(F)$
- We computed $g_1 \in F^{\perp}$
- To show that g_1 is a form of least degree in $F^{\perp}$ we computed the Hilbert function of $S/F^{\perp}$
- Depending upon g_1 is square-free or not we have determined the rank of F

Waring Problem over arbitrary field

- Let $S := K[x_0, \dots, x_n]$ where K is a field and F be a homogeneous polynomial in S of degree d. It is well-known that there exist linear forms L_i where $r \leq \binom{n+d}{n}$ such that $F = a_1 L_1^d + \cdots + a_r L_r^d$.

K-Waring rank of F

$\mathrm{rk}_K(F) := \min\{r : F = \sum_{i=1}^r a_i L_i^d,\ L_i \in S_1\}$

- Interesting: $K = \mathbb{R}$
- Generic rank over $\mathbb{R}$ is not known

Apolarity lemma

$F = a_1 L_1^d + \cdots + a_r L_r^d$ if and only if $I(X) \subseteq F^\perp$ where $X = \{[L_1], \dots, [L_r]\} \subseteq \mathbb{P}_K^n$ is a set of r distinct points.

Further problems

- Waring rank of real monomials in any number of variables is not known
- [Boij-Carlini-Geramita(2011)]: $\mathrm{rk}_{\mathbb{R}}(x^a y^b) = a + b$ where $a, b \neq 0$
- No algorithm in $\mathbb{R}[x, y]$!
- What about the Waring rank of real binary binomials?

Proposition

Consider a real binomial $F = x^r y^s(ay^\alpha + bx^\alpha)$ with $ab \neq 0$. For α odd, the real Waring rank of F does not depend on the coefficients a, b. For α even, there are at most two different real Waring ranks for F, depending on the sign of ab.

Examples

Example 1: Let $F = x^r y^r (x \pm y)$ where $r \geq 1$. Then $\mathrm{rk}(F) = r + 1$, but $\mathbb{R}$, $\mathrm{rk}_{\mathbb{R}}(F) = 2r + 1$.

Example 2: We have $\mathrm{rk}_{\mathbb{R}}(x^3 - xy^2) = 3$ and $\mathrm{rk}_{\mathbb{R}}(x^3 + xy^2) = 2$. But $\mathrm{rk}(x^3 \pm xy^2) = \mathrm{rk}(y^3 \pm x^2 y) = 2$ by our Theorem.

Further question

Reznick-Tokcan (2017): Does there exist a binary form of any degree with more than three different ranks (over different fields)

Thank you for the attention!

I wish Prof. Dilip Patil a very happy and healthy life ahead!

References

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