SOME RESULTS ON NUMERICAL SEMIGROUP RINGS

Symposium on superannuation of Prof. Dilip P. Patil

July 29-30, 2021

Indranath Sengupta
IIT Gandhinagar

My entry to the field of numerical semigroups is through the following 3 research manuscripts :

1. Generators for the derivation module and the defining ideals of certain affine curves — D. P. Pātil
Ph.D. thesis ; TIFR Bombay 1989.

2. Generators for the derivation modules and the relation ideals of certain curves — D. P. Patil & Balwant Singh.
Manuscripta Math. 68, 327–335 (1990).

3. Minimal sets of generators for the relation ideals of certain monomial curves — D. P. Patil.
Manuscripta Math. 80, 239–248 (1993).

BASICS OF NUMERICAL SEMIGROUPS

$\mathbb{N} = \{0, 1, 2, \ldots\}$

Defn. A numerical semigroup Γ is an additive submonoid of $\mathbb{N}$ such that $\mathbb{N} \setminus \Gamma$ is a finite set.

Theorem (1) Every numerical semigroup has a unique minimal generating set.

(2) A submonoid Γ of $\mathbb{N}$ is a numerical semigroup if and only if $\gcd(\Gamma) = 1$.

(3) Every nontrivial submonoid of $\mathbb{N}$ is isomorphic to a unique numerical semigroup.

Defn. There exists an integer $c \in \Gamma$, such that $c + i \in \Gamma$ $\forall i \in \mathbb{N}$ and $c-1 \notin \Gamma$. The number c is called the conductor and $c-1$ is called the Frobenius number of Γ.

Defn. Let $\Delta = \{\alpha \in \mathbb{Z}^+ \mid \alpha + \Gamma_+ \subseteq \Gamma\}$, where $\Gamma_+ = \Gamma \setminus \{0\}$. Let $\Delta' = \Delta \setminus F$. The set Δ' is called the pseudo-Frobenius of Γ.

Let $m_0 < m_1 < \cdots < m_{e-1}$ be e positive integers with gcd 1. Let Γ denote the numerical semigroup generated by these integers. Let us write $m = m_0$ — the multiplicity of Γ.

Defn. The Apéry set of S_m with respect to m is defined as

$$S_m = \{ x \in \Gamma \mid x - m \notin \Gamma \}.$$

This is precisely the set of m nonnegative integers giving for each $0 \le i \le m-1$ the smallest integer in Γ congruent to i modulo m.

[Patil–Singh; 1990] If $m_0 < \cdots < m_{e-1}$ form an almost arithmetic (AA) sequence then S can be described explicitly in terms of certain integers.

[Alfonsín & Rødseth; 2009] An approach through continued fractions to calculate the Apéry set of a numerical semigroup generated by AA sequence.

We can write the Apéry set S_m as

$$S_m = \{0,\ k_1 m + 1,\ k_2 m + 2, \ldots\ldots, k_{m-1} m + (m-1)\},$$

where $k_1, \ldots\ldots, k_{m-1}$ are natural numbers.

There are exactly k_i gaps of Γ equivalent to i modulo m.

The number $g = k_1 + \cdots + k_{m-1}$ is called the genus of Γ.

The Frobenius number of Γ is the largest Apéry set element minus m.

One can define the Apéry set w.r.t. any $0 \neq \alpha \in \Gamma$ and the definition is

$$S_\alpha = \{x \in \Gamma \mid x - \alpha \notin \Gamma\}.$$

[Patil - Singh] Let K be a field of characteristic 0. Let $\mathcal{O}$ be a curve in the affine e-space over K, with the relation ideal P. Given e, are there upper bounds on $\mu(\mathrm{Der}_K(\mathcal{O}))$ and $\mu(P)$?

They observed a "striking similarity" in several cases between the behaviours of $\mu(\mathrm{Der}_K(\mathcal{O}))$ and $\mu(P)$.

- $e=1$; then $\mu(P)=0$ and $\mu(\mathrm{Der}_K(\mathcal{O}))=1$
- [Kunz-Waldi; Patil-Singh] $e=2$; then $\mu(P)=1$ and $\mu(\mathrm{Der}_K(\mathcal{O}))\leq 2$.
- $e=3$; $\begin{cases} \text{[Moh; 1979] } \mu(P) \text{ is unbounded.} \\ \text{[Patil - Singh] } \mu(\mathrm{Der}_K(\mathcal{O})) \text{ is unbounded.} \end{cases}$

- Patil – Singh (1990) If $\mathcal{O}$ is a monomial curve defined by an almost arithmetic sequence then $\mu(\mathrm{Der}_K(\mathcal{O})) \leq 2e-3$ and $\mu(P) \leq \frac{e(e-1)}{2}$.
- Patil (1993) Explicit calculation of $\mu(P)$ - the first Betti number of $\mathcal{O}$.
- Patil – Sengupta 1999) $\mu(\mathrm{Der}_K(\mathcal{O}))$ was calculated explicitly for a monomial curve $\mathcal{O}$ defined by an almost arithmetic sequence. This also gives the type and hence the last Betti number of $\mathcal{O}$.
- Gimenez – Sengupta – Srinivasan (2013) Explicitly computed a minimal free resolution a monomial curve $\mathcal{O}$ defined by an arithmetic sequence.
- Roy – Sengupta – Tripathi (2015) Explicit minimal free resolution for a monomial curve $\mathcal{O}$ defined by an almost arithmetic sequence in $e=4$. The general case is still not solved.

- J. Kraft (Can. J. Math. 1985) For an affine monomial curve $\mathcal{O}$, $\mathrm{Der}_K(\mathcal{O})$ is minimally generated by the set

$$\left\{ T^{\alpha+1}\frac{d}{dT} \,\middle|\, \alpha \in \Delta' \cup \{0\} \right\}.$$

- J. Kraft (Thesis; 1983) If $\mathcal{O}$ is a monomial curve and its semigroup is symmetric then $\mu(\mathrm{Der}_K(\mathcal{O})) \leq 2$.

$\boxed{\text{Boundedness of } \mu(P) \text{ is still an open question for arbitrary } e.}$

- Counting the pseudo-Frobenius gives the type and the last Betti number of $\mathcal{O}$ as well.

- Knowledge of Apéry set is important!

An integer programming approach: Ongoing thoughts

Proposition (Rosales et. al.)

Consider the following set of inequalities

$$x_i \geq 1 \qquad \forall\, i \in \{1, \dots, m-1\}$$

$$x_i + x_j \geq x_{i+j} \quad \forall\, 1 \leq i \leq j \leq m-1, \quad i+j \leq m-1,$$

$$x_i + x_j + 1 \geq x_{i+j-m} \quad \forall\, 1 \leq i \leq j \leq m-1, \quad i+j > m,$$

$$x_i \in \mathbb{Z} \text{ for all } i \in \{1, \dots, m-1\}.$$

$$\sum_{i=1}^{m-1} x_i = g.$$

There is a one-one correspondence between semigroups with multiplicity m and genus g and solutions to the above inequalities, where we identify the solution $\{k_1, \dots, k_{m-1}\}$ with the semigroup that has Apéry set $\{k_1 m+1, \dots, k_{m-1} m+(m-1)\}$.

Gröbner basis technique for Integer Programming

"The standard form"

Minimize $c_1 A_1 + \cdots + c_n A_n$ subject to

$$\left.\begin{aligned} a_{11} A_1 + a_{12} A_2 + \cdots + a_{1n} A_n &= b_1 \\ a_{21} A_1 + a_{22} A_2 + \cdots + a_{2n} A_n &= b_2 \\ &\vdots \\ a_{m1} A_1 + a_{m2} A_2 + \cdots + a_{mn} A_n &= b_m \\ A_j \in \mathbb{Z}_{\geq 0}\ ;\ 1 \leq j &\leq n. \end{aligned}\right\} \text{Constraints}$$

n is the total number of variables (including slack variables).

The set of all real n tuples satisfying the constraint equations is called the feasible region.

Translation of the problem into a question about polynomials

Introduce z_i for each of the equations and obtain

$$z_i^{a_{i1}A_1 + \cdots + a_{in}A_n} = z_i^{b_i} \quad \forall \ i = 1, 2, \ldots, m.$$

$$\prod_{j=1}^{n} \left(\prod_{i=1}^{m} z_i^{a_{ij}} \right)^{A_j} = \prod_{i=1}^{m} z_i^{b_i}.$$

Theorem Let K be a field and define $\varphi : K[w_1, \ldots, w_n] \to K[z_1, \ldots, z_n]$ by setting $\varphi(w_j) = \prod_{i=1}^{m} z_i^{a_{ij}} \quad \forall \ j = 1, 2, \ldots, n;$

$$\varphi(g(w_1, \ldots, w_n)) = g(\varphi(w_1), \ldots, \varphi(w_n)).$$

Then $(A_1, \ldots, A_n)$ is an integer point in the feasible region if and only if φ maps the monomial $w_1^{A_1} w_2^{A_2} \cdots w_n^{A_n}$ to $z_1^{b_1} \cdots z_n^{b_n}$.

If we write $f_j = \prod_{i=1}^{m} z_i^{a_{ij}}$, then $K[f_1, \ldots, f_n]$ is the subring of $K[z_1, \ldots, z_m]$ which is the image of φ.
The problem has therefore reduced to a subring membership test.

* This is certainly not the most efficient technique as far as computing is concerned, however, it could be more effective for a detailed understanding of a numerical semigroup and related geometric objects.

Theorem Let $f_1, \ldots, f_n \in K[z_1, \ldots, z_m]$. Fix a monomial order in $K[z_1, \ldots, z_m, w_1, \ldots, w_n]$ with the elimination property: any monomial containing one of the z_i is greater than any monomial containing only the w_j. Let $\mathcal{G}$ be a Gröbner basis for the ideal

$$I = \langle f_1 - w_1, \ldots, f_n - w_n \rangle \subset K[z_1, \ldots, z_m, w_1, \ldots, w_n]$$

and for each $f \in K[z_1, \ldots, z_m]$, let $\overline{f}^{\mathcal{G}}$ be the remainder on division of f by $\mathcal{G}$. Then

(a) A polynomial f satisfies $f \in K[f_1, \ldots, f_n] \iff g = \overline{f}^{\mathcal{G}} \in K[w_1, \ldots, w_n]$.

(b) If $f \in K[f_1, \ldots, f_n]$ and $g = \overline{f}^{\mathcal{G}} \in K[w_1, \ldots, w_n]$ then $f = g(f_1, \ldots, f_n)$.

(c) If each f_j and f are monomials and $f \in K[f_1, \ldots, f_n]$, then g is also a monomial.

* If $z_1^{b_1} \cdots z_m^{b_m} \in \mathrm{im}(\varphi)$ then it is the image of some $w_1^{A_1} \cdots w_n^{A_n}$.

If some of the a_{ij} and b_i are negative then one has to use the ring of Laurent polynomials $K[z_1^{\pm 1}, \ldots, z_m^{\pm 1}]$

$$\simeq K[z_1, \ldots, z_m, t] / \langle t z_1 \cdots z_m - 1 \rangle$$

and an analogous formulation exists.

Question 1 Can one have analogous formulations for the Apéry set with respect to any non-zero element of Γ and also for the pseudo-Frobenius?

Question 2 Can one have analogous formulations for the Apéry table? Knowledge of the Apéry table helps us understand the tangent cone.

PATIL BASIS

Theorem (Patil; Thesis, Pg. 49)

The set $F := \bigcup_{i=1}^{t} \{f(\tau, i) \mid \tau \in \underline{S}_i'\}$ is a set of homogeneous generators for the prime ideal P of the affine monomial curve defined by a sequence of positive integers $m_0, m_1, \ldots, m_{e-1}$ (minimally generating Γ).

$\varepsilon_0 = (1, 0, \ldots, 0)$, $\varepsilon_1 = (0, 1, 0, \ldots, 0)$, $\ldots$, $\varepsilon_{e-1} = (0, 0, \ldots, 0, 1) \in \mathbb{N}^e$.

For $\alpha = (\alpha_0, \ldots, \alpha_{e-1}) \in \mathbb{N}^e$; $\deg(\alpha) = \sum_{i=0}^{e-1} \alpha_i m_i$, $\operatorname{Supp}(\alpha) = \{i \mid \alpha_i \neq 0\}$.

$X^\alpha = X_0^{\alpha_0} \cdots X_{e-1}^{\alpha_{e-1}}$; take the lexicographic order on $\mathbb{N}^{e-1}$.

For $z \in \mathbb{Z}$, let $\mathcal{E}(z) = \{\alpha \in \mathbb{N}^{e-1} \mid \deg(\alpha) = z\}$ - a finite subset of $\mathbb{N}^{e-1}$.

Let $\tau(z) = \max\{\tau(\alpha) \mid \alpha \in \mathcal{E}(z)\}$.

$\underline{S} = \{\tau(s) \mid s \in S_m\}$ and $0 \in \underline{S}$ because $0 = \tau(0)$.

$\underline{S}_i = \{\tau \in \underline{S} \mid \tau + \varepsilon_i \notin \underline{S}\}$ and $\underline{S}_i' = \underline{S}_i \setminus \bigcup_{j=0}^{t} (\underline{S}_i + \varepsilon_j)$.

Theorem (Patil; 1993) Extracted a minimal generating set from the set $\mathcal{F}$, when $m_0, m_1, \ldots, m_{e-1}$ form an almost arithmetic sequence.

Theorem (Sengupta; 2003) The set $\mathcal{F}$ is a Gröbner basis w.r.t. the reverse lexicographic monomial order when $m_0, m_1, \ldots, m_{e-1}$ form an almost arithmetic sequence.

Theorem (Bresinsky, Curtis & Stückrad; 2012) They have named the set $\mathcal{F}$ as Patil basis. They have a generalized notion of Patil basis in terms of a p-degree and have proved that it is a reduced and normalized Gröbner basis w.r.t. a suitable term order.

Wish Prof. Patil a very happy and successful life ahead !

Thank you !